Sum Perfect Square Graphs in context of some graph operations

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Abstract

A (p, q) graph $G = (V, E)$ is called sum perfect square if for a bijection $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ there exists an injection $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(G)$. Here f is called sum perfect square labeling of G. In this paper we prove that P_n^2, K_n^2, $K_{1,n} \cup K_{1,n+1}$, mC_n, $mK_{1,n}$, $spl(K_{1,n})$ and $D_2(K_{1,n})$ are sum perfect square graphs. Further we prove that the union of path graph with any sum perfect square graph is also sum perfect square graph.
Key words: Sum perfect square graph.
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1 Introduction

Sonchhatra and Ghodasara[4] initiated the study of sum perfect square graphs. Due to [4] it becomes possible to construct a graph, whose all edges can be labeled by different perfect square integers. In the same paper the authors proved that P_n, C_n, C_n with one chord, C_n with twin chords, tree, $K_{1,n}$, $T_{m,n}$ are sum perfect square graphs. Sonchhatra and Ghodasara[5] proved that several snakes related graphs are sum perfect square. The same authors found some new sum perfect square graphs in [6].

2 Literature survey and Previous work

Throughout this paper we consider graph $G = (p, q)$ (with p vertices and q edges) to be simple, finite and undirected. The set of vertices and edges of G are denoted by $V(G)$ and $E(G)$ respectively. For all other terminology and notations we follow Harary[1].

Definition 2.1 ([4]). *Let $G = (p, q)$ be a graph. A bijection $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ is called sum perfect square labeling of G, if the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$ is injective, $\forall uv \in E(G)$.*
A graph which admits sum perfect square labeling is called sum perfect square graph.

Definition 2.2 ([3]). *The square of a graph G is denoted by G^2, where $V(G^2) = V(G)$ and two vertices $u, v \in V(G^2)$ are adjacent in G^2 if and only if $d(u, v) \leq 2$ in G, where $d(u, v)$ denotes the distance between u and v.*

Definition 2.3 ([3]). *The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1 \cup G_2$, where $V(G_1 \cup G_2) = V_1 \cup V_2$ and $E(G_1 \cup G_2) = E_1 \cup E_2$.*

Definition 2.4 ([3]). *For a graph G the splitting graph $spl(G)$ is obtained from G by taking two copies of G say G' and G'' adding for each vertex v' of G' a new vertex v'' of G'' so that v' is adjacent to every vertex that is adjacent to v''.*

Definition 2.5 ([3]). *For a graph G the shadow graph is denoted by $D_2(G)$ is obtained from G by*

adding a new vertex v' corresponding to a vertex v of G, so that v' is adjacent to every vertex that is adjacent to v.

3 Main Results

Theorem 3.1. *P_n^2 is sum perfect square graph, $\forall n \in \mathbb{N} - \{1\}$.*

Proof. Let $V(P_n^2) = \{v_i; 1 \leq i \leq n\}$ and $E(P_n^2) = \{e_i^{(1)} = v_i v_{i+1}, 1 \leq i \leq n-1\} \cup \{e_i^{(2)} = v_i v_{i+2}, 1 \leq i \leq n-2\}$.
$|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n-3$.
We define a bijection $f : V(P_n^2) \to \{0, 1, 2, \ldots, n-1\}$ by
$f(v_i) = i-1, 1 \leq i \leq n$.
Let $f^* : E(P_n^2) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(P_n^2)$.
Injectivity for edge labels:
For $1 \leq i \leq n-1$, $f^*(e_i^{(1)})$ is increasing in terms of i.
$\Rightarrow f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2})$, $1 \leq i \leq n-3$.
Similarly $f^*(e_i^{(2)})$ is increasing for $1 \leq i \leq n-2$.
Further $f^*(e_i^{(1)})$ are odd perfect square integers and $f^*(e_i^{(2)})$ are even perfect square integers.
So $f^* : E(P_n^2) \to \mathbb{N}$ is injective and hence P_n^2 is sum perfect square graph, $\forall n \in \mathbb{N} - 1$. □

The below illustration provides better idea about the above defined labeling pattern.

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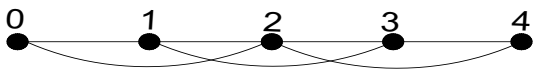

Figure 1 : Sum perfect square labeling of P_5^2.

Theorem 3.2. *K_n^2 is sum perfect square graph, $\forall n \in \mathbb{N} - \{1\}$, $n < 4$.*

Proof. We know that $K_n^2 \cong K_n$ and K_n is sum perfect square graph, for $n < 4$ (See Theorem 3.5 in [4]). Hence K_n^2 is sum perfect square graph, $\forall n \in \mathbb{N} - 1$, $n < 4$. □

Theorem 3.3 ([4]). *P_n is sum perfect square graph, $\forall n \in \mathbb{N}$.*

Corollary 3.4. *For any sum perfect square graph G, $G \cup P_m$ is also sum perfect square graph, $\forall m \in \mathbb{N}$.*

Proof. Let G be a sum perfect square graph of order n. Consider a sum perfect square labeling $g : V(G) \to \{0, 1, 2, \ldots, n-1\}$. By shifting the range set of sum perfect square labeling defined for P_m(See [4] lemma 3.1) to $\{n, n+1, \ldots, n+m-1\}$ and combination of above two labelings gives required sum perfect square labeling. □

Corollary 3.5. *$K_{1,m} \cup K_{1,n}$ is sum perfect square graph, $\forall m, n \in \mathbb{N}$.*

Proof. Let $V(K_{1,m} \cup K_{1,n}) = \{u, u_i; 1 \leq i \leq m\} \cup \{v, v_i; 1 \leq i \leq n\}$ and $E(K_{1,m} \cup K_{1,n}) = \{uu_i; 1 \leq i \leq m\} \cup \{vv_i; 1 \leq i \leq n\}$.
$|V(K_{1,m} \cup K_{1,n})| = m+n+2$ and $|E(K_{1,m} \cup K_{1,n})| = m+n$.
We define a bijection $f : V(K_{1,m} \cup K_{1,n}) \to \{0, 1, \ldots, m+n+1\}$ by
$f(u) = 0$.
$f(u_i) = i; 1 \leq i \leq m$.
$f(v) = m+n+1$.
$f(v_i) = m+i; 1 \leq i \leq n$.
Let $f^* : E(K_{1,m} \cup K_{1,n}) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(K_{1,m} \cup K_{1,n})$.
Injectivity for edge labels:
As f is strictly increasing(for increasing values of i) we get $\{f^*(uu_i); 1 \leq i \leq m\}$ and $\{f^*(vv_i); 1 \leq i \leq n\}$ are distinct.
Also $\max\{f^*(uu_i)\} = m^2 > \min\{f^*(vv_i)\} = (2m+n+2)^2$. Therefore f^* is injective and hence $K_{1,m} \cup K_{1,n}$ is sum perfect square graph, $\forall m, n \in \mathbb{N}$. □

The below illustration provides better idea about the above defined labeling pattern.

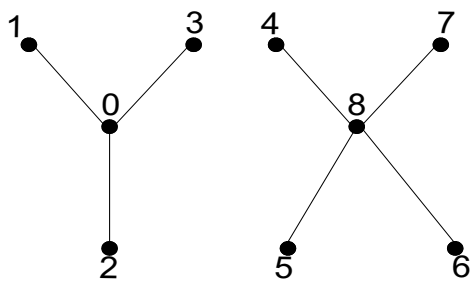

Figure 2 : Sum perfect square labeling of $K_{1,3} \cup K_{1,4}$.

However the authors strongly believe that the union of any two sum perfect square graphs is also a sum perfect square graph. So authors put the following conjecture.

Conjecture 3.6. *For any two sum perfect square graphs G and H, $G \cup H$ is sum perfect square.*

Theorem 3.7. *$mK_{1,n}$ is sum perfect square graph, $\forall m, n \in \mathbb{N}$.*

Proof. Let $V(mK_{1,n}) = \{u_i; 1 \leq i \leq m\} \cup \{v_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$, where $\{u_i; 1 \leq i \leq m\}$ is the apex vertex of ith copy of $K_{1,n}$ and $E(mK_{1,n}) = \{u_i v_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$.
$|V(mK_{1,n})| = m(n+1)$ and $|E(mK_{1,n})| = mn$.
We define a bijection $f : V(mK_{1,n}) \to \{0, 1, 2, \ldots, m(n+1)-1\}$ by

$f(u_i) = i - 1; 1 \leq i \leq m$.
$f(v_{i1}) = m + (i-1)n$, $1 \leq i \leq m$.
$f(v_{ij}) = f(v_{i1}) + (j-1)$, $1 \leq i \leq m$, $2 \leq j \leq n$.
Let $f^* : E(mK_{1,n}) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(mK_{1,n})$.
Injectivity for edge labels:
As f is increasing (for increasing values of i and j) all $\{f^*(u_i v_{ij}), 1 \leq i \leq m, 1 \leq j \leq n\}$ are distinct. Moreover the largest value of $f^*(u_t v_{tj})$ will be smaller than the smallest value of $f^*(u_{t+1} v_{(t+1)j})$, $1 \leq t \leq m-1$, $1 \leq j \leq n$. Hence the induced edge labeling $f^* : E(mK_{1,n}) \to \mathbb{N}$ is injective and so $mK_{1,n}$ is sum perfect square graph, $\forall n \in \mathbb{N}$. □

Theorem 3.8. *mC_n is sum perfect square graph, $\forall m, n \in \mathbb{N}$, $n \geq 3$.*

Proof. Let $V(mC_n) = \{v_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\}$, and $E(mC_n) = \{u_{ij} v_{(i+1)j}; 1 \leq i \leq n, 1 \leq j \leq m\}$. $|V(mC_n)| = mn$, $|E(mC_n)| = mn$.
We define a bijection $f : V(mC_n) \to \{0, 1, 2, \ldots, mn-1\}$ by

$$f(v_{ij}) = \begin{cases} 2i - 2 + (j-1)n; 1 \leq i \leq \lceil \frac{n}{2} \rceil. \\ 2(n-i) + 1 + (j-1)n; \lceil \frac{n}{2} \rceil + 1 \leq i \leq n. \end{cases}$$

and $1 \leq j \leq m$.
Injectivity for edge labels:
For any one copy of mC_n, it follows from theorem 3.2 [4] that induced edge labels are injective.
Moreover the largest edge label of tC_n is smaller than the smallest edge label of $(t+1)C_n$, $1 \leq t \leq m-1$.
Hence the induced edge labeling $f^* : E(mC_n) \to \mathbb{N}$ is injective.
Therefore mC_n is sum perfect square graph, $\forall m, n \in \mathbb{N}$, $n \geq 3$. □

Theorem 3.9. *$spl(K_{1,n})$ is sum perfect square graph, $\forall n \in \mathbb{N}$.*

Proof. Let $V(spl(K_{1,n})) = \{u, u_i; 1 \leq i \leq n\} \cup \{v, v_i; 1 \leq i \leq n\}$ and $E(spl(K_{1,n})) = \{uu_i; 1 \leq i \leq n\} \cup \{vv_i; 1 \leq i \leq n\} \cup \{uv_i; 1 \leq i \leq n\} \cup \{vu_i; 1 \leq i \leq n\}$.
$|V(spl(K_{1,n}))| = 2n + 2$, $|E(spl(K_{1,n}))| = 4n$.
We define a bijection $f : V(spl(K_{1,n})) \to \{0, 1, \ldots, 2n+1\}$ by
$f(u) = 0$.
$f(u_i) = i; 1 \leq i \leq n$.
$f(v) = 2n + 1$.
$f(v_i) = n + i; 1 \leq i \leq n$.
Let $f^* : E(spl(K_{1,n})) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(spl(K_{1,n}))$.
Injectivity for edge labels:
We note that $f^*(uu_i)$ is increasing for increasing values of i, $1 \leq i \leq n$ and hence all $f^*(uu_i)$ are distinct. Similarly $f^*(vv_i)$, $f^*(uv_i)$ and $f^*(vu_i)$ are also distinct.
Moveover

(1) $\max\{f^*(uu_i)\} \leq \min\{f^*(vv_i), f^*(uv_i), f^*(vu_i)\}$.

(2) $\min\{f^*(vv_i)\} \geq \max\{f^*(uv_i), f^*(vu_i)\}$.

(3) $\max\{f^*(uv_i)\} < \min\{f^*(vu_i)\}$.

$f^* : E(spl(K_{1,n})) \to \mathbb{N}$ is injective and hence $spl(K_{1,n})$ is sum perfect square graph, $\forall n \in \mathbb{N}$. □

The below illustration provides better idea about the above defined labeling pattern.

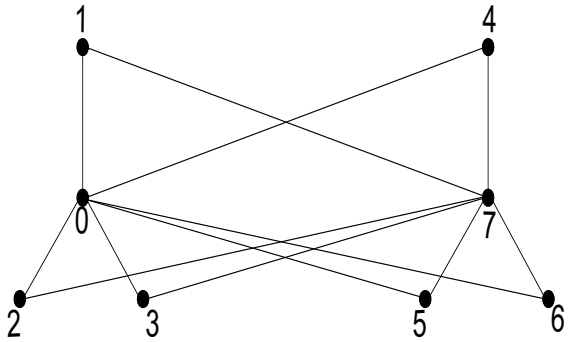

Figure 3 : Sum perfect square labeling of $spl(K_{1,3})$.

Theorem 3.10. *Shadow graph of a star graph $D_2(K_{1,n})$ is sum perfect square, $\forall n \in \mathbb{N}$.*

Proof. Let $V(D_2(K_{1,n})) = \{u, u_i; 1 \leq i \leq n\} \cup \{v, v_i; 1 \leq i \leq n\}$ and $E(D_2(K_{1,n})) = \{uu_i; 1 \leq i \leq n\} \cup \{uv_i; 1 \leq i \leq n\} \cup \{vu_i; 1 \leq i \leq n\}$.
$|V(D_2(K_{1,n}))| = 2n + 2$ and $|E(D_2(K_{1,n}))| = 3n$.
We define a bijection $f : V(D_2(K_{1,n})) \to \{0, 1, \ldots, 2n+1\}$ by
$f(u) = 0$.
$f(u_i) = i; 1 \leq i \leq n$.
$f(v) = 2n + 1$.
$f(v_i) = n + i; 1 \leq i \leq n$.
Let $f^* : E(D_2(K_{1,n})) \to \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(D_2(K_{1,n}))$.
Injectivity for edge labels:
We note that $f^*(uu_i)$ is increasing for increasing values of i, $1 \leq i \leq n$ and hence all $f^*(uu_i)$ are distinct. Similarly $f^*(vu_i)$ and $f^*(vu_i)$ are also distinct.
Moveover

(1) $\max\{f^*(uu_i)\} < \min\{f^*(uv_i), f^*(vu_i)\}$.

(2) $\max\{f^*(uv_i)\} < \min\{f^*(vu_i)\}$.

$f^* : E(D_2(K_{1,n})) \to \mathbb{N}$ is injective. Hence $D_2(K_{1,n})$ is sum perfect square graph, $\forall n \in \mathbb{N}$. □

The below illustration provides better idea about the above defined labeling pattern.

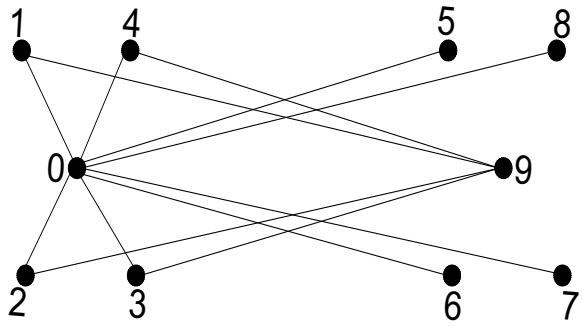

Figure 4 : Sum perfect square labeling of $D_2(K_{1,4})$.

4 Conclusion

In this paper several sum perfect square graphs have been found in context of the graph operations, which are union of graphs, splitting of a graph and shadow graph. A conjecture has been posed related to union of any two sum perfect square graphs.

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