Covering matrices of a graph

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Abstract *- Given a Graph G = ((V(G),E(G)), and a subset* $S \subseteq V(G)$*, S with a given property(covering set, Dominating set, Neighbourhood set), we define a matrix taking a row for each of the minimal set corresponding to the given property and a column for each of the vertex of G. The elements of the matrix are 1 or 0 respectively as the vertex is contained in minimal set or otherwise. That is matrix* (m_{ij}) *has elements* m_{ij} *and*

$m_{ij} = 1$ *if* i^{th} *row minimal set contains* j^{th} *vertex*
$= 0$ *otherwise*

This paper initiates a study on these new types of matrices of a graph and we characterize such matrices for some special classes of graphs.

Keywords *— Covering set, dominating set, neighbourhood number, Point covering number, matrix of a graph.*

I. INTRODUCTION

All graphs considered here are finite, undirected and without self loops and without multiple edges. The general notations and definitions conform to Harary[1].

For a graph G = ((V(G),E(G)) let S be a subset of V(G) having a property such as a covering set or a dominating set or a neighbourhood set. If S is dominating set, neighbourhood set, or a point covering set then every super set of S in G also has the corresponding property. Hence it is interesting to study minimal sets with a given property. We consider S to be a minimal set with respect to this property. That is S – {u} does not have this property for any $u \in S$.

We define a matrix corresponding to each of this property as follows: The matrix has a row corresponding to each of the minimal sets and a column for each of the vertices of V(G). The elements of this matrix are 1 if the i^{th} row contains j^{th} vertex, otherwise it is zero.

The purpose of this paper is to study such matrices, especially for the properties namely the minimal point covering set, minimal dominating set and the minimal neighbourhood set. Many matrices including adjacency matrix, incidence matrix, cycle matrix, path matrix are defined and studied in literature. See Harary[1].

A subset $S \subseteq V(G)$ is a point covering set if every edge of G has a vertex in S. The minimum cardinality among all minimal point covering sets is called the point covering number of the graph and is denoted by $\alpha_0(G)$.

A subset $S \subseteq V(G)$ is a dominating set if every vertex in V(G) – S is adjacent to a vertex in S. The minimum cardinality among all minimal dominating sets is called the domination number of the graph and is denoted by $\gamma(G)$. This concept was introduced by O. Ore[2] in 1962 and several authors have studied this concept. The first monograph on domination sets was written by Walikar et al[3]. Two books on this topic are by Haynes et al[7,8].

Sampathkumar and Neeralagi[4] define a set $S \subseteq V(G)$ as a neighbourhood set if G is the union of the graphs induced by the closed neighbourhoods of S i.e.

$$G = \bigcup_{u \in S} \langle N(u) \cup \{ u \} \rangle$$

The minimum cardinality among all minimal neighbourhood sets is called the neighbourhood number of the graph G, denoted by $n_0(G)$. This invariant is studied by Jayaram et al[6], Kulli and Sigarkanti[9]

Definition1.1:

For a parameter t where $t \in \{n_0, \alpha_0, \gamma\}$, $M_t(G)$ denotes the matrix of a graph G corresponding to the parameter t, defined by

$m_{ij} = 1$ if i^{th} row t-minimal set contains j^{th} vertex
$= 0$ otherwise

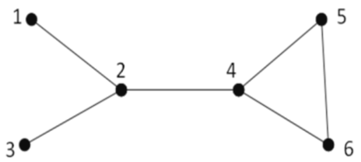
Thus for a graph G we have $M_{n_0}(G)$, $M_{\alpha_0}(G)$ and $M_{\gamma}(G)$ denote the neighbourhood set matrix, point covering set matrix and dominating set matrix respectively.

We consider an example. For the graph

G = ({1, 2, 3, 4, 5, 6}, {12, 23, 24, 45, 46, 56})

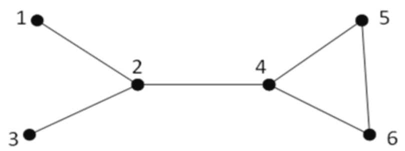
$M_{\gamma}(G)=$

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} \{2,4\} \\ \{2,5\} \\ \{2,6\} \\ \{1,3,4\} \\ \{1,3,5\} \\ \{1,3,6\} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



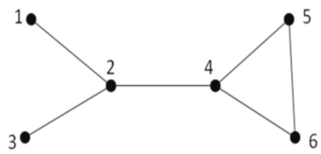
The minimal dominating sets are {1,3,4},{1,3,5},{1,3,6},{2,4},{2,5},{2,6}

$$M_{n_0}(G)= \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} \{2,4\} \\ \{2,5\} \\ \{2,6\} \\ \{1,3,4\} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$



The minimal neighbourhood sets are {1,3,4},{2,4},{2,5},{2,6}

$$M_{\alpha_0}(G)= \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} \{2,4,5\} \\ \{2,4,6\} \\ \{2,5,6\} \\ \{1,3,4,5\} \\ \{1,3,4,6\} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$



The minimal point covering sets are {1,3,4,5},{1,3,4,6},{2,4,5},{2,5,6},{2,4,6}

It is observed that the vertices of a graph need not be labelled. Change of label is equivalent to permuting rows or columns in the matrix representation and the graph itself is not altered by such permutations. Since each row in the matrix corresponds to a minimal set no two vertices of G with the same closed neighbourhood appear in the same row together. In the graph cited above for illustration since vertices 5 and 6 have same closed neighbourhood they do not appear in the same row together.

The following observations are immediate, and can be proved easily by the definitions:

1. Each column sum is the number of minimal sets (with a property) a vertex belongs to.
2. Each row sum is the cardinality of the corresponding minimal sets.
3. A row has only one non-zero entry if and only if $\gamma(G) = n_0(G) = \alpha_0(G) = 1$
4. Since an isolated vertex belongs to both minimal dominating set and minimal neighbourhood set the corresponding column has all entries as 1.
5. Each column has one non-zero entry, equivalently each vertex belongs to at least one minimal set with the said property.
6. If G is a totally disconnected graph on p vertices then $M_{n_0}(G) = M_{\gamma}(G)$ and it has order 1 x p with every entry as unity.

2 Results

In what follows below we obtain some results on these types of matrices for special classes of graphs.

Proposition 2.1

$M_\gamma(G)$ ($M_{n_0}(G)$) is a p x p identity matrix if and only if G $\cong$ K_p

Proof: Let G $\cong$ K_p. Then each vertex is a minimal dominating set and a neighbourhood set and $\gamma(G) = n_0(G) = 1$. So the matrix entries can be arranged such that m_{ii} th place is unity, resulting in an identity matrix.

Conversely if $M_{n_0}(G)$, $M_\gamma(G)$ is an identity matrix, then this implies that each vertex is a minimal dominating set and a minimal neighbourhood set, and so a vertex is adjacent to the remaining vertices hence G $\cong$ K_p.

Proposition 2.2

If G is a complete k-partite graph with partite sets p_1, p_2,…., p_k then $M_{n_0}(G)$ has k rows with sum of all the elements of each column equal to unity and conversely.

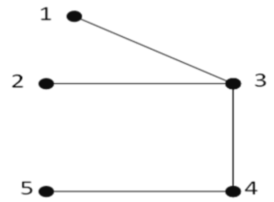
Proof: Let G be a complete k-partite graph with partite sets p_1, p_2,……., p_k, then $M_{n_0}(G)$ has k rows one corresponding to each of the k-partite sets. Also, since each partition set is a minimal neighbourhood set and any two sets are disjoint, each point of G belongs to only one minimal neighbourhood set. Hence G is a k x $\sum p_i$ matrix.

Conversely if $M_{n_0}(G)$ has k rows and the sum of all the elements of each column is unity, then G has k disjoint minimal neighbourhood sets, and hence G is a complete k-partite graph.

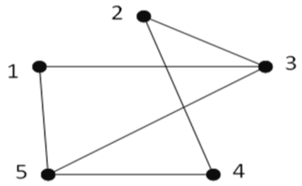
It is clear that if two non-isomorphic graphs should have the same matrix representation, then they should have same number of minimal sets with equal cardinalities. Further, they should be labeled in such a way that $M_\gamma(G)$ = $M_\gamma(H)$ and $M_{n_0}(G)$ = $M_{n_0}(H)$.

A necessary condition for two graphs to have the same neighbourhood set matrix representation is that the graphs should have the same number of minimal neighbourhood sets with equal cardinality.

The graphs G = ({ 1, 2, 3, 4, 5 }, { 13, 23, 34, 45}) and H = ({ 1, 2, 3, 4, 5 }, { 13, 15, 23, 24, 35, 45}) cannot be labelled to have the same neighbourhood set matrix representation, although both have minimal neighbourhood sets of cardinality two. The minimal neighbourhood sets of G are {1, 2, 4}, {3, 4}, {3, 5} and that of H are {1, 2, 4}, {2, 5}, {3, 4}, and hence the order of $M_\gamma(G)$ and $M_{n_0}(G)$ are same.



G: The minimal neighbourhood sets are : {3,4}, {3,5}, {1,2,4}



H: The minimal neighbourhood sets are {1,2,4} , {2,5}, {3,4}

The graph H has two disjoint minimal neighbourhood sets of cardinality two, but G has intersecting minimal neighbourhood sets of cardinality two.

The remarks made earlier lead to

Proposition 2.3

If G and H are two isomorphic graphs then they have the same neighbourhood matrix representation. Conversely if two graphs G and H have the same neighbourhood set matrix representation, then they have a label preserving isomorphism.

Sampathkumar and Neeralgi[4] have shown that there are large classes of graphs for which $\gamma(G) = n_0(G)$. So the next question to be considered here is if $\gamma(G) = n_0(G)$ will the neighbourhood set matrix representation be same as dominating set matrix representation. One can generate families of graphs to answer this negatively. (See earlier cited example.)

This raises the next question does there exist a graph G such that $M_{n_0}(G) = M_\gamma(G)$.

The answer follows.

Proposition 2.4

Following are equivalent for a graph G

i) $M_{n_0}(G) = M_\gamma(G)$

ii) G is one of the graphs K_p, $\overline{K_p}$ (the complement of complete graph on p vertices) $K_p \cup k K_1$ where k is a positive integer (k copies of K_1), $K_{1,p-1}$.

iii) G has disjoint minimal dominating, neighbourhood sets (except for isolated vertices).

Proof: i) $\Rightarrow$ ii) Let G be a graph such that it has same dominating and neighbourhood matrix representation. This is possible only if every minimal dominating set is also a minimal neighbourhood set. Then the structure of such graphs is K_p, $\overline{K_p}$, $K_{1,p}$ since the first graph has each vertex as a minimal dominating set and the second graph has only $V(\overline{K_p})$ as the minimal dominating set and minimal neighbourhood set. The star graph $K_{1,\,p-1}$ has only two disjoint minimal dominating sets and the result holds. Lastly for $K_p \cup k\,K_1$ each vertex of K_p along with all the vertices of $k\,K_1$ forms a minimal dominating set and also a minimal neighbourhood set.

ii) $\Rightarrow$ iii) If G is one of the graphs, then clearly except for the graphs $K_p \cup k\,K_1$ containing k copies of K_1, has isolated vertices contained in each of the minimal dominating, neighbourhood sets.

iii) $\Rightarrow$ i) If G has disjoint minimal dominating sets and also disjoint minimal neighbourhood sets, so they have same neighbourhood and dominating matrix representation hence $M_{n_0}(G) = M_\gamma(G)$

Sampathkumar and Neeralagi[4] proved that every neighbourhood set is also a dominating set. So if $S \subseteq V(G)$ is a neighbourhood set then it contains a dominating set. So it is interesting to see if there exist graphs G such that $M_\gamma(G)$ is a submatrix of $M_{n_0}(G)$.

It is easy to see that

1. For the path P_4 on four vertices, $\gamma(P_4) = 2$, $n_0(P_4) = 2$, $\alpha_0(P_4) = 2$. Also, $M_{n_0}(G) = M_{\alpha_0}(P_n)$, whereas dominating matrix is different.
2. For a cycle C_5 on five vertices, Minimal point covering sets and minimal neighbourhood sets are same and $M_{n_0}(G) = M_{\alpha_0}(P_n)$. But $\gamma(C_5) = 2$, $n_0(C_5) = 3$, $\alpha_0(C_5) = 3$.
3. For a cycle C_4 on four vertices, $\gamma(C_4) = 2$, $n_0(C_4) = 2$, $\alpha_0(C_4) = 2$, $M_{n_0}(G) = M_{\alpha_0}(P_n)$.
4. For a complete graph K_4 on four vertices, $\gamma(K_4) = n_0(K_4) = 1$ and $M_{n_0}(G) = M_\gamma(G)$. However $\alpha_0(K_4) = 3$ and has a different point covering matrix representation.

Given a graph G it is difficult to find all the minimal neighbourhood sets of G. However it can be done for some special classes of graphs. While studying intersection graphs of family of neighbourhood sets of a graph Jayaram et al[8] have enumerated the number of all minimal neighbourhood sets of the graphs P_n, C_n.

In what follows, we characterize matrix representations of P_n, C_n.

Proposition 2.5

The point covering set matrix of P_n, $M_{\alpha_0}(P_n)$ is a f(n) x n matrix, where f(n) is recursively defined as follows: $f(2) = f(3) = 2$, $f(4) = 3$ and for $n \geq 5$, $f(n) = f(n-2) + f(n-3)$.

Proof: Let $V(P_n) = \{1, 2, 3, \ldots, n\}$ and $E(P_n) = \{ i, i+1 \mid 1 \leq i < n\}$.
If n is odd, then it has a unique minimal point covering set namely $\{2, 4, 6, \ldots, 2n\}$ of cardinality n. Any other minimal point covering set has cardinality at least n+1. If n is even then the two disjoint minimal point covering sets are $\{2, 4, 6, \ldots, 2n\}$, and $\{1, 3, 5, \ldots, 2n+1\}$. Let f(n) denote the number of minimal point covering sets of a path on n vertices. It is clear that for n = 2, f(n) = 2, and for $n \geq 3$, a minimal point covering set S is characterized by the property that not both 1 and 2 belong to S, not both n, n-1 belong to S, S does not contain three continuous vertices and no two consecutive vertices can both be omitted from S. The following hold:
1. If S_1 is a minimal point covering set containing n, then n-1 does not belong to S_1 and $S_1 - \{n\}$ is a minimal point covering set of P_{n-1}.
2. If n does not belong to S_1, then n-1 belongs to S_1, the two possibilities are
$n-2 \in S_1$ or it does not belong to, In the former case S_1 is also a minimal point covering set of P_{n-1}.

The total number of minimal point covering sets for some n is listed below:

N	2	3	4	5	6	7	8
f(n)	2	2	3	4	5	7	9

Since all minimal point covering sets of P_n are enumerated, by definition of the point covering set matrix of P_n is a f(n) x n matrix.

Paths P_n are bipartite graphs and have one factors, it is known that for such graphs $n_0(G) = \alpha_0(G)$ (See [4]); also for bipartite graphs every minimal neighbourhood set is a minimal point covering set and conversely, hence the following result follows as a corollary

Corollary 2.6

The Neighbourhood set matrix $M_{n_0}(P_n)$ of P_n, is an f(n) x n matrix, where f(n) is recursively defined as follows:

f(2) = f(3) = 2, f(4) = 3 and for n ≥ 5, f(n) = f(n-2) + f(n-3).

Proposition 2.7

The Neighbourhood set matrix $M_{n_0}(C_n)$ of C_n, n ≥ 4 is a f(n) x n matrix, where f(n) is recursively defined as follows:

f(4) = 2, f(5) = f(6) = 5 and for n ≥ 7, f(n) = f(n-2) + f(n-3).

Proof: The proof is similar to proposition 5, so only outline of proof is given. Let $V(C_n) = \{ 1, 2, \dots, n\}$ and $E(C_n) = \{ i, i+1 \mid 1 \le i < n \} \cup \{ 1, n\}$. If S is a minimal neighbourhood set of C_n then the four types of S are

1. n ε S and 1, n-1 ∉ S
2. both n-1 and n ε S
3. both 1, n ε S
4. n ∉ S

The total number of minimal neighbourhood sets of C_n is f(n) and is listed in the following table for some n the number of vertices of G:

N	4	5	6	7	8	9	10
f(n)	2	5	5	7	10	12	17

So it is now easy to see that the neighbourhood set matrix of C_n is a f(n) x n (0,1) matrix where f(n) is recursively defined as above.

Proposition 2.8

The Neighbourhood set matrix $M_{n_0}(W_n)$ of W_n, n ≥ 4 the wheel on n vertices is an f(n) x n matrix, where f(n) is recursively defined as follows: f(4) = 4, f(5) =3, f(6) = 6 and for n ≥ 7, f(n) = f(n-2) + f(n-3) – 1.

Proof: The proof is similar to that of Proposition 6, Let $V(W_n) = \{ 1, 2, \dots, n-1, n\}$, where n is the central vertex adjacent to all the other n-1 vertices, and $E(W_n) = \{ i, i+1 / 1 \le i < n-1 \} \cup \{ 1, n-1\} \cup \{i, n / 1 \le i < n-1\}$. If S is a minimal neighbourhood set of W_n then the five types of S are

1. n-1 ∈ S and 1, n-2 ∉ S
2. both n-2, n-1 ∈ S
3. both 1, n-1 ∈ S
4. n-1 ∉ S
5. only n ∈ S

The total number of minimal neighbourhood sets of W_n is listed in the following table for some n, the number of vertices of G:

n	4	5	6	7	8	9	10	11	12
f(n)	4	3	6	6	8	11	13	18	23

So it is now easy to see that the neighbourhood set matrix of W_n, $M_{n_0}(W_n)$ is an f(n) x n matrix where f(n) is recursively defined as above.

Proposition 2.9

For K_n, the complete graph on n vertices, the order of $M_{n_0}(G)$ the neighborhood set matrix, $M_\gamma(G)$ the dominating set matrix, and $M_{\alpha_0}(G)$ the point covering set matrix, are same and is equal to n x n

Proof: For K_n each vertex is both a minimal neighborhood set and a minimal dominating set. Hence the corresponding matrices are also the same. Each vertex of K_n forms a minimal neighborhood set, and hence the total number of minimal neighborhood sets are equal to n, consequently the order of $M_{n_0}(G)$ and also the order of $M_\gamma(G)$ are equal to n.

Any subset of (n-1) vertices of K_n forms a minimal point covering set of K_n. There are n different ways of considering n-1 vertices out of n vertices, and hence the total number of minimal point covering sets is equal to n, as a result, the order of $M_{\alpha_0}(G)$ equal to n x n.

3.0 Complement of a Matrix

Definition 3.1: Given a graph G, let $M_t(G)$ be a (0, 1) matrix representation corresponding to a set S, S

$\subseteq$ V(G)with respect to a property. Then we obtain $\overline{M_t(G)}$ called as the complementary matrix of $M_t(G)$, by changing 1 to 0 and 0 to 1 in $M_t(G)$

Then we have the following observations:

Remark 3.1: For K_n, the dominating set matrix $M_\gamma(G)$ and the neighborhood set matrix $M_{n_0}(G)$ are n x n identity matrices and also $M_\gamma(G)$ $= M_{n_0}(G)$.

Let $M_\gamma(G) = M_{n_0}(G) = M(G)$, then $\overline{M(G)} = M_{\alpha_0}(G)$, the point covering matrix where all diagonal elements are zero. Also $\overline{M_{\alpha_0}(G)} = M(G)$.

Remark 3.2: In C_n, the cycles on n vertices, only C_5 & C_6 has the property $\overline{M_{n_0}(G)}$ [= $\overline{M_{\alpha_0}(G)}$] $= M_\gamma(G)$, because in C_n, only C_5 & C_6 has order of $M_{\alpha_0}(G)$, $M_\gamma(G)$ and $M_{n_0}(G)$ same.

Remark 3.3: For $K_{m,n}$, $M_{n_0}(G)$ has 2 rows, one with a set of cardinality m and the other with a set of cardinality n. Hence $\overline{M_{n_0}(G)} = M_{n_0}(G)$ for $K_{m,n}$.

Given a graph G if L(G) denotes the line graph of G , then

Remark 3.4: Since $C_n \cong L(C_n)$, n≥3, we have $M_{n_0}(C_n)$, $M_\gamma(C_n)$ and $M_{\alpha_0}(C_n)$ same as $M_{n_0}(L(C_n))$, $M_\gamma(L(C_n))$ and $M_{\alpha_0}(L(C_n))$ respectively.

Remark 3.5: For P_n, n≥3, we have $L(P_n)=P_{n-1}$, hence $M_{n_0}(L(P_n)) = M_{n_0}(P_{n-1})$, $M_\gamma(L(P_n)) = M_\gamma(P_{n-1})$ and $M_{\alpha_0}(L(P_n)) = M_{\alpha_0}(P_{n-1})$

A Few more observations:

Remark 3.6: For P_n, n ≥ 2 order of $M_{n_0}(G)$ (=order of $M_{\alpha_0}(G)$) ≤ order of $M_\gamma(G)$, since $M_{n_0}(G)$ and $M_{\alpha_0}(G)$ are same for P_n and the number of minimal neighborhood sets (= number of minimal point covering sets) ≤ number of minimal dominating sets

Remark 3.7: For C_n, n ≥ 3, order of $M_{n_0}(G)$ (=order of $M_{\alpha_0}(G)$) ≤ order of $M_\gamma(G)$

Since $M_{n_0}(G)$ and $M_{\alpha_0}(G)$ are same for C_n, and hence order of $M_{n_0}(G)$ = order of $M_{\alpha_0}(G)$ and the number of minimal neighborhood sets (= number of minimal point covering sets) ≤ number of minimal dominating sets

Remark 3.8: For W_n, n≥4 Order of $M_{\alpha_0}(G)$ ≤ order of $M_{n_0}(G)$ ≤ order of $M_\gamma(G)$.

Since number of minimal point covering sets ≤ number of minimal neighbourhood sets ≤ number of minimal dominating sets.

Hence $M_\gamma(G)$ has maximum order, and $M_{\alpha_0}(G)$ has minimum order.

Remark 3.9: For star graphs $K_{1,n-1}$

Order of $M_{n_0}(G)$ = order of $M_{\alpha_0}(G)$ = order of $M_\gamma(G) = 2 \times n$

For star graphs $K_{1,n-1}$ number of minimal neighbourhood sets = number of minimal point covering sets = number of minimal dominating sets, and each is 2

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