SP-RING AND ITS PROPERTIES

K.Shanmuga Priya[a], M.Mullai[b] *
Department of Mathematics, Alagappa University, Karaikudi, India

Abstract

Algebra is largely concerned with the study of abstract sets endowed with one or more binary operation. In this paper, an algebraic structure known as "SP-RING", which is an extension of SP-Algebra has been introduced. The definition of SP-Ring, integral domain, some theorems, lemmas, properties and unique factorization theorem are also defined and discussed briefly.

Keywords: SP-Ring, Integral domain, Euclidean domain, Unique Factorization theorem.

2010 Mathematics Subject Classification: 08A30, 08A35, 08A40

1 Introduction

In 1978 K.Iseki and S.Tanaka[4] introduced the concept of BCK-Algebra. K.iseki[5] introduced BCI-Algebra in 1980. The class of BCK-algebra is a proper subclass of BCI-algebra. Many authors introduced various algebras like d-algebra, TM-algebra, PS-algebra and Ku-algebra etc. M.Mullai and K.Shanmuga Priya[9] introduced a new notion of algebra known as "SP-Algebra", which is the generalization of BCK Algebra. In this paper, SP-Algebra is extended to "SP-RING". Some theorems, corollary, lemmas, Euclidean domain and Unique Factorization theorem are also established with suitable examples.

2 Preliminaries

Definition 2.1 *[9] An Algebra(X, $*$, e) of type (2,0) is said to be SP-Algebra if*

i). $x * x = e$.

ii). $x * e = x$.

iii). *if* $x * y = e$ *and* $y * x = e$, *then* $x = y$, *where* $*$ *is called a binary operation and e is any constant.*

Definition 2.2 *[9] A SP-Algebra (X, $*$, e) is said to be abelian if* $a * b = b * a$, $\forall$ $a, b \in X$ *and a SP-Algebra which is not abelian is called non-abelian SP-Algebra.*

Definition 2.3 *[9] Let X be a SP-Algebra and S is a subset of X. Then S is a SP-Subalgebra of X under the same operation defined on X if*

i). S is non empty.

ii). $\forall$ $a,b \in S$, $a * b \in S$.

Definition 2.4 *[9] Let* Z_n = 0,1,2,......,n-1 *and* $x, y \in Z_n$. *Define* $x \ominus y$ *as,*

$$x \ominus y = \begin{cases} (x-y) & \text{if } x-y \geq 0 \\ -(x-y) & \text{if otherwise.} \end{cases}$$

The binary operation $\ominus$ *is called subtraction modulo n.*

3 SP-Ring

Definition 3.1 *A non-empty set X together with two binary operations '$*$' and 'Δ', is called SP-Ring, if it satisfies the following axioms:*

i) (X, $*$) is a SP-Algebra.

ii) *'Δ' is associative on X.*

iii) $a \Delta (b * c) = (a \Delta b) * (a \Delta c)$.

iv) $(a * b) \Delta c = (a \Delta c) * (b \Delta c)$, $\forall a, b, c \in X$.

Example 3.2 *(R, -, .) and (Z, -, .) are SP-Rings.*

Definition 3.3 *A SP-Ring is said to be commutative or abelian, if* $\forall a,b \in X$

i) $a * b = b * a$.

ii) $a \Delta b = b \Delta a$.

Example 3.4 $(Z_n, \ominus, \odot)$ *is a commutative SP-Ring.*

Result 3.5 *For any SP-Ring* $(X, *, \Delta)$, $c \Delta (a * b) \Delta d = (c \Delta a \Delta d) * (c \Delta b \Delta d)$, $\forall a,b,c,d \in X$.

Definition 3.6 *Let* $(X, *, \Delta)$ *be a SP-Ring. X is called a SP-Ring with constant if there exists* $e' \in X$ *such that* $x \Delta e' = x$, $\forall x \in X$.

Example 3.7 *(Z, -, .) is a SP-Ring with constant 1.*

Theorem 3.8 *In every abelian SP-Ring, cancellation laws hold.*
(i.e) if $a \Delta b = a \Delta c$ *and* $b \Delta a = c \Delta a$ *then* $b = c$.

Proof:
(a) Right cancellation law:
Let $b \Delta a = c \Delta a$.
$\Longrightarrow b \Delta a \Delta a = c \Delta a \Delta a$
$\Longrightarrow b \Delta e' = c \Delta e'$
$\Longrightarrow b = c$

(b) Left cancellation law:
Let $a \Delta b = a \Delta c$
Since Δ *is commutative* $b \Delta a = c \Delta a$
$\Longrightarrow b = c$.
Therefore in every abelian SP-Ring, cancellation laws hold.

Theorem 3.9 *Let (X,$*$,Δ) be a SP-Ring with constant e'. Then the set of all elements satisfying $x \Delta x = e'$ is a SP-Algebra under Δ.*

Proof:

Given X is a SP-Ring with constant e'.
Let S be a set of all elements satisfying $x \Delta x = e'$
To prove that S is SP-Algebra under Δ, the following conditions are satisfied by the hypothesis.
i) $x \Delta x = e'$
ii)$x \Delta e' = x$.
It is enough to prove that, if $x \Delta y = e'$ and $y \Delta y = e'$, then $x = y$.
Since, $x \Delta x = e'$, $y \Delta y = e'$, $x \Delta y = x \Delta x$ and $y \Delta x = y \Delta y$
$\Longrightarrow x \Delta y = x \Delta x$
$\Longrightarrow (x \Delta y) \Delta y = (x \Delta x) \Delta y$
$\Longrightarrow x \Delta(y \Delta y) = (y \Delta y) \Delta y$
$\Longrightarrow x \Delta e' = y \Delta(y \Delta y)$
$\Longrightarrow x = y \Delta e'$
$\Longrightarrow x = y$
Hence, (S, Δ, e') is SP-Algebra.

4 Integral Domain

Definition 4.1 *A SP-Ring (X, $*$, Δ) is called integral domain if it has a constant e' and the set of all elements satisfying $x \Delta x = e'$ is a SP-Algebra under Δ.*

Example 4.2 *X = (Z, -, .) is an integral domain if S = {1,-1}. Here (S, ., 1) forms SP-Algebra under multiplication.*

Definition 4.3 *An integral domain (X, $*$, Δ) is called ordered integral domain if X contains a subset S with the following properties:*

i) $\forall$ a, $b \in S \Longrightarrow a * b \in S$.

i) $\forall$ a, $b \in S \Longrightarrow a \Delta b \in S$.

iii) $\forall$ *a, b* $\in$ *S,* $a < b$ *or* $a = b$ *or* $a > b$.

Example 4.4 *Consider (Z, -, .) is an integral domain. Then, Z is an ordered integral domain if S=(nZ, -, .).*

Definition 4.5 *Let* $(X, *, \Delta)$ *be an abelian SP-Ring and a, b* $\in X$, $a \neq 0$. *a divides b [write a/b], if there exists an element* $c \in X$ *such that* $b = a \Delta c$.

Example 4.6 *i) In (Z, -, .), 5/15 since* $15 = 5 \times 3$.
But in (5Z, -, .), 5 does not divide 15, since there is no $3 \in 5Z$ *such that* $15 = 5 \times 3$.
ii)In $(Z_8, \ominus, \odot)$, $4 = 2 \odot 2 \Longrightarrow 2/4$.

Definition 4.7 *Let* $(X, *, \Delta)$ *be an abelian SP-Ring and a, b are two non-zero elements of X. Then, a and b are said to be associates if a/b and b/a.*

Example 4.8 *i)In (Z, -, .), every a and -a are associates.*
ii)In $(Z_8, \ominus, \odot)$, *2 and 6 are associates, since* $2 = 6 \odot 3 \Longrightarrow 6/2$
$6 = 2 \odot 3 \Longrightarrow 2/6$.

Theorem 4.9 *In any commutative SP-Ring* $(X,*,\Delta)$, *an element* $e' \in X$ *satifying* $x\Delta e' = x$ *is unique.*

Proof:
To prove: e' *is unique*
(i.e) to prove $e' = e''$,
Suppose, there exists $e'' \in X$ *such that* $x \Delta e'' = x$
Since, $x \Delta e' = x$ *and* $x \Delta e'' = x$, *we have* $x \Delta e' = x \Delta e''$
$\Longrightarrow x \Delta (x \Delta e') = x \Delta (x \Delta e'')$
$\Longrightarrow (x \Delta x) \Delta e' = (x \Delta x) \Delta e''$
Consider e' *as a constant. Then*
$e' \Delta e' = e' \Delta e''$
$\Longrightarrow e' = e' \Delta e''$
Consider e'' *as a constant. Then*
$e'' \Delta e' = e'' \Delta e''$
$\Longrightarrow e'' \Delta e' = e''$
Since Δ *is commutative, then we have* $e' \Delta e'' = e''$
Hence, $e' = e''$.

Theorem 4.10 *Let $(X, *, \Delta)$ be an integral domain with commutative property and the non-zero elements a and b are associates. Then their constants satisfying $a = b \,\Delta\, c_1$ and $b = a \,\Delta c_2$ are equal.*

Proof:

Given X is an integral domain with commutative property a/b and b/a, since a and b are associates.

Therefore by definition, there exist c_1 and $c_2 \in X$ such that

$a = b \,\Delta\, c_1$ and $b = a \,\Delta\, c_2$

$\Longrightarrow a = b \,\Delta\, c_1$

$\Longrightarrow a = (a \,\Delta\, c_2) \,\Delta\, c_1 = a \,\Delta\, (c_2 \,\Delta\, c_1)$

$\Longrightarrow (c_2 \,\Delta\, c_1) = e'$, by theorem 4.9

$\Longrightarrow c_2 = c_1$.

Theorem 4.11 *Let $(X, *, \Delta)$ be an abelian SP-Ring. In X-$\{0\}$, we define $a \sim b$ if a and b are associates then prove that $\sim$ is an equivalence relation.*

Proof:

Given X is an abelian SP-Ring with constant e'.

In X-$\{0\}$, define $a \sim b = a/b$ and b/a.

To prove: $\sim$ is an equivalence relation

i)Reflexive:

Clearly, for every $a \in X$,

$a \,\Delta\, e' = a$

$\Longrightarrow a/a$

Therefore $a \sim a$.

ii)Symmetric:

Let $a \sim b$. Then a/b and b/a

$\Longrightarrow b/a$ and a/b

$\Longrightarrow b \sim a$

iii)Transitive:

Let $a \sim b$ and $b \sim c$.

Now,

$a \sim b \Longrightarrow a/b$ and b/a

and

$b \sim c \Longrightarrow b/c$ and c/b.

Also,
a/b and $b/a \Longrightarrow$ there exist c_1 $c_2 \in X$ such that $a = b \ \Delta \ c_1$ and $b = a \ \Delta \ c_2$.
b/c and $c/b \Longrightarrow$ there exist c_3 and $c_4 \in X$ such that $b = c \ \Delta \ c_3$ and $c = b \ \Delta \ c_4$.
Then, $a = b \ \Delta \ c_1$
$a=(c \ \Delta \ c_3) \ \Delta \ c_1$
$a=c \ \Delta \ (c_3 \ \Delta \ c_1)$
$\Longrightarrow a/c$.
Similarly,
$c = b \ \Delta \ c_4$
$\Longrightarrow c = (a \ \Delta \ c_2)\Delta \ c_4$
$\Longrightarrow c = a \ \Delta(c_2 \ \Delta \ c_4)$
$\Longrightarrow c/a$
$\Longrightarrow$ '$\sim$' is transitive
Hence '$\sim$' is equivalence relation.

Definition 4.12 *Let* $a, \ b \in X$. *Then* $d \in X$ *is said to be a greatest common divisor of* a *and* b *if i)* d/a *and* d/b.
ii)whenever c/a *and* c/b *then* c/d.
It can be written as $d = (a,b)$.

Definition 4.13 *If* $a, \ b \in (X, *, \Delta)$ *and* d *is the greatest common divisor of* a *and* b, *then there exist* $s, \ t \in X$ *such that* $(a * s)\Delta(b * t) = d$.

Definition 4.14 *Let* $(X, *, \Delta)$ *be a SP-Ring. If* $a, \ b \in X$ *are relatively prime, then* $(a,b) = e^{'}$.

Definition 4.15 *An integral domain* $(X, *, \Delta)$ *is said to be an Euclidean domain if for every non zero* a *in* X, *there is a non-negative integer* $d(a)$ *such that*

i) $\forall \ a, \ b \in X(a \neq 0$ and $b \neq 0)$, $d(a) \leq d(a\Delta b)$.

ii) For any $a, \ b \in X,(a \neq 0$ and $b \neq 0)$,there exist $t, \ r \in X$ such that $a = (t \ \Delta \ b)* \ r$, either $r = 0$ or $d(r) < d(b)$.

Example 4.16 *(Z, -, .) is an Euclidean domain, where* $d(a) = a^2$.
Proof:

$$\begin{aligned} d(a\Delta b) &= d(a.b) \\ &= (ab)^2 \\ &= a^2.b^2 \geq a^2 \\ &= d(a) \\ d(a\Delta b) &= d(a) \end{aligned} \tag{1}$$

Let a, b be two non-zero elements of Z.
Let q be the quotient and r be the -(remainder of a/b).
Then, $a = (q \Delta b) * r = q.b\text{-}r$, *and* $d(r) = r^2$,
which is positive for all r, where r = 0 or $d(r) < d(b)$.

Definition 4.17 *In an Euclidean domain* $(X, *, \Delta)$, *an element* $a \in X$ *is said to be prime if 'a' cannot be expressed as* $a = b\Delta c$, *where* $b, c \neq e' \in X$, *and* e' *is constant in X corresponding to* Δ.

Example 4.18 *In (Z, -, .), every prime number is prime element.*

Definition 4.19 *Let* $a \neq 0$ *and b are in an abelian SP-Ring* $(X, *, \Delta)$. *Then, a divides b(a/b) if there exists an element* $c \in X$ *such that* $b = a\Delta c$.

Problem 4.20 *1.If a/b and b/c, then a/c.*
$a/b \implies$ *there exist a constant* $u_1 \in X$ *such that* $b = a\Delta u_1$.
$b/c \implies$ *there exist a constant* $u_2 \in X$ *such that* $c = b\Delta u_2$.
Then,

$$\begin{aligned} c &= (a\Delta u_1)\Delta u_2 \\ &= a\Delta(u_1\Delta u_2) \\ c &= a\Delta u, where u = u_1\Delta u_2 \end{aligned} \tag{2}$$

$\implies a/c$.

Problem 4.21 *If a/b and a/c, then a/(b∗c).*
Solution: *Given* $a/b \Longrightarrow$ *there exist a constant* u_1 *such that* $b = a \; \Delta \; u_1$.
Given $a/c \Longrightarrow$ *there exist a constant* u_2 *such that* $c = a \; \Delta \; u_2$.
Then,

$$\begin{aligned} b * c &= (a\Delta u_1) * (a\Delta u_2) \\ &= a\Delta(u_1 * u_2) \\ b * c &= a\Delta u \end{aligned} \tag{3}$$

$\Longrightarrow a/(b{*}c)$.

Theorem 4.22 *Let (X, ∗, Δ) be an Euclidean domain with commutative property. Suppose that p, a,* $b \in X$, $p/(a\Delta b)$ *and* $(p,a) = e'$, *then p/b.*
Proof:
Given (X, ∗, Δ) is an Euclidean domain and p/(a Δ b) and p does not divide a.
Since, $p/(a \; \Delta \; b)$, *there exists* $c \in X$ *such that* $a \; \Delta \; b = p\Delta c$
Given $(p,a) = e'$. *Then, there exist* $s, t \in X$ *such that* $(p \; \Delta \; s){*}(a \; \Delta \; t) = e'$
$b \; \Delta[(p \; \Delta \; s){*}(a \; \Delta \; t)] = b \; \Delta \; e'$
$\Longrightarrow (b \; \Delta \; p \; \Delta \; s){*}(b * a \; \Delta t) = b \; \Delta \; e'$
$\Longrightarrow (p \; \Delta \; b \; \Delta \; s){*}(a * b \; \Delta \; t) = b \; \Delta \; e' = b$
$\Longrightarrow (p \; \Delta \; b \; \Delta \; s){*}(p * c \; \Delta \; t) = b$
$\Longrightarrow p \; \Delta[(b \; \Delta \; s){*}(c \; \Delta \; t) = b$
$\Longrightarrow p/b$.

Lemma 4.23 *Let P be a prime element in an Euclidean domain(X, ∗, Δ). If* $p/(a \; \Delta \; b)$ *where a,* $b \in X$, *then p divides either a or b.*

Corollary 4.24 *If p is a prime element in an Euclidean domain (X, ∗, Δ) and* $p/(a_1 \; \Delta \; a_2 \; ....\Delta \; a_n)$, *then p divides atleast one* a_i.

Lemma 4.25 *Let (X, ∗, Δ) be an Euclidean domain and a,* $b \in X$. *If b is not a constant satisfying* $x \; \Delta \; b = x, \; \forall \; x \in X$, *then* $d(a) < d(a \; \Delta \; b)$.
Proof:
Given (X, ∗, Δ) is an Euclidean domain.

Let a,b $\in$ X.
By the first condition of Euclidean domain,
$d(a) \leq d(a \,\Delta\, b)$
Now,
$x \,\Delta\, b = x, \forall\, x$, *since b is not constant*
$\Longrightarrow a \,\Delta\, b \neq a$
$\Longrightarrow d(a \,\Delta\, b) \neq d(a)$
$\Longrightarrow d(a) < d(a \,\Delta\, b)$.

Lemma 4.26 *Let $(X, *, \Delta)$ be an Euclidean domain with $d(e^{'}) = d(1)$ and X is abelian SP-Algebra under Δ. Every element in X can be either a constant $e^{'}$ in X or can be written as the product of a finite number of prime elements in X.*
Proof:
It can be proved by induction on d(a).
If d(a)= $d(e^{'})$, a is the constant element $e^{'}$ in X.
Assume that, this is true for all elements in X satisfying $d(x) < d(a)$
To prove it for 'a':
Suppose 'a' is a prime, there is nothing to prove.
If not, 'a' can be written as $a = b \,\Delta\, c$, where b and c are not constants in X.
$\Longrightarrow d(a) = d(b\Delta c)$
Then by definition of Euclidean domain,
$d(b) \leq d(b \,\Delta\, c)$
Also, $d(b) < d(b \,\Delta\, c)$, $d(c) < d(c \,\Delta\, b)$, *since 'c' is not constant.*
$d(c) < d(b \,\Delta\, c) = d(a)$, *since Δ is commutative.*
$\Longrightarrow d(c) < d(a)$
$\Longrightarrow d(b) < d(a)$
By our induction hypothesis, b and c can be written as the product of prime elements.
(i.e) $b = b_1 \,\Delta\, b_2 ....\Delta\, b_n$, *where b_i are prime elements and* $c = c_1 \,\Delta\, c_2 ....\Delta\, c_n$, *where c_i are prime elements in X.*
We know that, $a = b\Delta c = (b_1 \,\Delta\, b_2 ....\Delta\, b_n)\Delta\, (c_1 \,\Delta\, c_2 ....\Delta\, c_n)$.
Therefore, 'a' can be written as the product of finite number of prime elements.

Theorem 4.27 *UNIQUE FACTORIZATION THEOREM*
Let $(X,*, \Delta)$ be an Euclidean domin with commutative property and a$\neq$ 0 be a

non-constant in X. If $a = \pi_1 \ \Delta \ \pi_2......\Delta \ \pi_r \ = \pi_1^{'} \ \Delta \ \pi_2^{'}......\Delta \ \pi_s^{'}$*, where* π_i *and* $\pi_j^{'}$ *are prime elements of X,*
then $r = s$ *and each* π_i *is an associates of some* $\pi_j^{'}$*, where* $1 \leq i \leq r$*,* $1 \leq j^{'} \leq s$*.*

Proof:

Given (X, ∗, Δ) is an Euclidean domain and $a = \pi_1 \ \Delta \ \pi_2......\Delta \ \pi_r \ = \pi_1^{'} \ \Delta \ \pi_2^{'}......\Delta \ \pi_s^{'}$*,*
where π_i *and* $\pi_j^{'}$ *are prime elements of X.*
$\pi_1^{'} \ \Delta \ \pi_2^{'}...\Delta \ \pi_s^{'} = \pi_1\Delta(\pi_2...\Delta\pi_r)$.
$\Longrightarrow \pi_1/(\pi_1^{'} \ \Delta \ \pi_2^{'}......\Delta \ \pi_s^{'})$
Also, $\pi_1/$*some* $\pi_j^{'}$ *(by corollary 4.25)*
Without loss of generality, assume that $\pi_1/\pi_1^{'}$
Since π_1 *and* $\pi_1^{'}$ *are both prime elements of X,* π_1 *and* $\pi_1^{'}$ *must be associates.*
$\Longrightarrow \pi_1^{'} \ = \pi_1 \ \Delta \ u_1$*, where* $u_1 \in X$*.*
Eqn(4) $\Longrightarrow \pi_1 \ \Delta \ u_1 \ \Delta \ \pi_2^{'} \ ... \ \Delta \ \pi_s^{'} \ = \pi_1 \ \Delta \ (\pi_2... \ \Delta \ \pi_r)$
$\Longrightarrow \pi_2......\Delta \ \pi_r \ = u_1 \ \Delta \ \pi_2^{'}......\Delta \ \pi_s^{'}$
Now if $r < s$*, then repeating the above process r times,the left side become* $e^{'}$ *and right side contains the product of some prime elements.*
Which is a contradiction to our assumption.
Therefore, $r \geq s$
Similarly, $s \leq r$
$\Longrightarrow s = r$
Also every π_i *associates with some* $\pi_j^{'}$ *and conversly, since each* $\pi_i/\pi_j^{'}$ *and* π_i *and* $\pi_j^{'}$ *are prime elements.*

5 Conclusion

In this paper, the concept of SP-Ring is introduced with examples. Also, some properties of SP-Ring, Integral and ordered integral domains, theorems and Unique Factorization Theorem are established. In future, using the concept of SP-Ring Polynomials in SP-Ring will be extented with suitable examples.

Acknowledgement

The authors would like to express their thanks to the referees for their valuable comments and suggestions which improved the paper.

References

[1] Borzooei.R.A and Zahiri.O, *Prime Ideals in BCI and BCK-Algebras*, Annals of the University of Craiova, Mathematics and Computer Science Series. Volume 39(2), 2012, pages 266-276, ISSN: 1223-6934.

[2] Eric Jespesrs, *Associtive Algebra*, Website: http: homepages.vub.ac.be/efjesper HOC: donderdag 09-11uur, F.4.113(2016-2017)

[3] Herstein.N, *Topics in Algebra*, Second Edition(2011).

[4] Iseki.T and Tanaka.S, *An Introduction to the theory Of BCK-Algebra*, Math Japonica23(1978).

[5] Iseki.K, *On BCI-Algebra*, Math. Seminar Notes 8(1980), 125-130.

[6] Jiang Hao and Chen Xue li, *On Ideals of an Ideal in a BCI-Algebra*, Scientiae Mathematicae Japonicae, No.10,(2004), 493-500.

[7] Mahmut Kuzucuoglu, *Exercises and Solutions in Groups Rings and Fields*, Middle East Technical University, matmah@metu.edu.tr, Ankara, Turkey April 18, 2012.

[8] Marek Palasinski, *Some Remarks on BCK-Algebra*, Bulletin of the section of logic. Volume 9/2(1980), pp 85-87, Re-edition 2010[Original edition,pp.85-87].

[9] Mullai.M, Shanmuga Priya.K, *A Note on SP-Algebra*, International Conference on Mathematical Modelling and Computational Methods in Science and Engineering february-2017.

[10] Priya, Ramachandran.T, *Classification of PS- Algebra*, The International Journal of Science and Technology, vol.1 Issue 7, September 2014.

[11] Sung Min Hong, Young Bae Jun and Mehmet Ali Ooturk, *Generalization of BCK-Algebras*, Sientiae Mathematica Japnonicae Online, Vol. 8,(2003), 549-557.

[12] Zahra Samaei.M, *A Class of BCK-Algebra*, International Journal of Algebra, vol.5, 2011, no.28, 1379-1385.

[13] Huang Yisheng, *BCI-Algebra*, Science Press(2006).