

Generalized derivations in prime rings

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Abstract

Let R be a prime ring and I be a non zero ideal of R . Suppose that $F, G, H : R \rightarrow R$ are generalized derivations associated with derivations d, g, h respectively. If the following holds (i) $F(xy) + G(x)H(y) + [\alpha(x), y] = 0$; for all $x, y \in I$, where α is any map on R , then R is commutative.

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1 Introduction

Throughout this paper, R denotes an associative ring with center $Z(R)$. A ring R is said to be prime ring if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$. We denote operation \circ as a Jordan product which is defined on R as $x \circ y = xy + yx$, for all $x, y \in R$ and Lie product of x, y is defined as $[x, y] = xy - yx \forall x, y \in R$. A mapping $f : R \rightarrow R$ is said to be additive if $f(x + y) = f(x) + f(y)$, for all $x, y \in R$. An additive mapping d from R to R is said to be a derivation, if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Let S be a subset of R , then a mapping $f : R \rightarrow R$ is said to be commuting on S if $[f(x), x] = 0$ for all $x \in S$. A mapping $F : R \rightarrow R$ is said to be left multiplier if $F(xy) = F(x)y$ for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. The concept of generalized derivation introduced by Bresar [8]. Obviously, the class

of generalized derivation is bigger than the class of derivation as every derivation is generalized derivation but not conversely. The concept of generalized derivation also covers the concept of left multiplier maps.

Firstly, E.C. Posner [17] proved pioneer results on derivation in prime rings. He established relation between derivation on a ring and structure of that ring. Many authors have generalized Posner's theorems, for suitable subsets of R , as ideal, left ideal, Lie ideal and Jordan ideal, further information can be found in ([2], [5],[9],[12],[15],[16], [18],[20]). In [6] Bell and Kappe proved that if a derivation d of a prime ring R which acts as homomorphisms or anti-homomorphisms on a nonzero right ideal of R then $d = 0$ on R . In [14] Nadeem-ur rehman generalized Bell and Kappe result by taking generalized derivation instead of derivation. Precisely he proved, let R be a 2-torsion free prime ring and I be a non zero ideal of R . Suppose $F : R \rightarrow R$ is a nonzero generalized derivation with non zero derivation d . If F acts as a homomorphism or anti-homomorphism on I then R is commutative. In this sequence, in 2001 Ashraf and Rehman [4], proved that if R is a prime ring with a non-zero ideal I of R and d is a derivation of R such that either $d(xy) \pm xy \in Z(R)$ for all $x, y \in I$ or $d(xy) \pm yx \in Z(R)$ for all $x, y \in I$, then R is commutative. Again, Asraf et al. [3] proved that if R is a prime ring which is 2 torsion free and F is a generalized derivation associated with derivation d on R . If F satisfies any one of the following conditions: (i) $F(xy) - xy \in Z(R)$; (ii) $F(xy) - yx \in Z(R)$; (iii) $F(x)F(y) - xy \in Z(R)$; (iv) $F(x)F(y) - yx \in Z(R)$, for all $x, y \in I$, where I is an ideal of R , then R is commutative. Recently, Albas [17] studied following identities in prime rings: (i) $F(xy) \pm F(x)F(y) \in Z(R)$; (ii) $F(xy) \pm F(y)F(x) \in Z(R)$, for all $x, y \in I$, a non zero ideal of R .

In 2015, S.K Tiwari et al. [19] considered the following situations: (i) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$; (ii) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$; (iii) $G(xy) \pm F(y)F(x) \pm xy \in Z(R)$; (iv) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$; (v) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$; (vi) $G(xy) \pm F(x)F(y) \pm [\alpha(x), y] \in Z(R)$, for all $x, y \in I$, where I is a non-zero ideal in prime ring R , $\alpha : R \rightarrow R$ is any mapping and F, G are two generalized derivations associated with derivations d, g respectively.

Motivated by above results, in this paper we are considering following situations: (i) $F(xy) + G(x)H(y) + [\alpha(x), y] = 0$; for all $x, y \in I$, where I , a non zero ideal of R , where α is any map on R and F, G, H are generalized derivations

associated with derivations d, g, h respectively.

2 Main Result

Theorem 2.1. *Let R be a prime ring and I be a nonzero ideal of R . Suppose that $F, G, H : R \rightarrow R$ are generalized derivations, associated with derivations $d, g, h : R \rightarrow R$ respectively and $\alpha : R \rightarrow R$ is any map such that $F(xy) + G(x)H(y) + [\alpha(x), y] = 0$, for all $x, y \in I$. If g, h are non zero derivation then R is commutative.*

Proof. We have

$$F(xy) + G(x)H(y) + [\alpha(x), y] = 0 \quad (1)$$

for all $x, y \in I$. Replacing y by yz in (1), we get

$$(F(xy) + G(x)H(y) + [\alpha(x), y])z + xyd(z) + G(x)yh(z) + y[\alpha(x), z] = 0 \quad (2)$$

for all $x, y, z \in I$. Using equation (1), we get

$$xyd(z) + G(x)yh(z) + y[\alpha(x), z] = 0 \quad (3)$$

for all $x, y, z, w \in I$. Replacing y by wy in (3), we get

$$xwyd(z) + G(x)wyh(z) + wy[\alpha(x), z] = 0 \quad (4)$$

for all $x, y, z, w \in I$. Multiplying equation (3), by w from the right and subtracting from the equation (4), we get

$$[x, w]yd(z) + [G(x), w]yh(z) = 0 \quad (5)$$

for all $x, y, z, w \in I$. Replacing $x = w$ in (5), we get $[G(x), x]yh(z) = 0$, for all $x, y, z \in I$. Since R is prime then either $h = 0$ or $[G(x), x] = 0$, for all $x \in I$. As h is a non zero derivation on R , therefore

$$[G(x), x] = 0 \quad (6)$$

for all $x \in I$. Linearising above equation we get $[G(x), y] + [G(y), x] = 0$, for all $x, y \in I$. Replacing x by xy , we obtain, $[xg(y), y] = 0$, for all $x, y \in I$. Again replacing x by tx , we obtain $[t, y]xg(y) = 0$, for all $x, t, y \in I$. Using primness of R and let $A = \{y \in I | g(y) = 0\}$ and $B = \{y \in I | [t, y] = 0\}$. Then A and B are proper additive subgroup of I and $I = A \cup B$. Since a group can not be union of two proper subgroups therefore either $I = A$ or $I = B$. If $I = A$, we get $g = 0$, contradiction therefore $I = B$, hence R is commutative. This proves the theorem. \square

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