Numerical Simulation by Galerkin Method of 2D Nonlinear Convection-Diffusion

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Abstract: *The objective of this paper is to numerically solve a 2D Transient Nonlinear Convection-Diffusion Equation using the Galerkin Method. For numerical formulation, the Crank-Nicolson Method was used for temporal discretization, the Newton Method for linearization of the nonlinear terms, the Galerkin Method for spatial discretization and the Finite Differences Method for calculating the derivatives. Finally, to analyze the results obtained in the applications presented in this work the* L_∞ *norm was used from the comparison with exact solutions.*

Keywords: *Nonlinear Convection-Diffusion, Galerkin Method, Newton Method, Numerical Simulation.*

1. Introduction – Model Equation

In this work we intend to solve numerically the problem of 2D transient nonlinear convection-diffusion governed by the equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial y^2} \quad (1)$$

where u is velocity component in x direction, v is the cinematic viscosity (m²/s). The Equation (1) it is a simplification for only one variable of the 2D Burgers Equation [1-3].

2. Formulation

The following will be presented the methodological sequence for the formulation of the problem governed by Equation (1).

2.1 Temporal Discretization

For temporal discretization, the traditional Cranck-Nicolson method [4] will be used in Equation (1) as follows,

$$\left(\frac{u^{n+1}-u^{n}}{\Delta t}\right) \cong \frac{1}{2}\left(v\frac{\partial^2 u^{n+1}}{\partial x^2}+v\frac{\partial^2 u^{n+1}}{\partial y^2}-u^{n+1}\frac{\partial u^{n+1}}{\partial x}-u^{n+1}\frac{\partial u^{n+1}}{\partial y}\right) + \frac{1}{2}\left(v\frac{\partial^2 u^{n}}{\partial x^2}+v\frac{\partial^2 u^{n}}{\partial y^2}-u^{n}\frac{\partial u^{n}}{\partial x}-u^{n}\frac{\partial u^{n}}{\partial y}\right) \quad (2)$$

2.2 Linearization – Newton Method

Note in Equation (2), in the current step of time n + 1, two non-linear terms. To linearize such terms will be used Newton's Method [5-7] that from the reasoning of the expression,

$$u^{n+1}\frac{\partial u^{n+1}}{\partial x} \cong u^{n+1}\frac{\partial u^{n}}{\partial x}+u^{n}\frac{\partial u^{n+1}}{\partial x}-u^{n}\frac{\partial u^{n}}{\partial x} \quad (3)$$

that when substituted in Equation (2) is obtained,

$$\left(\frac{u^{n+1}-u^{n}}{\Delta t}\right) = \frac{1}{2}\left(v\frac{\partial^2 u^{n+1}}{\partial x^2}+v\frac{\partial^2 u^{n+1}}{\partial y^2}-u^{n+1}\frac{\partial u^{n}}{\partial x}-u^{n}\frac{\partial u^{n+1}}{\partial x}-u^{n+1}\frac{\partial u^{n}}{\partial y}-u^{n}\frac{\partial u^{n+1}}{\partial y}\right) + \frac{1}{2}\left(v\frac{\partial^2 u^{n}}{\partial x^2}+v\frac{\partial^2 u^{n}}{\partial y^2}\right) \quad (4)$$

2.3 Spatial Discretization – Galerkin Method

For spatial discretization of Equation (4) the Galerkin Method [8] will be used by the expression,

$$\int_{\Omega} RW d\Omega = 0 \quad (5)$$

where *W* is adopted as the interpolation functions and *R* is the residual equation. In other words, applying Equation (5) in the result obtained in Equation (4), we obtain,

$$\int_{\Omega}\frac{u^{n+1}}{\Delta t}N_i d\Omega + \frac{1}{2}\int_{\Omega}\left(-v\frac{\partial^2 u^{n+1}}{\partial x^2}-v\frac{\partial^2 u^{n+1}}{\partial y^2}+u^{n+1}\frac{\partial u^{n}}{\partial x}+u^{n}\frac{\partial u^{n+1}}{\partial x}+u^{n+1}\frac{\partial u^{n}}{\partial y} + u^{n}\frac{\partial u^{n+1}}{\partial y}\right)N_i d\Omega = \int_{\Omega}\frac{u^{n}}{\Delta t}N_i d\Omega + \frac{1}{2}\int_{\Omega}\left(v\frac{\partial^2 u^{n}}{\partial x^2}+v\frac{\partial^2 u^{n}}{\partial y^2}\right)N_i d\Omega \quad (6)$$

Using the principle of part integration of Differential and Integral Calculus, the second-order terms of Equation (6) can be simplified as follows,

$$\int_{\Omega} v \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) N_i \, d\Omega = \oint_{\Gamma} v \frac{\partial u}{\partial x} N_i \cos\theta \, d\Gamma - \int_{\Omega} v \frac{\partial N_i}{\partial x}\frac{\partial u}{\partial x} d\Omega \quad (7)$$

$$\int_{\Omega} v \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) N_i \, d\Omega = \oint_{\Gamma} v \frac{\partial u}{\partial y} N_i \operatorname{sen}\theta \, d\Gamma - \int_{\Omega} v \frac{\partial N_i}{\partial y}\frac{\partial u}{\partial y} d\Omega \quad (8)$$

In addition to replacing the results presented in Equations (7-8) in Equation (6) we also replace the approximation

$$u \cong u^e = \sum_{j=1}^{Nnodes} N_j(x,y) u_j^e \qquad (9)$$

(where *Nnodes* is number of nodes in element) results in,

$$\left(\frac{1}{\Delta t} + \frac{1}{2}\frac{\partial u_j^{n,e}}{\partial x} + \frac{1}{2}\frac{\partial u_j^{n,e}}{\partial y}\right) \int_{\Omega^e} N_i N_j \, d\Omega u_j^{n+1,e} + \frac{v}{2}\int_{\Omega^e} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} d\Omega u_j^{n+1,e}$$
$$+ \frac{v}{2}\int_{\Omega^e} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} d\Omega u_j^{n+1,e} + \frac{u_j^{n,e}}{2}\int_{\Omega^e} N_i \frac{\partial N_j}{\partial x} d\Omega u_j^{n+1,e}$$
$$+ \frac{u_j^{n,e}}{2}\int_{\Omega^e} N_i \frac{\partial N_j}{\partial y} d\Omega u_j^{n+1,e}$$
$$= \frac{1}{\Delta t}\int_{\Omega^e} N_i N_j \, d\Omega u_j^{n,e} - \frac{v}{2}\int_{\Omega^e} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} d\Omega u_j^{n,e} - \frac{v}{2}\int_{\Omega^e} \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} d\Omega u_j^{n,e} +$$
$$+ \frac{v}{2}\oint_{\Gamma} \frac{\partial u^n}{\partial x} N_i \cos\theta \, d\Gamma + \frac{v}{2}\oint_{\Gamma} \frac{\partial u^n}{\partial y} N_i \operatorname{sen}\theta \, d\Gamma + \frac{v}{2}\oint_{\Gamma} \frac{\partial u^{n+1}}{\partial x} N_i \cos\theta \, d\Gamma$$
$$+ \frac{v}{2}\oint_{\Gamma} \frac{\partial u^{n+1}}{\partial y} N_i \operatorname{sen}\theta \, d\Gamma \quad (10)$$

In the form of matrix system,

$$[K]\{u_j^{n+1,e}\} = [F] - [H]\{u_j^{n,e}\} \quad (11)$$

where,

$$[K_{ij}] = \left(\frac{1}{\Delta t} + \frac{1}{2}\frac{\partial u_j^{n,e}}{\partial x} + \frac{1}{2}\frac{\partial u_j^{n,e}}{\partial y}\right)\int_{\Omega^e} N_i\, N_j\, d\Omega + \frac{\nu}{2}\int_{\Omega^e}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}d\Omega + \frac{\nu}{2}\int_{\Omega^e}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}d\Omega$$
$$+ \frac{u_j^{n,e}}{2}\int_{\Omega^e} N_i \frac{\partial N_j}{\partial x}d\Omega + \frac{u_j^{n,e}}{2}\int_{\Omega^e} N_i \frac{\partial N_j}{\partial y}d\Omega \quad (12)$$

$$[H_{ij}] = -\frac{1}{\Delta t}\int_{\Omega^e} N_i\, N_j\, d\Omega + \frac{\nu}{2}\int_{\Omega^e}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}d\Omega + \frac{\nu}{2}\int_{\Omega^e}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}d\Omega \quad (13)$$

$$[F_i] = \frac{\nu}{2}\oint_{\Gamma}\frac{\partial u^n}{\partial x}N_i \cos\theta\, d\Gamma + \frac{\nu}{2}\oint_{\Gamma}\frac{\partial u^n}{\partial y}N_i \operatorname{sen}\theta\, d\Gamma + \frac{\nu}{2}\oint_{\Gamma}\frac{\partial u^{n+1}}{\partial x}N_i \cos\theta\, d\Gamma$$
$$+ \frac{\nu}{2}\oint_{\Gamma}\frac{\partial u^{n+1}}{\partial y}N_i \operatorname{sen}\theta\, d\Gamma \quad (14)$$

4. Numerical Applications

To analyze the efficiency of the proposed numerical formulation two applications with exact solutions to perform the comparison are presented. From the solution of the linear system in (11), through a computer code constructed via Fortran Language the numerical results will be presented next. It is important to note that for the construction of the linear system in (11), knowing that this is highly sparse, it used a technique proposed by [9] where three vectors are constructed assuming only the nonzero coefficients of matrix K of system (11). Thus, from these three vectors we construct two integers vectors (one containing the row position and the other the column position of each non-zero coefficient) and a "ordered" real vector of nonzero coefficients. Further details of the advantages of constructing the system in this way can be found in [9]. To analyze the numerical efficiency of the proposed formulation will be used the L_∞ norm that presents the larger absolute error committed in the computer domain.

4.1 First Application

In this first application, it was adopted as spatial domain a square of side 0.1 (L = 0.1) and a total time of 0.1 (L_t = 0.1). The viscosity was varied at values 1, 5 and 10, and the spatial ($\Delta x = \Delta y$) and temporal (Δt) meshes were varied according to Tables 1 to 3. The spatial mesh was constructed from quadrilateral elements with four nodes and the interpolation functions used can be found in [10]. For analysis of the error was used as exact solution the expression,

$$u(x, y, t) = e^{\frac{x-y+2t}{\nu}}$$

Table 1. L_∞ norm for $v = 1$.

$\Delta x = \Delta y$	Δt			
	$L_t/10$	$L_t/20$	$L_t/30$	$L_t/50$
L/10	1.04E-02	2.06E-02	3.01E-02	4.72E-02
L/20	1.03E-02	2.05E-02	2.99E-02	4.69E-02
L/30	1.03E-02	2.04E-02	2.99E-02	4.69E-02
L/50	1.03E-02	2.04E-02	2.99E-02	4.69E-02

Table 2. L_∞ norm for $v = 5$.

$\Delta x = \Delta y$	Δt			
	$L_t/10$	$L_t/20$	$L_t/30$	$L_t/40$
L/10	3.84E-04	7.83E-04	1.18E-03	1.94E-03
L/20	3.81E-04	7.78E-04	1.17E-03	1.93E-03
L/30	3.81E-04	7.77E-04	1.17E-03	1.93E-03
L/50	3.81E-04	7.76E-04	1.17E-03	1.93E-03

Table 3. L_∞ norm for $v = 10$.

$\Delta x = \Delta y$	Δt			
	$L_t/10$	$L_t/20$	$L_t/30$	$L_t/40$
L/10	9.50E-05	1.94E-04	2.93E-04	4.88E-04
L/20	9.44E-05	1.93E-04	2.91E-04	4.85E-04
L/30	9.43E-05	1.93E-04	2.91E-04	4.84E-04
L/50	9.42E-05	1.93E-04	2.91E-04	4.84E-04

From the results presented in Tables 1 to 3 it can be noted that for the spatial refinements adopted, the numerical results are already stagnant, while with the increase in refinement in time, the numerical results decrease accuracy. Next, another application will be presented to assess whether these situations recur.

4.2 Second Application

For this application will only be modified the exact solution proposed that in this case is given by the expression,

$$u(x,y,t) = \frac{1}{1 + e^{\frac{x+y-t}{2v}}}$$

After some numerical tests it was noticed that the behavior of the numerical solution for this case was similar to that presented in application 1, that is, the greater the number of time steps, the lower the numerical precision. However, from detailed studies it was noted that the computational behavior of the first order derivatives of *u* did not show the same numerical efficiency as the numerical solution of *u*, as can be seen in Tables 4 and 5.

Table 4. L_∞ norm for $\nu = 1$, $\Delta x = \Delta y = L/20$, $\Delta t = L_t/10$.

Time steps	**L_∞ norm - *u***	**L_∞ norm - $\partial u/\partial x$**
1	7.28E-05	1.83E-02
2	3.63E-06	1.73E-02
3	8.60E-05	1.64E-02
4	1.71E-04	1.54E-02
5	2.56E-04	1.44E-02
6	3.43E-04	1.38E-02
7	4.29E-04	1.73E-02
8	5.16E-04	2.08E-02
9	6.03E-04	2.43E-02
10	6.89E-04	2.78E-02

Note in the first term on the right side of Equation (12) that there is a need to calculate the derivative of *u*, $\frac{\partial u_j^{n,e}}{\partial x}$ and $\frac{\partial u_j^{n,e}}{\partial y}$, that for the calculations performed and presented in Tables 4 and 5 the Finite Difference Method (Central Difference Method of order 2) was used. After that, we used finite difference expressions of order 4, and the numerical results in the calculation of the first derivative did not evolve.

It is important to note that the numerical results of *u* are good, however, the need to find an alternative to improve the first derivative calculation is clear.

Table 5. L_∞ norm for $\nu = 1$, $\Delta x = \Delta y = L/40$, $\Delta t = L_t/10$.

Time steps	**L_∞ norm - *u***	**L_∞ norm - $\partial u/\partial x$**
1	7.27E-05	1.83E-02
2	3.60E-06	1.73E-02
3	8.59E-05	1.64E-02
4	1.70E-04	1.54E-02
5	2.56E-04	1.44E-02
6	3.42E-04	1.50E-02
7	4.28E-04	1.87E-02
8	5.15E-04	2.25E-02
9	6.01E-04	2.63E-02
10	6.88E-04	3.00E-02

5. Conclusion

The numerical results presented by the use of the Galerkin Method have proved to be efficient, however, it is noted that the calculation of derivatives the *u* at each time step should be improved. For this, a first proposal for future work is to use quadrilateral elements with nine nodes (quadratic elements) and to calculate the derivatives using finite element approximation.

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