Lecture Notes on Frame Operators

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Abstract

In this paper, we recall basic definitions of topological algebras, normed algebras, Banach algebras, involutive algebras, and C^*-algebras. We also give many elementary examples for these algebras. Different types of frame operators is explained. Norms on frame operator is defined. Some results on eigen values and eigen vectors is proved.

Introduction

***Definition**: (i) A topological vector space is a vector space endowed with a topology such that both the scalar multiplication and the addition are continuous maps.*

(ii) A topological algebra is a topological vector space A with a jointly continuous multiplication, that is the multiplication $A \times A \to A$ is a continuous map.

(iii) A normed algebra is a normed space $(A, \| - \|)$ with a sub-multiplicative multiplication, that is

$$\|ab\| \leq \| a \| \| b \|, \forall a, b \in A.$$

(iv) A normed algebra $(A, \| - \|)$ is called a Banach algebra if A is complete with respect to its norm.

The key point in topological algebras it that the multiplication is always assumed to be jointly continuous. Let A be a ring or an algebra. We denote the algebra of $n \times n$ matrices with entries in A by $M_n(A)$.

***Definition** : Let A be an algebra. A involution over A is a map * : $A \to A$*

Satisfying the following conditions for all $x, y \in A$ and $\lambda \in \mathbb{C}$:

(i) $(x^*)^* = x$,
(ii) $(x + y)^* = x^* + y^*$,
(iii) $(\lambda x)^* = \bar{\lambda}\, x^*$.
(iv) $(xy)^* = y^* x^*$.

When A is a normed algebra, we also assume

(v) $\| x^* \| = \| x \|$.

*An algebra A equipped with an involution * is called an involutive algebra and is denoted as an ordered pair by $(A,*)$. Involutive normed algebras and involutive Banach algebras are defined similarly and are denoted by $(A, \| - \|, *)$. A subalgebra of an involutive algebra is called an involutive subalgebra or a *subalgebra if it is closed under the involution.*

***Definition:** An involutive Banach algebra $(A, \| - \|, *)$ is called a C^*-algebra if*

$$\| x^* x \| = \| x \|^2, \forall x \in A.$$

We call the above identify the C^*-identity. A norm satisfying this identity is called a C^*-norm.

Let us start by recalling the basic definition and notions of frame theory.

***Definition**: Given a family of vectors* $\{f_i\}_{i=1}^{M}$ *in an* $n-dimensional$ $C^*algebra$ A_N, *we say that* $\{f_i\}_{i=1}^{M}$ *is a frame if there exist constants* $0 < A \leq B < \infty$ *satisfying that*

$$A\|f\|^2 \leq \sum_{i=1}^{M} |f, f_i|^2 \leq B\|f\|^2 \text{ for all } f \in A_N.$$

The numbers A, B are called lower and upper frame bounds (respectively) for the frame. A = B

***Definition** :* $\{f_i\}_{i=1}^{M}$ *is called an* $A-$*tight frame, and if* $A = B = 1$, *it is referred to as a Parseval frame.*

***Defition** : The frame is an equal norm frame, if* $\|f_i\| = \|f_k\|$ *for all* $1 \leq i, k \leq M$, *and a unit norm frame, if*

$\|f_i\| = 1$ *for all* $1 \leq i \leq M$.

***Definition**: The analysis operator* $F: A_N \to \ell_2(M)$ *of the frame is defined by*

$$F(f) = \{\langle f, f_i \rangle\}_{i=1}^{M}.$$

***Definition:** The adjoint of the analysis operator is referred to as the synthesis operator* $F^* = \ell_2(M) \to A$, *which is*

$$F^*(\{a_i\}_{i=1}^{M}) = \sum_{i=1}^{M} a_i f_i.$$

***Definition**: The frame operator is then the positive, self-adjoint invertible operator* $S: A_N \to A_N$ *given by* $S = F^*F$, *i.e.,*

$$Sf = \sum_{i=1}^{M} \langle f, f_i \rangle f_i, \text{ for all } f \in A$$

Definition*: Two frames* $\{f_i\}_{i=1}^{M}$ *and* $\{g_i\}_{i=1}^{M}$ *in A are called equivalent (unitarily equivalent), if there exists an invertible (a unitary) operator* $T: A \to A$ *such that* $g_i = Tf_i$ *for all* $i = 1, \ldots\ldots, M$.

***Theorem**: Let* T_1, T_2 *be positive, self-adjoint invertible operators on a* $C^*algebra$ *A, and let S be an invertible operator on A, Then the following conditions are equivalent.*

1. $T_2 = ST_1S^*$

2. There exists a unitary operator U on A such that $S = T_2^{1/2}\, U\, T_1^{-1/2}$

Proof: (1) ⇒(2)

$$\text{Take } U = T_2^{-1/2}\, S\, T_1^{1/2} \quad \text{which is a unitaryOperator}$$

$$\text{Since } \left(T_2^{-1/2}\, S\, T_1^{1/2}\right)\left(T_2^{-1/2}\, S\, T_1^{1/2}\right)^*$$

$$= T_2^{-1/2}\, S\, T_1\, S^* T_1^{-1/2}$$

$$= T_2^{-1/2}\, T_2\ T^{-1/2} = I$$

$$\text{More over} \quad T_2^{1/2}\, U\, T_1^{-1/2},\, T_2^{1/2}\left(T_2^{-1/2}\, S\, T_1^{1/2}\right) T_1^{-1/2} = S$$

(2) ⇒ (1) Since $S = T_2^{1/2}\, U\, T_1^{-1/2}$

$$ST_1\, S^* = T_2^{1/2}\, U\, T_1^{-1/2}\, T_1\, T_1^{-1/2}\, U^*\, T_2^{1/2}$$

$$= T_2^{1/2}\, UU^*\, T_2^{1/2} = T_2^{1/2}\, I\, T_2^{1/2} = T_2$$

Definition: *Let* $F = \{f_i\}_{i=1}^{M}$ *be frame for* A_N*, and let* $\varepsilon = \{e_j\}_{j=1}^{N}$ *be an orthonormal basis for* $\boldsymbol{A_N}$*. Then we define*

$$\boldsymbol{H}(\boldsymbol{F},^{\varepsilon}) = span\left\{\left\|\boldsymbol{f},\boldsymbol{e_j}\right\|_{j=1}^{\boldsymbol{n}}\right\} \subseteq \boldsymbol{A_N}$$

Theorem*: Let* $\{\boldsymbol{f_i}\}_{i=1}^{M}$ *be a frame for*$\boldsymbol{A_N}$ *and let* $\{\boldsymbol{C_i}\}_{i=1}^{M}$ *be positive scalars. Further let T be an invertible operator to* $\boldsymbol{A_N}$*.*

Let $\{\boldsymbol{e_j}\}$ *be the eigen vectors for* $\boldsymbol{T^*T}$ *with respective to eigen values* $\{\boldsymbol{\lambda_i}\}_{i=1}^{M}$*. Then following are equivalent*

1. $\boldsymbol{C_i^2}\|\boldsymbol{f_i}\|^2 = \|\boldsymbol{Tf_i}\|^2$

2. $\left\|\left(\boldsymbol{\lambda_j} - \boldsymbol{C_j}\right)^2\boldsymbol{e_j}, \sum_{j=1}^{N}\left\|\boldsymbol{f_j}\boldsymbol{e_j}\right\|\boldsymbol{e_j}\right\| = \boldsymbol{0}$

Proof: Since

$$\|\boldsymbol{T^*T} - \boldsymbol{C^2Id}\|\boldsymbol{f^2} = \|\boldsymbol{T^*Tf^2} - \boldsymbol{C^2Id}\|$$

$$= \|\boldsymbol{T^*Tf^2}\| + \|\boldsymbol{C^2Idf^2}\|$$

$$= \|\boldsymbol{Tf}\|^2 - \boldsymbol{C^2}\|\boldsymbol{f}\|^2$$

Hence $\|\boldsymbol{T^*T} - \boldsymbol{C^2d}\|\boldsymbol{f^2} = \boldsymbol{0}$ *for all* $\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2} \ldots\ldots \boldsymbol{N}$

But for all $\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2} \ldots\ldots \boldsymbol{N}$

$$\boldsymbol{0} = \|\boldsymbol{T^*T} - \boldsymbol{C^2Id}\|\boldsymbol{f_i^2}$$

$$= \left\|\sum_{j=1}^{N}\left(\boldsymbol{\lambda_j} - \boldsymbol{C_j}\right)^2\left\|\boldsymbol{f_j}\boldsymbol{e_j}\right\|, \sum_{j=1}^{N}\left\|\boldsymbol{f_j}\boldsymbol{e_j}\right\|\boldsymbol{e_j}\right\|$$

$$= \sum_{j=1}^{N}\left(\boldsymbol{\lambda_j} - \boldsymbol{C_j}\right)^2\left\|\boldsymbol{f_j}\boldsymbol{e_j}\right\|^2$$

$$= \left\|\sum_{j=1}^{N}\left(\boldsymbol{\lambda_j} - \boldsymbol{C_j}\right)^2\boldsymbol{e_j}, \sum_{j=1}^{N}\left\|\boldsymbol{f_j}\boldsymbol{e_j}\right\|^2\boldsymbol{e_j}\right\|$$

Arthemetic/Geometric Mean Inequality*: Let* $\{\boldsymbol{x_j}\}_{j=1}^{N}$ *be a sequence of positive real numbers then*

$$\left(\overset{N}{\underset{j=1}{11}}\boldsymbol{x_j}\right)^{1/N} \leq \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x_j}$$

With equality if and only if $\boldsymbol{x_j} = \boldsymbol{x_k}$ *for every* $\boldsymbol{j}, \boldsymbol{k} = \boldsymbol{1}, \boldsymbol{2} \ldots\ldots \boldsymbol{N}$

Theorem: *Let* $\{\boldsymbol{f_i}\}_{i=1}^{M}$ *be a frame for* $\boldsymbol{A_N}$*with frame operator S. It det(S)* $\geq \boldsymbol{1}$ *and*

$\sum_{i=1}^{M}\|\boldsymbol{f_i}\|^2 = \boldsymbol{N}$*, then* $\{\boldsymbol{f_i}\}_{i=1}^{M}$ *constitutes parseval frame*

Proof: Set $\{\boldsymbol{g_i}\}_{i=1}^{M} = \left\{\frac{f_i}{\boldsymbol{det(S)}}\right\}_{i=1}^{M}$

Let $\{\lambda_i\}_{i=1}^N$ *denotes eigen values of S and let eigen values of frame operator for* $\{g_i\}_{i=1}^M$ *be denoted by* $\{\lambda_j^1\}_{i=1}^N$ *Then we obtain*

$$\sum_{j=1}^N \lambda_j^1 = \frac{\sum_{j=1}^N \lambda_j}{det(S)^2} = \frac{N}{det(S)^2}$$

Which implies $\frac{\sum_{j=1}^N \lambda_j}{N} = \frac{1}{det(S)^2} \leq 1 = \underset{j=1}{\overset{N}{11}} \lambda_j^1$

This contradicts the arithmetic-Geometric mean inequality unless λ_j^1 *= 1 for all* $j = 112 - N$

That is unless $\{f_i\}_{i=1}^M$ *constitutes parseval frame*

Example *: Let* $f_1, \ldots, f_4$ *be the vectors in* R^3 *defined by*

$$f_1 = \frac{1}{2}(1,1,1), f_2 = (-1,1,1), \quad f_3 = \frac{1}{2}(1,-1,1), \quad f_4 = \frac{1}{2}(1,1,-1)$$

$\{f_i\}_{i=1}^4$ *is an equal norm Parseval frame for* R^3. *If we let* $\{e_j\}_{i=1}^3$ *denote the standard unit vector basis, then*

$$H(F,\varepsilon) = span\,\{(1,1,1)\}$$

We next choose a vector g *such that* $\perp H(F,\varepsilon\;)$ ***by*** $g = (1,-1,0)$. *Let now, for instance ,* $c = 2$, *and set*

$$\lambda_1 = 1 + c^2 = 5, \ \lambda_2 = -1 + c^2 = 3, \ \lambda_3 = 0 + c^2 = 4.$$

Then define an operator T *such that* $\{Te_i\}_{i=1}^3$ *is an orthogonal set and* $\|Te_j\| = \lambda_j$. *One example of such an operator is defined by*

$$T(1,0,0) = (5,0,0), T(0,1,0) = \left(0, \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right), T(0,0,1) = (0, \sqrt{2}, -\sqrt{2}).$$

Thus

$$Tf_1 = \frac{1}{2}\left(\sqrt{5}, \ \sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}\right),$$

$$Tf_2 = \frac{1}{2}\left(-\sqrt{5}, \ \sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}\right),$$

$$Tf_3 = \frac{1}{2}\left(\sqrt{5}, \ -\sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}\right),$$

$$Tf_4 = \frac{1}{2}\left(\sqrt{5}, \ \sqrt{\frac{3}{2}} - \sqrt{2}, \sqrt{\frac{3}{2}} + \sqrt{2}\right),$$

and we indeed obtain

$$\|Tf_i\|^2 = \frac{1}{4}(5 + 3 + 4) = 3$$

As desired

Corollary *: Every invertible operator T on a Hilbert space H_N maps some equal norm Parseval frame to an equal norm frame.*

Proof. *Let T be an invertible operator on H_N and let $\{e_i\}_{i=1}^{N}$ be an eigenbasis for T^*T with respective eigen values $\{\lambda_i\}_{i=1}^{N}$. Set*

$$c^2 = \frac{1}{N}\sum_{j=1}^{N}\lambda_j \text{ and } f = \sum_{j=1}^{N} e_j.$$

Then

$$\langle T^*T - c^2 Id\, f, f\rangle = \sum_{j=1}^{N}(\lambda_j - c^2).1 = 0,$$

Which means

$$(1, 1, \ldots\ldots, 1) \perp (\lambda_1 - c^2, \lambda_2 - c^2, \ldots\ldots, \lambda_N - c^2).$$

Next, consider the frame

$$\{f_i\}_{i=1}^{2^N} = \left\{\sum_{j=1}^{N}\varepsilon_j e_j\right\}_{\{\varepsilon_j\}\in(1,-1)^N}$$

For every $g = \sum_{j=1}^{N} a_j e_j$, we obtain

$$\sum_{i=1}^{2^N}|\langle g, f_i\rangle|^2 = \sum_{i=1}^{2^N}\left|\sum_{j=1}^{N}\varepsilon_j a_j\right|^2 = 2^N\sum_{j=1}^{N}|a_j|^2 = 2^N\|g\|^2$$

Thus $\left\{\frac{1}{\sqrt{2^N}}f_i\right\}_{i=1}^{2^N}$ forms an equal norm Parseval frame, and we have $H(F, \varepsilon\ \ )$ = span $\{(1, 1, \ldots\ldots, 1)\}$.

, this implies that $\{Tf_i\}_{i=1}^{2^N}$ is an equal norm frame with $\|Tf_i\|^2 = c^2$ for all

$i = 1, 2, \ldots\ldots, 2^N$.

We now provide an example of an equal norm Parseval frame and a non-unitary operator T, which maps it to a unit norm frame.

Theorem *: Let $\{f_i\}_{i=1}^{M}$ be a frame for A_N with frame operator T if $det(T) \geq 1$ and $\sum_{i=1}^{M}\|f_i\|^2 = N$*

Then $\{f_i\}_{i=1}^{M}$ constitutes a parseval frame.

Proof: Set $\{g_i\}_{i=1}^{M} = \left\{\frac{f_2}{det(s)}\right\}_{i=1}^{m}$

Now let $\{\lambda_j\}_{j=1}^{N}$ denote the eigen values of T and let eigen values of the frame operator for

$\{g_i\}_{i=1}^{M}$ be denoted by $[\lambda_j^1]_{j=1}^{N}$ then we obtain

$$\sum_{j=1}^{N}\lambda_j^1 = \frac{\sum_{j=1}^{N}\lambda_j}{det(S)^2} = \frac{N}{det(S)^2}$$

Which implies

$$\frac{\sum_{j=1}^{N}\lambda_j^1}{N} = \frac{1}{det(S)^2} \leq 1 = \prod_{j=1}^{N}{}^{11}\lambda_j^1$$

However this contradicts the orithmetic-Geometric mean inequality unless $\lambda_j^1 = 1$

For all $j = 1, 2, \ldots\ldots N$ *i.e unless* $\{f_i\}_{i=1}^{M}$ *constitute a parse val frame.*

Theorem: *Let* $N \geq 2, 0 \leq x_j \leq N$ *for all* $j = 1, 2, \ldots\ldots N$

If $\frac{\sum_{j=1}^{N} x_j}{N} - \left(\prod_{j=1}^{N} x_j\right)^{1/N} < \in$

Then there exists a function $f: R^+ \to R^+$ *with*

$|x_j - x_k| \leq fC \in$ *for all* $j, K = 1, 2, -N$

And $1 - fC \in \leq x_j \leq 1 + fC \in$ *for all* $j = 1, 2, -N$

More over f *is bounded by*

$fC \in \leq 2 \in^{1/2} \ N^{3/2}$

Proof: Since

$$\frac{1}{N(N-1)} \sum_{1 \leq j \leq k \leq N} \left(x_j^{1/2} - x_k^{1/2}\right)^2 \leq \frac{\sum_{j=1}^{N} x_j}{N} - \left(\prod_{j=1}^{N} x_j\right)^{1/N}$$

There fore, for all $1 \leq j < k \leq N$

$$\left|x_j^{1/2} - x_k^{1/2}\right|^2 \leq \sum_{1 \leq j,k \leq N} \left(x_j^{1/2} - x_k^{1/2}\right)^2 \leq N(N-1) < \in$$

Since $x_j \leq N$, *it follows that* $x_j^{1/2} \leq N^{1/2}$ *for all* $j = 112. N.$ *thus*

$$|x_j - x_k|^2 = \left|x_j^{1/2} - x_k^{1/2}\right|^2 \left|x_j^{1/2} + x_k^{1/2}\right|^2 \leq N(N-1) \in^2 4N \leq 4N^2 \in$$

Which implies

$$|x_j - x_k| \leq 2N^{3/2} \ \in^{1/2}$$

Further, for any $1 \leq j \leq N$, *we obtain*

$$x_j = \frac{\sum_{k=1}^{N} x_k}{N} \leq \frac{\sum_{k=1}^{N} x_k}{N} + \frac{\sum_{k=1}^{N} |x_j - x_k|}{N} \leq 1 + \frac{N 2N^{3/2} \in^{1/2}}{N} = 1 + 2N^{3/2} \ \in^{1/2}$$

The inequality $x_j \geq 1 - 2N^{3/2} \ \in^{1/2}$ *can be similarly proved.*

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