

# Regions of Motion in the R3BP with Finite Straight Segment and Oblateness

Dinesh Kumar<sup>1</sup>, Mamta Jain<sup>\*2</sup>, Satyendra Kumar Satya<sup>3</sup>, Govind Kumar Jha<sup>4</sup>

<sup>1</sup>Department of Mathematics, Shyama Prasad Mukherji College (For Women), University of Delhi, New Delhi, India.

<sup>2</sup>Department of Mathematics, Bhagini Nivedita College, University of Delhi, New Delhi, India

<sup>3</sup>Department of Mathematics, Sri Aurobindo College, University of Delhi India.

<sup>4</sup>Department of Mathematics, Vinoba Bhave University, Hazaribagh, Jharkhand, India.

**Abstract:** In this paper, we have studied the regions of motion in the restricted three body problem (R3BP). The smaller primary is taken as a finite straight segment and bigger one as an oblate spheroid. There exist five libration points in this problem, out of which three are collinear and two are non-collinear, with the primaries. The collinear libration points are unstable for all values of mass parameter  $\mu$  and the non-collinear libration points are stable for a critical value of  $\mu$ . The location of libration points for different values of mass parameter, length of straight segment and oblateness parameter are also discussed numerically and shown graphically. The regions of motion of the infinitesimal mass are also studied.

**Key Words:** Restricted three body problem, Libration points, Straight segment, Oblate Spheroid, Zero Velocity Curve.

## I. Introduction

The restricted problem of three bodies describes the motion of the infinitesimal mass  $m_3$  moving in the gravitational field of two massive primaries in the same plane or out of plane called two dimensional or three dimensional problem accordingly. The primaries are revolving around their center of mass either in circular or elliptical orbits under the influence of their mutual gravitational attraction. If the orbit of the primaries around their center of mass is circular, problem is said to be circular restricted three body problem or restricted three body problem (R3BP). The problem possesses five libration points out of which three are collinear and two are non-collinear.

Bhatnagar and Chawla (1977) have studied the effect of oblateness on the collinear libration points in the restricted three body problem. They found that the location of libration points and the roots of the characteristic equation at these points also depend on the oblateness term of the bigger primary. Sharma (1987) studied the linear stability

of stationary solutions of the photogravitational restricted three body problem when the more massive primary is a source of radiation and the smaller primary is an oblate spheroid. He found that the collinear equilibria have conditional retrograde elliptical periodic orbits around them in the linear sense, while the triangular points have long or short periodic retrograde elliptical orbits. Jain and Sinha (2014) have studied stability and regions of motion in the restricted three body problem by taking both primaries as finite straight segments. They observed that the collinear equilibrium points are unstable for all values of mass parameter and the triangular equilibrium points are conditionally stable for  $0 < \mu < \mu_c$  and unstable in the range  $\mu_c < \mu \leq 1/2$ , where  $\mu$  is the mass ratio. They have also discussed the regions of motion of the infinitesimal mass and found that the Jacobian constant decreases in comparison to the classical case of the R3BP, for a fixed value of  $\mu$  and lengths  $l_1$  and  $l_2$  of the segments. In 1999, Riaguas et al. have studied periodic orbits around a massive straight segment and observed many periodic orbits and bifurcations in their study. Non-linear stability of the equilibria in the gravity field of a finite straight segment has been studied by Riaguas et al. in 2001, by considering one body as a finite straight segment and other is a point mass. They have observed that there exist four equilibrium points. Kumar et al. (2016) studied existence and stability of libration points in the restricted three body problem under the combined effects of finite straight segment and oblateness. They found that, there exist five libration points, out of which three are collinear and two are non-collinear with the primaries. The collinear libration points are unstable for all values of mass parameter  $\mu$ , and the non-collinear libration points are stable if  $\mu < \mu_c$ , where

$$\mu_c = 0.038521 - 0.00735612 - 0.285002 A.$$

Jain and Aggarwal (2015) determined the existence and stability of libration points in the restricted problem under the effect of Poynting Robertson

Light Drag and conclude that both the non-collinear libration points are unstable. By considering smaller primary as an oblate spheroid, the existence and stability of the non-collinear libration points with Stokes drag effect have examined by Jain and Aggarwal (2015). They found that the non-collinear libration points are unstable. Khanna and Bhatnagar (1999) studied R3BP by taking one of the primary as an oblate spheroid. They found that the collinear libration points are unstable and the triangular libration points are stable for a critical value of mass parameter. Aggarwal et al. (2006) discussed the non-linear stability of the triangular libration point  $L_4$  of the restricted three body problem under the presence of the third and fourth order resonances by taking bigger primary as an oblate body and the smaller one as a triaxial body and both are source of radiation. They found that  $L_4$  is always unstable. The equilibrium solutions and linear stability of  $m_3$  and  $m_4$  considering one of the primaries as an oblate spheroid have been examined by Aggarwal and Kaur (2014). They concluded that there are no non-collinear equilibrium solutions of the system.

## II. Equations of Motion

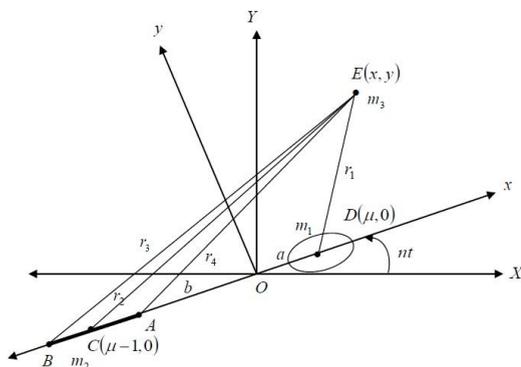


Fig.1. The configuration of R3BP when  $m_1$  is an oblate spheroid and  $m_2$  is a finite straight segment

Let  $m_1, m_2$  be the masses of an oblate spheroid and finite straight segment (called primaries), that are moving with angular velocity  $n$  (say) in circular orbits about their common centre of mass  $O$ . Suppose there is an infinitesimal mass  $m_3$ , which is moving in the plane of motion of  $m_1$  and  $m_2$  ( $m_1 \geq m_2$ ).  $O(XYZ)$  and  $O(xyz)$  are inertial and synodic coordinate system respectively. The line joining  $m_1$  and  $m_2$  is taken as  $X$ -axis and  $O$  their centre of mass as origin and the line passing through  $O$  and perpendicular to  $OX$  and lying in

the plane of motion of  $m_1$  and  $m_2$  is taken as  $Y$ -axis.  $O(xyz)$  initially coincident with the inertial coordinate system  $O(XYZ)$ . The synodic axes are rotating with angular velocity  $n$  (say) about  $Z$ -axis (the  $z$ -axis is coincident with  $Z$ -axis) (Fig. 1). The equations of motion of  $m_3$  in the dimensionless synodic coordinate system are (Kumar et al. (2016))

$$\ddot{x} - 2n\dot{y} = \Omega_x, \tag{1}$$

$$\ddot{y} + 2n\dot{x} = \Omega_y, \tag{2}$$

where

$$\Omega = \frac{1}{2}n^2(x^2 + y^2) + \frac{1-\mu}{r_1} \left( 1 + \frac{A}{2r_1^2} \right) + \frac{\mu}{2l} \log \left( \frac{r_3 + r_4 + 2l}{r_3 + r_4 - 2l} \right),$$

$$\mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2}, \Rightarrow m_2 = \mu, m_1 = (1 - \mu),$$

$$r_1^2 = (x - \mu)^2 + y^2, r_2^2 = (x - \mu + 1)^2 + y^2,$$

$$r_3^2 = \{x - (\mu - 1 - l)\}^2 + y^2,$$

$$r_4^2 = \{x - (\mu - 1 + l)\}^2 + y^2,$$

$$A = \frac{a^2 - c^2}{5},$$

$$n^2 = (1 + l^2 + \frac{3}{2}A), l \ll 1, 0 < A \ll 1,$$

$2l$  = Dimensionless length of the straight segment  $AB$ .

Here, we are taking terms containing  $l$  up to second order, and terms containing  $A$  up to first order.

## III. Libration points

The libration points are the solution of the equations

$$n^2 x - (1 - \mu)(x - \mu) \left( \frac{1}{r_1^3} + \frac{3A}{2r_1^5} \right) - \frac{2\mu(x - \mu + 1)}{r_3 r_4 (r_3 + r_4)} = 0, \tag{3}$$

$$y \left[ n^2 - (1 - \mu) \left( \frac{1}{r_1^3} + \frac{3A}{2r_1^5} \right) - \frac{2\mu(r_3 + r_4)}{r_3 r_4 \{(r_3 + r_4)^2 - 4l^2\}} \right] = 0. \tag{4}$$

### Collinear libration points:

The collinear libration points are obtained from Eqs. (3) and (4) by taking  $y = 0$ . The abscissa of the collinear libration points are the roots of the equation

$$f(x,0) = n^2 - (1 - \mu) \left( \frac{1}{r_1^3} + \frac{3A}{2r_1^5} \right) - \frac{2\mu(r_3 + r_4)}{r_3 r_4 \left\{ (r_3 + r_4)^2 - 4l^2 \right\}} = 0.$$

There exist three collinear libration points  $L_1, L_2$  and  $L_3$ . In Table 1, we have calculated numerical values of these collinear libration points

$\mu$	$L_1$	$L_2$	$L_3$	$L_{4,5}$
0.05	(-1.21570, 0)	(-0.727161, 0)	(1.02100, 0)	(-0.514603, $\pm 0.822535$ )
0.09	(-1.23508, 0)	(-0.649120, 0)	(1.03978, 0)	(-0.474671, $\pm 0.822571$ )
0.15	(-1.24198, 0)	(-0.549070, 0)	(1.06796, 0)	(-0.414787, $\pm 0.822631$ )
0.20	(-1.23798, 0)	(-0.472851, 0)	(1.09142, 0)	(-0.364896, $\pm 0.822688$ )
0.25	(-1.22894, 0)	(-0.400162, 0)	(1.11477, 0)	(-0.315020, $\pm 0.822753$ )
0.30	(-1.21652, 0)	(-0.329698, 0)	(1.13797, 0)	(-0.265162, $\pm 0.822827$ )

Table 1. Location of Libration points, when  $A = 0.15, l = 0.1$ .

$A$	$L_1$	$L_2$	$L_3$	$L_{4,5}$
0.00001	(-1.27665, 0)	(-0.511016, 0)	(1.05913, 0)	(-0.351590, $\pm 0.861617$ )
0.00005	(-1.27654, 0)	(-0.511163, 0)	(1.05916, 0)	(-0.351790, $\pm 0.861504$ )
0.0010	(-1.27640, 0)	(-0.511345, 0)	(1.05920, 0)	(-0.352040, $\pm 0.861363$ )
0.0100	(-1.27399, 0)	(-0.514528, 0)	(1.05993, 0)	(-0.356480, $\pm 0.858841$ )
0.1000	(-1.25236, 0)	(-0.539062, 0)	(1.06566, 0)	(-0.396013, $\pm 0.834934$ )
0.1500	(-1.24198, 0)	(-0.549070, 0)	(1.06796, 0)	(-0.414787, $\pm 0.822631$ )

Table 2. Location of Libration points, when  $\mu = 0.15, l = 0.1$ .

$l$	$L_1$	$L_2$	$L_3$	$L_{4,5}$
0.001	(-1.23420, 0)	(-0.558627, 0)	(1.07025, 0)	(-0.413271, $\pm 0.826272$ )
0.005	(-1.23422, 0)	(-0.558605, 0)	(1.07025, 0)	(-0.413274, $\pm 0.826263$ )
0.010	(-1.23428, 0)	(-0.558533, 0)	(1.07023, 0)	(-0.413286, $\pm 0.826236$ )
0.050	(-1.23614, 0)	(-0.556243, 0)	(1.06968, 0)	(-0.413650, $\pm 0.825360$ )
0.100	(-1.24198, 0)	(-0.549070, 0)	(1.06796, 0)	(-0.414787, $\pm 0.822631$ )
0.150	(-1.25181, 0)	(-0.537073, 0)	(1.06514, 0)	(-0.416680, $\pm 0.818106$ )

Table 3. Location of Libration points, when  $\mu = 0.15, A = 0.15$ .

Libration points	Coordinates	Jacobian constant
$L_1(x_1, 0)$	(-1.253290, 0)	$C_1=3.58226$
$L_2(x_2, 0)$	(-0.290965, 0)	$C_2=3.95603$
$L_3(x_3, 0)$	(1.124790, 0)	$C_3=3.32301$
$L_4(x_4, y_4)$	(-0.20502, 0.863097)	$C_4=3.80895$

Table 4. Coordinates and value of Jacobian constant of all libration points when  $\mu = 0.3, l = 0.01, A = 0.001$

for fixed values of  $A = 0.15, l = 0.1$  and different values of  $\mu = 0.05, 0.09, 0.15, 0.20, 0.25, 0.30$ . In Table 2, we have calculated numerical values of these collinear libration points for fixed values of  $\mu = 0.15, l = 0.1$  and different values of  $A = 0.0001, 0.0005, 0.001, 0.01, 0.1, 0.15$ . In

Table 3, we have calculated numerical values of these collinear libration points for fixed values of  $\mu = 0.15, A = 0.15$  and different values of  $l = 0.001, 0.005, 0.01, 0.05, 0.1, 0.15$ .

**Non-collinear libration points:**

The non-collinear libration points are obtained from Eqs. (3) and (4) by taking  $y \neq 0$ . There exist two non-collinear libration points namely  $L_4$  and  $L_5$ . The numerical values of non-collinear libration points for different values of parameters are given in Table 1, 2 and 3.

**IV. Regions of Motion**

To find the possible regions of motion, we find the Jacobian integral. On multiplying Eq. (1) by  $\dot{x}$ , Eq. (2) by  $\dot{y}$  and add them, we have

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\Omega_x + \dot{y}\Omega_y \tag{5}$$

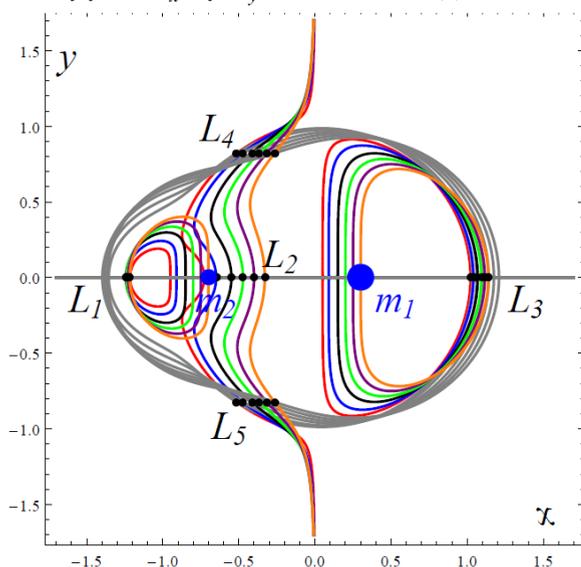


Fig. 2(a) The location of libration points for  $\alpha = 0.05, k = 10^{-5}, l = 0.05$  and  $\mu = 0.05, 0.09, 0.15, 0.20, 0.25, 0.30$ .

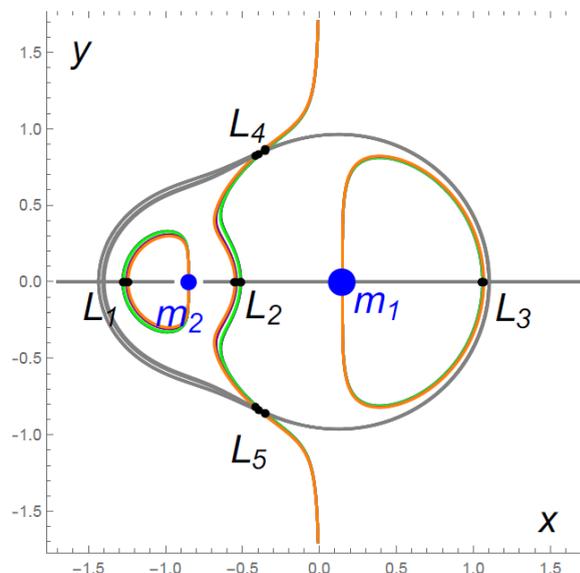


Fig. 2(b) The location of libration points for  $\mu = 0.15, l = 0.1$  and  $A = 0.0001, 0.0005, 0.001, 0.01, 0.1, 0.15$ .

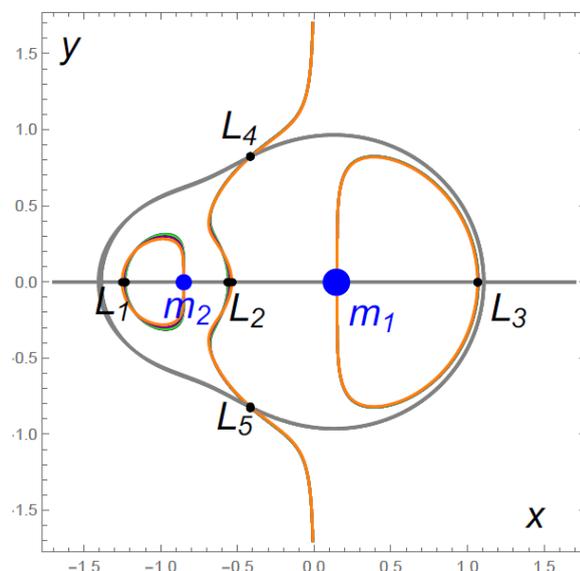


Fig. 2(c) The location of libration points for  $\mu = 0.15, A = 0.15$  and  $l = 0.001, 0.005, 0.01, 0.05, 0.1, 0.15$ .

On integrating both sides of Eq. (5), we get the Jacobian integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{6}$$

where  $C$  is Jacobi constant. We have calculated coordinates of libration points, for fixed values of  $\mu = 0.3, l = 0.01$  and  $A = 0.001$ . Then, for these libration points we have determined the values of various Jacobian constant  $C$ , by using the zero-velocity curve  $\dot{x}^2 + \dot{y}^2 = 2\Omega - C$ , which are given in Table 4.

The zero velocity curves are shown on the  $xy$  plane. The four possibilities are (a)  $C_2 < C$ , (b)  $C_1 < C < C_2$ , (c)  $C_3 < C < C_1$ , (d)  $C_4 = C_5 < C < C_3$ . Now, we discuss all these cases. The values of Jacobian constant are given in Table 4, and we have chosen some value of  $C$  arbitrarily.

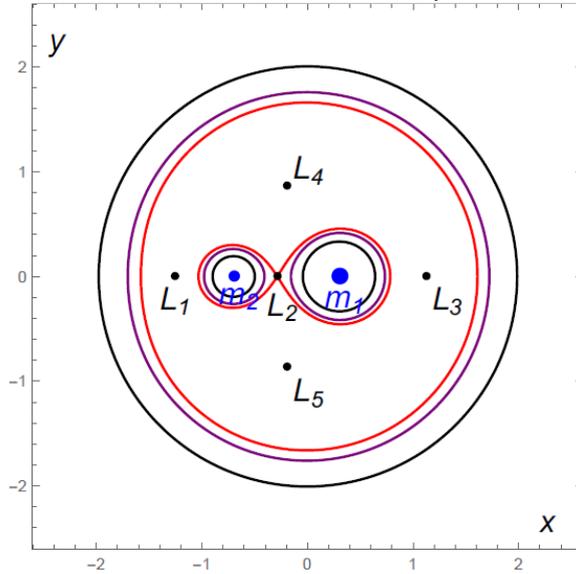


Fig. 3(a) Curves of zero velocity for  $\mu = 0.3, C=5$ (black),  $C=4.2$ (purple),  $C=C_2=3.92404$ (red).

ovals shrink. We have also seen that when  $C = C_2$ , inside oval enlarge, meet at the point  $L_2$  and form a figure eight. In this process, the ovals never enclose  $L_1$  and  $L_3$ , since the corresponding values of  $C_1$  and  $C_3$  are smaller than  $C_2$ .

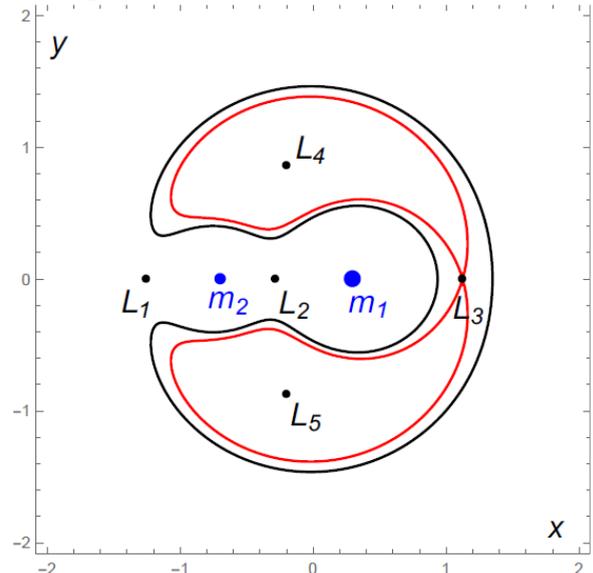


Fig. 3(c) Curves of zero velocity for  $\mu = 0.3, C=3.45$ (black),  $C=C_3=3.29463$ (red).

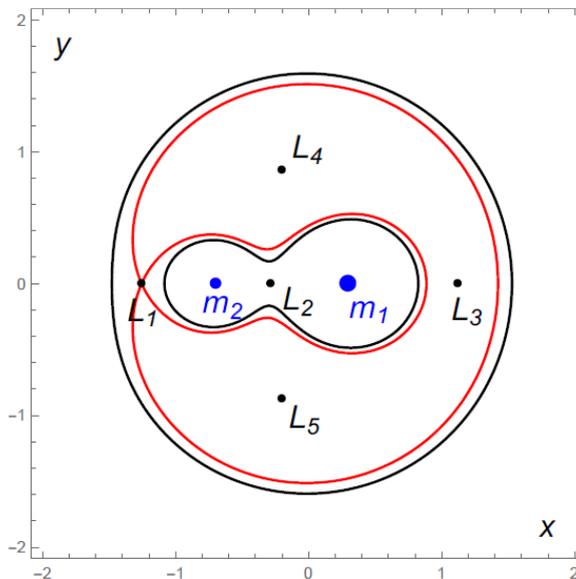


Fig. 3(b) Curves of zero velocity for  $\mu = 0.3, C=3.75$ (black),  $C=C_1=3.55924$ (red).

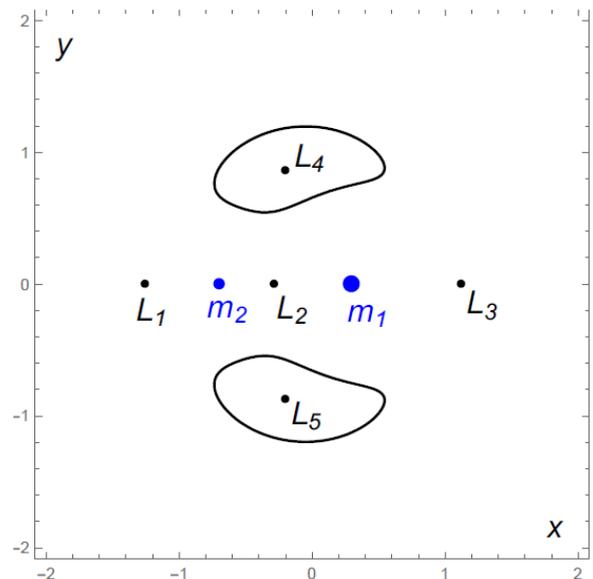


Fig. 3(d) Curves of zero velocity for  $\mu = 0.3, C=3.0$ (black),  $C=C_4=2.79196$ .

**Case (a)** In Fig. 3(a), the zero velocity curves are shown for  $C = 5$  (black),  $C = 4.2$  (blue) and  $C = C_2 = 3.95603$  (red). We observe that as  $C$  decreases, the size of the zero velocity ovals surrounding  $m_1$  and  $m_2$  increase while the outer

**Case (b)** In Fig. 3(b), we observe that when  $C_1 < C < C_2$ , the curves of zero velocity constitutes two branches. The first branch is

dumbbell or pear-shaped figure inside of which motion is possible. This curve encloses  $m_1, m_2$  and  $L_2$ , but  $L_1$  and  $L_3$  are outside. In this case we have taken  $C = 3.75$ . Secondly, we see that as the value of  $C$  decreases and reach at  $C = C_1 = 3.58226$ , the dumbbell shape ends at  $L_1$  between the inner and outer area. In this case motion is always possible outside these ovals.

**Case (c)** In Fig. 3(c), we observe that as the value of  $C$  decreases from  $C = C_1$ , the point on the curve is not on the curve of zero velocity. The cusp at  $L_1$  exists when  $C = C_1$ , disappears and the curve does not intersect the  $x$ -axis. We see a horseshoe shaped curve which encloses only  $L_3$  as  $C \rightarrow C_3$ . Here we have taken  $C = 3.45$ . If we take  $C = C_3 = 3.32301$ , we observe that the curve forms a cusp at  $L_3$ . The horseshoe shaped curve begins at  $C_1$  when a cusp is formed at  $L_1$  and end with  $C_3$  when the cusp is at  $L_3$ . Motion is possible everywhere outside the area enclosed by the horseshoes.

**Case (d)** In Fig. 3(d), we observe that as the value of  $C$  decreases, the cusp disappears and the curve leaves  $L_3$ . The zero-velocity curve form two branches, one enclosing  $L_4$  and the other  $L_5$ . In this case, we take  $C = 2.9$ . If we take  $C = C_4 = 2.80895$ , then these curves shrink to the point  $L_4$  and  $L_5$ . Motion is possible everywhere outside the tadpole like area.

## V. Conclusion

We have investigated the motion of the infinitesimal mass in the restricted three body problem when the bigger primary is an oblate spheroid and smaller one as a finite straight segment. It is observed that, there exist five libration points, three collinear and two non-collinear. The numerical values of libration points for different values of parameters  $\mu, A$  and  $l$  are given in Table 1, 2 and 3. We observed that, as we increase the parameters  $\mu, A$  and  $l$ ,  $L_1$  moves away from the smaller primary along  $x$ -axis.  $L_2$  moves towards the center of mass of the primaries along  $x$ -axis.  $L_3$  moves away from the bigger primary along  $x$ -axis.  $L_{4,5}$  moves towards  $y$ -axis, parallel to  $x$ -

axis. The locations of libration points are shown graphically for different values of parameters,  $\mu, A$  and  $l$  in Fig.2. We have also discussed the regions of motion of the infinitesimal mass  $m_3$ . In our case for  $\mu = 0.3$ , the value of Jacobian constants

are  $C_1 = 3.58226$ ,  $C_2 = 3.95603$ ,  $C_3 = 3.32301$ ,  $C_4 = 2.80895$ . And in the classical case for  $\mu = 0.3$ , Jacobian constant are  $C_1 = 4.033$ ,  $C_2 = 4.130$ ,  $C_3 = 3.501$ ,  $C_4 = 3.0$  (Szebehely 1967). We observed that in our case for a fixed value of mass parameter  $\mu$  due to the presence of  $A$  and  $l$  in the Jacobi integral the value of the Jacobian constant decreases in comparison to classical case while increases in comparison to Jain and Sinha (2014).

## REFERENCES

- [1] A. Riaguas, A. Elipe and M. Lara, Periodic Orbits around a massive Straight Segment, "*Celestial Mechanics and Dynamical Astronomy*," 73(1), 169-178, 1999.
- [2] A. Riaguas, A. Elipe, and T. Lopez-Moratalla, "Non-Linear Stability of the equilibria in the gravity field of a finite straight segment," *Celestial Mechanics and Dynamical Astronomy*, 81, 235-248, 2001.
- [3] D. Kumar, R. Aggarwal and M. Jain, "Combined effects of finite straight segment and oblateness on the libration points in the restricted three body problem," *Int. J. Tech.* 6(2), 185-190, 2016.
- [4] K.B. Bhatnagar, J.M. Chawla, "The effect of oblateness of the bigger primary on collinear libration points in the restricted problem of three bodies," *Celestial Mechanics*, 16, 129-136, 1977.
- [5] M. Jain and R. Aggarwal, "Existence and Stability of Non-collinear libration points in the restricted problem with Poynting Robertson Light Drag Effect," *Int. J. of Mathematics Trends and Technology*, 19, 20-33, 2015.
- [6] M. Jain, R. Aggarwal, "A study of non-collinear libration points in the restricted three body problem with Stokes drag effect when smaller primary is an oblate spheroid," *Astrophys and space Science*, 358, 51, 2015.
- [7] M. Khanna, K.B. Bhatnagar, "Existence and stability of libration points in the restricted three body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate spheroid," *Indian J. Pure Appl. Math.*, 30(7), 721-733, 1999.
- [8] R. Aggarwal, Z.A. Taqvi and I. Ahmad, "Non-linear stability of  $L_4$  in the Restricted Three Body Problem for radiated axes symmetric primaries with resonances," *Bulletin of Astronomical Society of India*, 34, 327-356, 2006.
- [9] R. Aggarwal, B. Kaur, "Robes restricted problem of 2+2 bodies with one of the primaries an oblate body," *Astrophys and Space Science*, 352(2), 467-479, 2014.
- [10] R. Jain, D. Sinha, "Stability and regions of motion in the restricted three-body problem when both the primaries are finite straight segments," *Astrophys and Space Science*, 351, 87-100, 2014.
- [11] R.K. Sharma, "The linear stability of libration points of the photogravitational restricted three body problem

- when the smaller primary is an oblate spheroid,”  
*Astrophys and Space Science*, 135(2), 271-281, 1987.
- [12] V. Szebehely, “Theory of Orbits, the Restricted  
Problem of Three Bodies,” *Academic Press, New  
York*, 1967.