Certain Results on η-Ricci Solitons in α-Sasakian Manifolds

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Abstract

In this paper we study η- Ricci Solitons in α-Sasakian Manifolds. It is shown that a symmetric parallel second order covariant tensor in a α-Sasakian manifold is a constant multiple of the metric tensor. In the last, we discuss η- Ricci Solitons in conharmonically flat α-Sasakian manifold.

Keywords — *η-Ricci Solitons, α-Sasakian Manifolds, Ricci-semisymmetric Manifold, Conharmonically flat Manifolds.*

I. INTRODUCTION

In 1982, Ricci solitons were introduced by R.S. Hamilton as natural generalizations of Einstein metrics[12]. A Ricci soliton (g,V,λ) is defined on a pseudo-Riemannian manifold (M, g) by

$$£_V g + 2S + 2\lambda g = 0 \tag{1.1}$$

where $£_V g$ denoted the Lie derivative of Riemannian metric g along a vector field V, λ is a constant and X, Y are arbitrary vector fields on M.

A Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Theoretical physicists have also been taking interest in the equation of Ricci soliton in relation with string theory and the fact that equation (1.1) is a special case of Einstein field equations. The Ricci soliton in Riemannian Geometry was introduced [13] as self similar solution of the ricci flow. Recent progress on Riemannian Ricci solitons may be found in [9], Also, Ricci solitons have been studied extensively in the context of Pseudo-Riemannian Geometry. We may refer to [4, 6, 7, 8, 14].

As a generalization of Ricci solitons, the notion of η-Ricci soliton was introduced by Cho and Kimura [10]. This notion has also been studied in [5] for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ ,μ), where λ and μ are constants, V is a vector field on M and g is a Riemannian (or Pseudo-Riemannian) metric satisfying the equation

$$£_V g + 2S + 2\lambda g + 2\mu\eta\otimes\eta = 0 \tag{1.2}$$

where S is the Ricci tensor associated to g. In this connection we mention the works of Blaga [1, 2, 3] with η-Ricci solitons. In particular, if μ = 0 then the notion η-Ricci soliton (g,V,λ,μ) reduces to the notion of Ricci soliton (g, V, λ).

In the present paper, we study α-Sasakian η-Ricci solitons. The paper is organised as follows: Section 2 is devoted to preliminaries on α-Sasakian manifolds. In section 3, It is shown that a symmetric parallel second order covariant tensor in a α-Sasakian manifold is a constant multiple of the metric tensor. Next, we discuss η-Ricci Solitons in conharmonically flat α-Sasakian manifold.

II. PRELIMINARIES

Let M be an almost contact metric manifold of dimension n, equipped with an almost contact metric structure (φ, ξ, η, g) consisting of a (1,1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfy

$$\Phi^2 = -I + \eta\otimes\xi,\ \ \eta(\xi) = 1,\ \ \eta\circ\varphi = 0,\ \ \varphi\xi = 0 \tag{2.1}$$

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),\ \ \eta(X) = g(X,\xi) \tag{2.2}$$

for all X,Y∈T(M). An almost contact metric manifold M(φ, ξ, η, g) is said to be *α*-Sasakian manifold if the following condition hold:

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) \tag{2.3}$$

$$(\nabla_X \phi)Y = \alpha\phi X, \qquad (\nabla_X \eta)Y = \alpha g(X,\phi Y) \tag{2.4}$$

for some nonzero constant α on M.

In an α- Sasakian manifold, the following relations hold:

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y] \tag{2.5}$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X] \tag{2.6}$$

$$\eta(R(X,Y),Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \tag{2.7}$$

$$S(X,\xi) = \alpha^2(n-1)\eta(X) \tag{2.8}$$

$$S(\xi,\xi) = \alpha^2(n-1) \tag{2.9}$$

$$Q\xi = \alpha^2(n-1)\xi \tag{2.10}$$

for all $X,Y,Z \in T(M)$, where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

Definition 1. The conharmonic curvature tensor C of type (1,3) on M of dimension n is defined by

$$C(X,Y)Z = R(X,Y)Z - (1/(n-2))[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{2.11}$$

Definition 2. The projective curvature tensor P of type (1,3) on M of dimension n is defined by

$$P(X,Y)Z = R(X,Y)Z - (1/(n-1))[S(Y,Z)X - S(X,Z)Y] \tag{2.12}$$

Definition 3. An almost contact metric manifold M is said to be η- Einstein if its Ricci tensor S satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{2.13}$$

For all $X, Y \in T(M)$,

For some real constants α and β.

Such notion was first introduced and studied by Okumura [19] and named by Sasaki [20] in the lecture notes 1965. In particular, if $\beta = 0$, we say that M is Einstein.

III. ETA- RICCI SOLITON ON ALPHA- SASAKIAN MANIFOLD

Let $M(\varphi,\xi,\eta,g)$ be an n-dimensional α-Sasakian manifold and let $(M,(g,\xi,\lambda,\mu))$ be a α-Sasakian η-Ricci soliton. Then the relation (1.2) implies

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

where £_{V} is the Lie derivative operator along the vector field V, S is the Ricci curvature tensor field of the metric g and λ and μ are real constants. The above equation can be written as:

$$2S(X,Y) = -(\pounds_V g)(X,Y) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y) \tag{3.1}$$

for any $X,Y \in T(M)$.

An important geometrical object in studying Ricci solitons is well-known to be a symmetric (0,2)- tensor field which is parallel with respect to the Levi-Civita connection, some of its geometrical properties being described in [3],[4],[18] etc. In the same manner as in [5] we shall state the existence of η-Ricci solitons in our settings.

Consider now π such a symmetric (0,2)- tensor field which is parallel with respect to the Levi-Civita connection ($\nabla\pi = 0$). Applying the Ricci commutation identity $\nabla^2\pi(X,Y;Z,W) - \nabla^2\pi(X,Y;W,Z) = 0$, we obtain for any $X,Y,Z,W \in T(M)$ [17].

$$\pi(R(X,Y)Z,W) + \pi(Z,R(X,Y)W)=0 \tag{3.2}$$

Replacing Z = W = ξ in (3.2) and by the symmetry of π, we have

$$\pi(R(X,Y)\xi,\xi) = 0 \tag{3.3}$$

Taking X = ξ in (3.3), then by virtue of (2.5), we have

$$\alpha^2[\pi(Y,\xi) - \pi(\xi,\xi)\eta(Y)] = 0 \tag{3.4}$$

The above equation yields:

$$\pi(Y,\xi) - \pi(\xi,\xi)\eta(Y) = 0 \tag{3.5}$$

Again by taking X = Z = ξ in (3.2), we obtain

$$\alpha^2[\eta(Y)\pi(\xi,W) - \pi(Y,W) + g(Y,W)\pi(\xi,\xi) - \eta(W)\pi(\xi,Y)] = 0 \tag{3.6}$$

Since $\alpha^2 \neq 0$, we have

$$\pi(Y,W) = \eta(Y)\pi(\xi,W) + g(Y,W)\pi(\xi,\xi) - \eta(W)\pi(\xi,Y) \tag{3.7}$$

By using (3.5) in (3.7), we have

$$\pi(X,Y) = \pi(\xi,\xi)g(X,Y) \tag{3.8}$$

The above equation gives the conclusion:

Theorem 1: Let $M(\varphi,\xi,\eta,g)$ be a α-Sasakian manifold with non-vanishing ξ- sectional curvature and endowed with a symmetric tensor field π of type (0,2). If π is parallel with respect to the Levi-Civita connection ∇ then it is a constant multiple of the metric tensor g.

On a α-Sasakian manifold M, from (2.5) and the skew-symmetric property of φ, we obtain

$$(\pounds_V g)(X,Y) = g(\nabla_V\xi,Y) + g(X,\nabla_V\xi) = 0 \tag{3.9}$$

Now, From equation (3.1) and (3.9), we get

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y) \tag{3.10}$$

thus, we conclude that $(M,(g,\xi,\lambda,\mu))$ is an η- Einstein manifold.
The equation (3.10) yields

$$S(X,\xi) = -(\lambda+\mu)\eta(X) \tag{3.11}$$

$$QX = -\Lambda x - \mu\eta(X)\xi \tag{3.12}$$

$$Q\xi = -(\lambda+\mu)\xi \tag{3.13}$$

From (3.10) we state :

Theorem 2: A α-Sasakian η-Ricci soliton $(M,(g,\xi,\lambda,\mu))$ is an η- Einstein manifold.

In particular, if $\mu = 0$ in (3.10), then it reduces to

$$S(X,Y) = -\lambda g(X,Y) \tag{3.14}$$

Thus the pair $(M,(g,\xi,\lambda))$ is an Einstein manifold. So we have the following corollary.

Corollary 1: A α-Sasakian Ricci soliton $(M,(g,\xi,\lambda))$ is an Einstein manifold.

Consider a α-Sasakian η-Ricci soliton $(M,(g,\xi,\lambda,\mu))$ which is projective Ricci-semisymmetric. Then the condition P(X,Y).S = 0 implies that

$$S(P(\xi,X)Y,Z) + S(Y,P(\xi,X)Z) = 0 \tag{3.15}$$

For any vector fields X,Y,Z on M and P denotes the projective curvature tensor defined by

$P(X,Y)Z = R(X,Y)Z - (1/(n-1))(S(Y,Z)X - S(X,Z)Y)$

Putting $X = \xi$ in above, we get

$$P(\xi,Y)Z = R(\xi,Y)Z - (1/(n-1))(S(Y,Z)\xi - S(\xi,Z)Y) \quad (3.16)$$

using (3.10) in (3.15), we get

$$-\lambda[g(P(\xi,X)Y,Z) + g(Y,P(\xi,X)Z)] - \mu[\eta(P(\xi,X)Y)\eta(Z) + \eta(Y)\eta(P(\xi,X)Z)] = 0 \quad (3.17)$$

using (2.6) and (3.10) in (3.16), we obtain

$$P(\xi,X)Y = [\alpha^2 + (\lambda/(n-1))]g(X,Y)\xi - [((\lambda+\mu)/(n-1)) + \alpha^2]\eta(Y)X + (\mu/(n-1))\eta(X)\eta(Y)\xi \quad (3.18)$$

and

$$\eta(P(\xi,X)Y) = (\alpha^2 + (\lambda/(n-1)))(g(X,Y) - \eta(X)\eta(Y)) \quad (3.19)$$

using (3.18) and (3.19) in (3.17), we get

$$-((\alpha^2\mu)/(n-1))[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0 \quad (3.20)$$

Putting $Z = \xi$ in (3.20), we get

$$-((\alpha^2\mu)/(n-1))[g(X,Y) - \eta(X)\eta(Y)] = 0 \quad (3.21)$$

This follows that $\mu = 0$. Again, a contradiction. Hence $(M,(g,\xi,\lambda,\mu))$ $(n>1)$ can not be projective Ricci-symmetric. Thus, it follows the following results:

Theorem 3: A α-Sasakian η-Ricci soliton $(M,(g,\xi,\lambda,\mu))$ $(n>1)$ can not be projective Ricci-symmetric.

IV. CONHARMONICALLY FLAT ALPHA-SASAKIAN ETA-RICCI SOLITONS

It is known that in case of a conharmonically flat α-Sasakian manifold $M(\phi,\xi,\eta,g)$ $(n>3)$ the curvature $'R$ of type (0,4) has the following form [16].

$$'R(X,Y,Z,W) = (1/(n-2))[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \quad (4.1)$$

Contracting (3.10) over X and Y we get $r = -(n\lambda+\mu)$, where r denotes the scalar curvature of the manifold. By making use of the value of r and (3.10) equation (4.1) can be written as

$$'R(X,Y,Z,W) = ((2\lambda)/((n-1)(n-2)))[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] - (\mu/(n-2))[\eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W) + \eta(X)\eta(W)g(Y,Z) - \eta(Y)\eta(W)g(X,Z)] \quad (4.2)$$

Let $n = \sqrt{[(2\lambda)/((n-1)(n-2))]}$ and $n' = (\mu/(n-2)) \sqrt{[((n-1)(n-2))/(2\lambda)]}$

Then equation (4.2) can be written as

$$'R(X,Y,Z,W) = H(X,Z)H(Y,W) - H(X,W)H(Y,Z) \quad (4.3)$$

where

$$H(X,Y) = ng(X,Y) + n'\eta(X)\eta(Y) \quad (4.4)$$

The above result is stated in the following theorem:

Theorem 4: A conharmonically flat α-Sasakian η-Ricci soliton $M(\varphi,\xi,\eta,g)$ is $(\Phi H)_n$ with associated symmetric tensor H given by (4.4).

REFERENCES

[1] A.M. Blaga,: η-Ricci solitons on Lorentzian para-Sasakian manifolds, Filomat 30(2),489-496(2016).

[2] A.M. Blaga,: η-Ricci solitons on para Kenmotsu manifolds. Balkan J, Geom. Appl. 20(1),1-13(2015).

[3] A.M. Blaga,: M. Crasmareanu, "Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry, Filomat 31:2 (2017), 499–504, doi: 10.2298/FIL1702499B.

[4] C.L. Bejan, M. Crasmareanu,: Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry, Ann. Glob. Anal. Geom. doi.:10.1007/s10455-014-9414-4.

[5] C. Calin, M. Crasmareanu,: Eta-Ricci solitons on Hopf hypersurfaces in complex space forms, Revue Roumaine de Mathematiques pures et appliques 57(1), 55-63(2012).

[6] G. Calvaruso, D. Perrone,: Geometry of H-paracontact metric manifolds (2013), arXiv:1307.7662v1.

[7] G. Calvaruso, A. Zaeim,: A complete classification of Ricci and Yamabe solitons of non-reductive homogeneous 4-space, J. Geom. Phys.80,15-25(2014).

[8] G. Calvaruso, D. Perrone,: Ricci solitons in three-dimensional paracontact geometry (2014), arXiv:1407.3458v1.

[9] H.D. Cao,: Recent progress on Ricci solitons Adv. Lec. Math. (ALM) 11,1-38(2009), arXiv:0908:2006v1.

[10] J.T. Cho, M. Kimura,: Ricci solitons and real hypersurfaces in a complex space form, Tohoku Math. J. 61(2),205-212(2009).

[11] I. Gurupadavva, C.S. Bagewadi,: Ricci solitons in α-Sasakian manifolds, International Scholarly Research Network, ISRN Geometry, Vol.2012, Article ID 421384, doi:10.5402/2012/421384.

[12] R.S. Hamilton,: Three-manifolds with positive Ricci curvature, Journal of Differential geometry, Vol. 17, no.2, pp. 255-306, 1982.

[13] R.S. Hamilton,: The Ricci flow on surfaces, In:Mathematics and general relativity (Santa cruz, CA, 1986), Contemp Math. Vol. 71, pp. 237-262, Amr. Math. Soc., Providence (1988).

[14] R. Pina, K. Tenenblat,: On solutions of the Ricci curvature and the Einstein equation, Isr. J. Math., 171,61-76(2009).

[15] D.G. Prakasha,: Torseforming vector fields in a 3-dimensional para-Sasakian manifold, Sci. Stud. Res. Ser. Math. Inform. 20(2), 61-66(2010).

[16] K. Yano, M. Kon,: Structures on Manifolds, Series in Pure Mathematics, Vol. 3, Word Scientific, Singapore (1984).

[17] R. Sharma,: Second order parallel tensor in real and complex space forms, Internat. J. Math. Math. Sci. 12,4(1989),787-790.

[18] M. Crasmareanu,: Parallel tensors and Ricci Solitons in N(k)- quasi Einstein manifolds, Indian J. Appl. Math. 43, 4 (2012).

[19] M. Okumura, Some remarks on space with a certain contact structure, Tohoku Math.J. (2)14(1962), 135-145.

[20] S. Sasaki, Almost-contact manifolds Part I Lecture Notes, Mathematical Institute, Tohoku University, 1965.