# Stability of Stratified Compressible Shear Flow

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**Abstract**

*In the present chapter we have studied the effect of a magnetic field on linear stability of stratified horizontal flows of an in viscid compressible fluid by the generalized progressing wave expansion method. Here we have discussed the different cases and have established the conditions for the stability. It is found that the magnetic field stabilizes the system.*

**Keywords:** *Linear stability, Inviscid compressible fluid shear flows.*

**Introduction:-**

In the present chapter we have studied the effect of a magnetic field on linear stability of stratified horizontal flows of an inviscid compressible fluid by the generalized progressing wave expansion method. Here we have discussed the different cases and have established the conditions for the stability. It is found that the magnetic field stabilizes the system.

[11] and [12] has studied the linear stability of stratified horizontal flows of an inviscid compressible fluid. He has shown that a shear in the horizontal directions always gives rise to instabilities and also that all shear flows are unstable if the external force field vanishes. The same results are also obtained for homogeneous incompressible fluid in [2] and [8]. Previously this problem was discussed by [7] for incompressible fluid, and also by [6], [4] and [19] under various restrictions. Stability in the rotating Bfenard problem with Newton-Robin and fixed heat flux boundary conditions was studied by [3]. Straughan, B. also discussed the problem of Stability and wave motion in porous media [15].

**2. Formulation of the problem:-**

The basic equations for governing the motion of an ideally conducting inviscid, compressible and adiabatic fluid in the presence of a magnetic field [13, 15] are

$$\frac{\partial \nu}{\partial t} + \nu.\nabla\nu = -\frac{1}{\rho}\nabla p + \nabla v + (\nabla \times B) \times B \qquad (2.1)$$

$$\frac{\partial B}{\partial t} = \nabla \times (\nu \times B) \qquad (2.2)$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \nu) = 0 \qquad (2.3)$$

$$\frac{\partial}{\partial t}(p\rho^{-\gamma}) + \nu.\nabla(p\rho^{-\gamma}) = 0 \qquad (2.4)$$

Where $\nu$ denotes the velocity, $\rho$, the density; B, the magnetic field; V, the external potential force; p, the pressure; and $\gamma$ a constant.

The potential for the external forces is assumed to depend on the height z only. Let us introduce a unit vector $\hat{e}$ (z) such that

$$\hat{e}(z) = \cos\phi\ \hat{\imath} + \sin\phi\ \hat{j} \tag{2.5}$$

Where $\hat{\imath}$ and $\hat{j}$ be the unit vectors along the x-axis and y-axis respectively. The angle of $\phi$ depends on z only and

$$\nu = U(x, y, z)\hat{e}(z),\ \ \rho = \rho_0(z), p = p_0(z) \tag{2.6}$$

Since U is constant on every stream line, it follows that

$$\hat{e}(z).\nabla U = \frac{\partial u_0}{\partial x} + \frac{\partial \nu_0}{\partial y} = 0 \tag{2.7}$$

Where $y_0 = U\cos\phi, \nu_0 = U\sin\phi, w_0 = 0$ are the components of the velocity along x, y and z-axes respectively. We obtain from eq. (2.1)

$$\frac{dp_0}{dz} = \rho_0 \frac{dV}{dz} - \nabla^2 B_{01} \tag{2.8}$$

Where we choose $B = B_0(x, y, z)\hat{e}(z)$ and $B_{01} = B_0 \cos\phi$ is the component of magnetic field.

**3. Stability Analysis:-**

To study the stability of basic equations, the following perturbations have been applied:

$$U = \left[u_0 + \left(\frac{1}{\rho_0^{1/2}}\right)u_1\right] \times \left[\nu_0 + \left(\frac{1}{\rho_0^{1/2}}\right)\nu_1\right]\hat{j} + \left[\left(\frac{1}{\rho_0^{1/2}}\right)w_1\right]\hat{k}$$

$$B = \left[B_{01} + \left(\frac{1}{\rho_0^{1/2}}\right)b_1\right]\hat{\imath} + \left[\left(\frac{1}{\rho_0^{1/2}}\right)b_2\right]\hat{j} + \left[\left(\frac{1}{\rho_0^{1/2}}\right)b_3\right]\hat{k} \tag{3.1}$$

$$\rho = \rho_0 + \left[\left(\frac{1}{C_0}\right)\rho_0^{1/2}(S_1 + S_2)\right]$$

$$p = p_0 + C_0 \rho_0^{1/2} S_2$$

Where $C_0 = \left[\frac{\gamma p_0}{\rho_0}\right]^{1/2}$ denotes the local sound speed.

After linearization, the equations for the perturbation are found to be

$$\frac{\partial w}{\partial t}+A_1\frac{\partial w}{\partial x}+A_2\frac{\partial w}{\partial y}+A_3\frac{\partial w}{\partial z}\times Bw=0 \qquad (3.2)$$

Where $w=\begin{bmatrix}u_1 & v_1 & w_1 & b_1 & b_2 & b_3 & S_2 & S_2\end{bmatrix}^T$

is a transpose matrix and coefficient square matrices $A_1$, $A_2$, $A_3$ and B are defined as

$$A_1=\begin{bmatrix}u_1 & 0 & 0 & -B_{01} & 0 & 0 & 0 & C_0\\ 0 & u_0 & 0 & 0 & -B_{01} & 0 & 0 & C_0\\ 0 & 0 & u_0 & 0 & 0 & -B_{01} & 0 & 0\\ 0 & 0 & 0 & u_0 & 0 & 0 & 0 & 0\\ 0 & -B_{01} & 0 & 0 & u_0 & 0 & 0 & 0\\ 0 & 0 & -B_{01} & 0 & 0 & u_0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & u_0 & 0\\ C_0 & 0 & 0 & 0 & 0 & 0 & 0 & u_0\end{bmatrix} \qquad (3.3)$$

$$A_2=\begin{bmatrix}v_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & v_0 & 0 & 0 & 0 & 0 & 0 & C_0\\ 0 & 0 & v_0 & 0 & 0 & 0 & 0 & 0\\ 0 & B_{01} & 0 & v_0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & v_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & v_0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & v_0 & 0\\ 0 & C_0 & 0 & 0 & 0 & 0 & 0 & v_0\end{bmatrix} \qquad (3.4)$$

$$A_3=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_0\\ 0 & 0 & B_{01} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & C_0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \qquad (3.5)$$

and

$$B=\begin{bmatrix} (\partial/\partial x)u_0 & (\partial/\partial y)u_0 & (\partial/\partial z)u_0 & -(\partial/\partial x)B_{01} & -(\partial/\partial y)B_{01} & -(\partial/\partial z)u_0 & 0 & 0 \\ (\partial/\partial x)v_0 & (\partial/\partial y)v_0 & (\partial/\partial z)v_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & G \\ (\partial/\partial x)B_{01} & (\partial/\partial y)B_{01} & (\partial/\partial z)B_{01} & -(\partial/\partial x)u_0 & -(\partial/\partial y)u_0 & -(\partial/\partial z)u_0 & 0 & 0 \\ 0 & 0 & 0 & -(\partial/\partial x)v_0 & -(\partial/\partial y)v_0 & -(\partial/\partial z)v_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

where $\alpha = C_0\left[(1/\rho_0)\left(\frac{d\rho_0}{dz}\right)-\left(\frac{1}{C_0^2}\right)\left(\frac{dV}{dz}\right)\right]$,

$$\beta = -(1/C_0)(dV/dz)$$

$$G=(1/C_0)\left(\frac{1}{2}\gamma-1\right)(dV/dz)$$

And

$$H=(1/C_0)(dV/dz)-\frac{1}{2}\left[(C_0/\rho_0)(d\rho_0/dz)\right]$$

We see that eq. (3.2) is a symmetric hyperbolic system. The characteristic equation associated with eq. (3.2)

$$\det\left[E_1A_1+E_2A_2+E_3A_3-\lambda I\right]=0 \tag{3.7}$$

The above equation gives

$$\left.\begin{aligned} \lambda_1 &= E_1u_0+E_2v_0 \\ \lambda_2 &= E_2u_0+E_2v_0 \\ \lambda_3 &= E_1u_0+E_2v_0-C_0K \\ \lambda_4 &= E_1u_0+E_2v_0+C_0K \\ \lambda_5 &= E_1u_0+E_2v_0-E_1B_{01} \\ \lambda_6 &= E_1u_0+E_2v_0-E_1B_{01} \\ \lambda_7 &= E_1u_0+E_2v_0+E_1B_{01} \\ \lambda_5 &= E_1u_0+E_2v_0+E_1B_{01} \end{aligned}\right\} \tag{3.8}$$

Where the roots $\lambda_3$ and $\lambda_4$ correspond to acoustic waves $\lambda_5$, $\lambda_6$, $\lambda_7$ and $\lambda_8$ correspond to magnetic waves and $\lambda_1$, $\lambda_2$ correspond to interna; gravity waves. We now explore whether gravity and magnetic waves have had impact on the stability of the basic flow.

The ray equations for the gravity waves associated with the characteristic roots $\lambda_1$ and $\lambda_2$ are

$$\left.\begin{aligned}
\frac{dE_1}{dt} &= -\left[E_1\left(\frac{\partial u_0}{\partial x}\right) + E_2\left(\frac{\partial v_0}{\partial x}\right)\right] \\
\frac{dE_2}{dt} &= -\left[E_1\left(\frac{\partial u_0}{\partial y}\right) + E_2\left(\frac{\partial v_0}{\partial y}\right)\right] \\
\frac{dE_3}{dt} &= -\left[E_1\left(\frac{\partial u_0}{\partial z}\right) + E_2\left(\frac{\partial v_0}{\partial z}\right)\right]
\end{aligned}\right\} \qquad (3.9)$$

Transport equations for gravity waves show the stability of the system. The ray equations for the magnetic waves associated with the characteristic roots ($\lambda_5$, $\lambda_6$) and ($\lambda_7$, $\lambda_8$) are

$$\left.\begin{aligned}
\frac{dE_1}{dt} &= -\left[E_1\left(\frac{\partial u_0}{\partial x}\right) \mp \left(\frac{\partial B_{01}}{\partial x}\right) + E_2\left(\frac{\partial v_0}{\partial x}\right)\right] \\
\frac{dE_2}{dt} &= -\left[E_1\left(\frac{\partial u_0}{\partial y}\right) \mp \left(\frac{\partial B_{01}}{\partial y}\right) + E_2\left(\frac{\partial v_0}{\partial y}\right)\right] \\
\frac{dE_3}{dt} &= -\left[E_1\left(\frac{\partial u_0}{\partial z}\right) \mp \left(\frac{\partial B_{01}}{\partial z}\right) + E_2\left(\frac{\partial v_0}{\partial z}\right)\right]
\end{aligned}\right\} \qquad (3.10)$$

Where we also assume that the generalized progressive wave solution is given as

$$a_0\,(x, y, z\ t)\ \exp\,[iw\phi(x, y, z, t)] \qquad (3.11)$$

Where w denotes the frequency parameter and solar function $\phi(x, y, z, t)$ is called the phase function. In addition to eqns. (3.9) and (3.10), we also have

$$\frac{dx}{dt} = u_0, \qquad \frac{dy}{dt} = v_0, \qquad \frac{dz}{dt} = 0 \qquad (3.12)$$

The solution of eqns. (3.9), (3.10) are found to be

$$E_1 = E_{01} - \left[E_{01}\left(\frac{\partial u_0}{\partial x}\right) + E_{02}\left(\frac{\partial v_0}{\partial x}\right)\right]t$$

$$E_2 = E_{02} - \left[E_{01}\left(\frac{\partial u_0}{\partial y}\right) + E_{02}\left(\frac{\partial v_0}{\partial y}\right)\right]t \qquad (3.13)$$

$$E_3 = E_{03} - \left[E_{01}\left(\frac{\partial u_0}{\partial z}\right) + E_{02}\left(\frac{\partial v_0}{\partial z}\right)\right]t$$

$$+\frac{1}{2}\left[E_{01}\left(\frac{\partial v_0}{\partial x}\right) + E_{02}\left(\frac{\partial v_0}{\partial x}\right)\right]\left(\frac{\partial u_0}{\partial z}\right)$$

$$+\left[E_{01}\left(\frac{\partial v_0}{\partial y}\right) + E_{02}\left(\frac{\partial v_0}{\partial y}\right)\right]\left(\frac{\partial u_0}{\partial z}\right)t^2$$

For gravity waves and

$$\left.\begin{aligned}
E_1 &= E_{01} - \left[E_{01}\left[\left(\frac{\partial u_0}{\partial x}\right)+\left(\frac{\partial B_{01}}{\partial x}\right)\right]+E_{02}\left(\frac{\partial v_0}{\partial x}\right)\right]t \\
E_2 &= E_{02} - \left[E_{01}\left[\left(\frac{\partial u_0}{\partial y}\right)+\left(\frac{\partial B_{01}}{\partial y}\right)\right]+E_{02}\left(\frac{\partial v_0}{\partial y}\right)\right]t \\
E_3 &= E_{03} - \left[E_{01}\left[\left(\frac{\partial u_0}{\partial z}\right)+\left(\frac{\partial B_{01}}{\partial z}\right)\right]+E_{02}\left(\frac{\partial u_0}{\partial z}\right)\right]t \\
&+\frac{1}{2}\left[E_{01}\left[\left(\frac{\partial u_0}{\partial x}\right)+\left(\frac{\partial B_{01}}{\partial x}\right)\right]+E_{02}\left(\frac{\partial v_0}{\partial x}\right)\right]\left[\left(\frac{\partial u_0}{\partial z}\right)\right) \\
&+\left[E_{01}\left[\left(\frac{\partial u_0}{\partial y}\right)+\left(\frac{\partial B_{01}}{\partial y}\right)\right]+E_{02}\left(\frac{\partial v_0}{\partial x}\right)\right]\left(\frac{\partial v_0}{\partial z}\right)\right]t^2
\end{aligned}\right\} \qquad (3.14)$$

For magnetic waves, Here

$x = x_0 + u_0\,(x_0, y_0, z_0)\,t,\; y = y_0 + v_0\,(x_0, y_0, z_0)\,t,\; z$

$= z_0$ (3.15)

Where $x_0, y_0, z_0, E_{01}, E_{02}, E_{03}$ denotes the initial values at $t = 0$.

We obtain the transport equations for the magnetic waves by following [13], where the amplitude $a_0$ must have the form

$a_0 = a_1\gamma_1 + \sigma_2\gamma_2$ (3.16)

where$\sigma_1$ and $\sigma_2$ are determined by the transport equation which we shall give below, and the orthogonal eigenvectors $\gamma_1$ and $\gamma_2$ are given by

$$\gamma_1 = K^{-1}\begin{bmatrix} 0 \\ -E_3 \\ E_3 \\ 0 \\ -E_3 \\ E_2 \\ \sqrt{2}E_1 \\ 0 \end{bmatrix}, \gamma_2 = K^{-1}\begin{bmatrix} -E_1 \\ E_1 \\ 0 \\ -E_2 \\ E_1 \\ 0 \\ \sqrt{2}E_3 \\ 0 \end{bmatrix} \qquad (3.17)$$

Where $K = \sqrt{2\left(E_1^2+E_2^2+E_3^2\right)}$

With the help of eqns. (3.6), (3.16) and (3.17), we obtain the transport equations [12] for the magnetic waves

$$\frac{d\sigma_1}{dt} = -\left[\left(\frac{1}{K^2}\right)\left[\sqrt{2}E_1E_2(\alpha+\beta)\right]\sigma_1 + \left(\sqrt{2}E_3E_2\beta - K_1\right)\sigma_2\right] \tag{3.18}$$

$$\frac{d\sigma_2}{dt} = \left[\left(\frac{1}{K^2}\right)\left[\sqrt{2}E_3E_2\alpha + K_1\right]\sigma_1\right] \tag{3.19}$$

Where

$$K_1 = 2\left[E_1^2\left(\frac{\partial u_0}{\partial z}\right) + E_1E_2\left(\frac{\partial v_0}{\partial z}\right) - E_3E_1\left(\frac{\partial u_0}{\partial x}\right) - E_3E_2\left(\frac{\partial v_0}{\partial x}\right)\right]$$

We can rewrite the above set of eqns. (3.18) and (3.19) in the form

$$(d/dt)\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix}$$

$$= -\left(\frac{1}{K^2}\right)\begin{bmatrix}\sqrt{2}E_1E_2(\alpha+\beta) & \sqrt{2}E_3E_2\beta - K_1\\ \sqrt{2}E_3E_2\alpha + K_1 & 0\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} \tag{3.20}$$

Eq. (3.20) is nothing but simply the form of the ordinary differential equations

$$\frac{d\sigma}{dt} = [A]\sigma \tag{3.21}$$

For the amplitude $\sigma = [\sigma_1, \sigma_2]$ of the magnetic waves.

**4. Discussion and Conclusions:-**

Stability properties can be determined after finding eigen values of the coefficient matrix [A], i.e.

$$\eta_{1,2} = \frac{1}{2}\Big[[\sqrt{2}E_1E_2(\alpha+\beta) \pm [2E_1^2E_2^2(\alpha+\beta)^2 + 4[E_1^2E_2^2(\alpha+\beta)^2]$$

$$-\sqrt{2}K_1E_3E_2(\alpha-\beta) - K_1^2]\Big] \tag{4.1}$$

To determine the condition for stability of the system [9], the following different cases have to be considered.

Case (i)

Let $E_1 = 0 = \left(\frac{\partial v_0}{\partial x}\right)$ and $E_2 \neq 0, E_3 \neq 0.$ Eq. (4.1) is reduced to the form

$$\eta_{1,2} = E_2E_3\sqrt{\alpha\beta} \tag{4.2}$$

The system would be stable if $N^2 > 0$, $N^2 = -\alpha\beta$

Case (ii)

If $E_2 = 0$ and $E_1 \neq 0$, $E_3 \neq 0$, we obtain

$\eta_1 = iK_1$ and $\eta_2 = -\eta_1$ (4.3)

This shows that the system would be stable if $K_1 > 0$, i.e.

$$E1\left(\frac{\partial u_0}{\partial z}\right) > E_3\left(\frac{\partial u_0}{\partial x}\right) \tag{4.4}$$

Case (iii)

Let $E_3 = 0$ and $E_1 \neq 0$, $E_2 \neq 0$. Eq. (4.1) is reduced to the form

$$\eta_{1,2} = \frac{1}{2}\left[\sqrt{2}E_1E_2(\alpha+\beta) \pm [2E_1^2E_2^2(\alpha+\beta)^2 - 4K^2]^{1/2}\right] \tag{4.5}$$

If $\alpha + \beta = 0$, then the system reduces to case (ii).

Case (iv)

If $E_1 = 0 = E_2$ and $E_3 \neq 0$. It gives a stable solution as the coefficient matrix [A] becomes zero.

Case (v)

If $E_3 = 0 = E_1$ and $E_2 \neq 0$. If reduces to case (iv).

Case (vi)

Let $E_3 = 0 = E_2$ and $E_1 \neq 0$. We obtain

$$\eta_1 = -2iE_1\left(\frac{\partial u_0}{\partial z}\right), \eta_2 = -\eta_1 \tag{4.6}$$

It becomes stable if $E_1\left(\frac{\partial u_0}{\partial z}\right) > 0.$

Finally we conclude that basic flow shall be stable if $\alpha\beta \geq 0$ (4.7)

and

$$\left(\frac{\partial u_0}{\partial x}\right) = \left(\frac{\partial v_0}{\partial x}\right) = \left(\frac{\partial u_0}{\partial y}\right) = \left(\frac{\partial v_0}{\partial y}\right) = \left(\frac{\partial \rho_0}{\partial z}\right) = \left(\frac{\partial v}{\partial z}\right)$$

$$= \left(\frac{\partial u_0}{\partial z}\right) = \left(\frac{\partial v_0}{\partial z}\right) = \left(\frac{\partial B_{01}}{\partial x}\right) = \left(\frac{\partial B_{01}}{\partial y}\right) = \left(\frac{\partial B_{01}}{\partial z}\right)$$

$$= 0 \tag{4.8}$$

This result agrees with that obtained by Storesletten (1982).

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