Numerical Simulation of 1D Heat Conduction in Spherical and Cylindrical Coordinates by Fourth-Order Finite Difference Method

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Abstract. This paper aims to apply the Fourth Order Finite Difference Method to solve the one-dimensional Convection-Diffusion equation with energy generation (or sink) in in cylindrical and spherical coordinates.

Keywords. Central Difference Method, Cylindrical and Spherical coordinates, Numerical Simulation, Numerical Efficiency.

1. Introduction

According to [1-2] heat conduction refers to the transport of energy in a medium due to the temperature gradient. To represent the physical phenomena of three-dimensional heat conduction in steady state and in cylindrical and spherical coordinates, respectively, [1] present the following equations,

$$\rho c_p v_r \frac{\partial T}{\partial r} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q} \tag{1}$$

$$\rho c_p v_r \frac{\partial T}{\partial r} = k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right) + \dot{q} \tag{2}$$

where, T is the temperature, r, z and θ are the spatial coordinates, ρ is the specific mass, c_p is the specific heat, v_r is the velocity, k the thermal conductivity, $\dot{q}$ is a heat flux.

In this work the numerical solution will be proposed by using the Fourth Order Finite Difference Method, of the reduction of the problems described in Equations (1-2) for only one spatial dimension, according to the following equations,

$$\rho c_p v_r \frac{\partial T}{\partial r} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right) + \dot{q} \tag{3}$$

$$\rho c_p v_r \frac{\partial T}{\partial r} = k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right) + \dot{q} \tag{4}$$

This proposal is a numerical evolution in the work proposed in [3] where the Second Order Finite Difference Method is used to solve the problems governed by Equations (3-4). It is important to emphasize that the idea of using the Fourth Order Finite Difference Method has already been successful in [4-8] for problems in cartesian coordinates, and thus, the same idea of solution to problems in cylindrical and spherical coordinates is now proposed.

2. Numerical Formulation – Spatial Discretization

Before starting the spatial discretization, here it will be realized a reorganization of Equations (3-4), respectively, as follows (adopting $\alpha = k/(\rho c_p)$),

$$v_r \frac{\partial T}{\partial r} = \alpha\left(\frac{1}{r}\right)\left(\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2}\right) + \dot{q} \Rightarrow$$

$$\alpha\frac{\partial^2 T}{\partial r^2}+\left(\frac{\alpha}{r}-v_r\right)\frac{\partial T}{\partial r}=-\dot{q} \tag{5}$$

$$v_r\frac{\partial T}{\partial r}=\alpha\left(\frac{1}{r^2}\right)\left(2r\frac{\partial T}{\partial r}+r^2\frac{\partial^2 T}{\partial r^2}\right)+\dot{q}\Rightarrow$$

$$\alpha\frac{\partial^2 T}{\partial r^2}+\left(\frac{2\alpha}{r}-v_r\right)\frac{\partial T}{\partial r}=-\dot{q} \tag{6}$$

Internal nodes

In the internal nodes of the computational mesh, the following fourth order central finite differences were used to discretize the first and second order partial derivatives, respectively [9-10],

$$\frac{\partial T_i}{\partial r}=\frac{-T_{i+2}+8T_{i+1}-8T_{i-1}+T_{i-2}}{12\Delta r} \tag{7}$$

$$\frac{\partial^2 T_i}{\partial r^2}=\frac{-T_{i+2}+16T_{i+1}-30T_i+16T_{i-1}-T_{i-1}}{12\Delta r^2} \tag{8}$$

After replacing the approximations (7-8) in Equations (5-6), the following expressions are obtained,

$$\alpha\left(\frac{-T_{i+2}+16T_{i+1}-30T_i+16T_{i-1}-T_{i-2}}{12\Delta r^2}\right)+\left(\frac{\alpha}{r}-v_r\right)\left(\frac{-T_{i+2}+8T_{i+1}-8T_{i-1}+T_{i-2}}{12\Delta r}\right)=-\dot{q}\Rightarrow$$

$$\left(\frac{-\alpha}{12\Delta r^2}+\frac{\alpha}{12r\Delta r}-\frac{v_r}{12\Delta r}\right)T_{i-2}+\left(\frac{4\alpha}{3\Delta r^2}-\frac{2\alpha}{3r\Delta r}+\frac{2v_r}{3\Delta r}\right)T_{i-1}+\left(\frac{-5\alpha}{2\Delta r^2}\right)T_i$$
$$+\left(\frac{4\alpha}{3\Delta r^2}+\frac{2\alpha}{3r\Delta r}-\frac{2v_r}{3\Delta r}\right)T_{i+1}+\left(\frac{-\alpha}{12\Delta r^2}-\frac{\alpha}{12r\Delta r}+\frac{v_r}{12\Delta r}\right)T_{i+2}=-\dot{q} \tag{9}$$

and

$$\alpha\left(\frac{-T_{i+2}+16T_{i+1}-30T_i+16T_{i-1}-T_{i-2}}{12\Delta r^2}\right)+\left(\frac{2\alpha}{r}-v_r\right)\left(\frac{-T_{i+2}+8T_{i+1}-8T_{i-1}+T_{i-2}}{12\Delta r}\right)=-\dot{q}$$

$$\left(\frac{-\alpha}{12\Delta r^2}+\frac{\alpha}{6r\Delta r}-\frac{v_r}{12\Delta r}\right)T_{i-2}+\left(\frac{4\alpha}{3\Delta r^2}-\frac{4\alpha}{3r\Delta r}+\frac{2v_r}{3\Delta r}\right)T_{i-1}+\left(\frac{-5\alpha}{2\Delta r^2}\right)T_i$$
$$+\left(\frac{4\alpha}{3\Delta r^2}+\frac{4\alpha}{3r\Delta r}-\frac{2v_r}{3\Delta r}\right)T_{i+1}+\left(\frac{-\alpha}{12\Delta r^2}-\frac{\alpha}{6r\Delta r}+\frac{v_r}{12\Delta r}\right)T_{i+2}=-\dot{q} \tag{10}$$

Nodes distant Δx and/or Δy of the boundary

For discretization of nodes near the boundary it is not possible to use the expressions (7-8), for example, a node at a distance Δx from the boundary will not have two nodes to its left. Thus, for these nodes will be used to discretize the Equations (5-6) the following second order central finite difference,

$$\frac{\partial T}{\partial r}=\frac{T_{i+1}-T_{i-1}}{\Delta r} \tag{11}$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2} \tag{12}$$

what resulted in the following expressions,

$$\alpha\left(\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2}\right) + \left(\frac{\alpha}{r} - v_r\right)\left(\frac{T_{i+1} - T_{i-1}}{2\Delta r}\right) = -\dot{q} \Rightarrow$$

$$\left(\frac{\alpha}{\Delta r^2} - \frac{\alpha}{2r\Delta r} + \frac{v_r}{2\Delta r}\right)T_{i-1} + \left(\frac{-2\alpha}{\Delta r^2}\right)T_i + \left(\frac{\alpha}{\Delta r^2} + \frac{\alpha}{2r\Delta r} - \frac{v_r}{2\Delta r}\right)T_{i+1} = -\dot{q} \tag{13}$$

and

$$\alpha\left(\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2}\right) + \left(\frac{2\alpha}{r} - v_r\right)\left(\frac{T_{i+1} - T_{i-1}}{2\Delta r}\right) = -\dot{q} \Rightarrow$$

$$\left(\frac{\alpha}{\Delta r^2} - \frac{\alpha}{r\Delta r} + \frac{v_r}{2\Delta r}\right)T_{i-1} + \left(\frac{-2\alpha}{\Delta r^2}\right)T_i + \left(\frac{\alpha}{\Delta r^2} + \frac{\alpha}{r\Delta r} - \frac{v_r}{2\Delta r}\right)T_{i+1} = -\dot{q} \tag{14}$$

In summary, Equations (9) and (13) constructs the linear system that solves the problem governed by Equation (5) (cylindrical coordinates) and Equations (10) and (14) solves Equation (6) (spherical coordinates).

3. Numerical Applications

To analyze the numerical efficiency of the formulation presented in this work will be presented two applications. The first makes a comparison of numerical results with the presented in [3] while the second application presents an exact solution to analyze the numerical efficiency. In both applications it was considered $\alpha = 1$ and $v_r = 1$.

Aplicação 1: The analytical solutions for the cylindrical and spherical coordinates that will be used for comparison with the numerical results are, respectively,

$$T(r) = A \ln r + B \qquad \text{e} \qquad T(r) = \frac{A}{r} + B$$

where A and B are constant.

In this case, the following boundary conditions were considered $T(0.5) = 0$ and $T(1) = C$, with C constant. From the analytical solutions and using the boundary conditions, the following solutions were obtained, respectively:

$$T(r) = C(\ln(r)/\ln(2)) + C \quad \text{and} \quad T(r) = -(C/r) + 2C.$$

Table 1. Maximum Error for various C values (cylindrical coordinates).

C	**1**	**2**	**3**	**4**	**5**
Fourth Order	1.85E-11	1.85E-11	1.85E-11	1.85E-11	1.85E-11
Second Order [2]	7.59E-05	1.51E-04	2.27E-04	3.03E-04	3.79E-04

Table 2. Maximum Error for various *C* values (spherical coordinates).

C	**1**	**2**	**3**	**4**	**5**
Fourth Order	6.95E-10	1.39E-09	2.08E-09	2.78E-09	3.47E-09
Second Order [2]	8.94E-07	1.78E-06	2.62E-06	3.57E-06	4.05E-06

Analyzing Tables 1 and 2 it is evident that the use of a discretization by Fourth Order Finite Difference Method presents an evolution in numerical precision.

Aplicação 2: In this case, the exact solution $T(r)$ = e^r was used for comparison with the numerical solution. For this, the values of Δr were varied to analyze how much it improved the numerical efficiency of the proposed formulation (see Table 3). It is evident the improvement of the numerical accuracy as Δr decreases.

Table 3. Maximum error of the numerical solution in Application 2.

Δr	**Cylindrical**	**Spherical**	Δr	**Cylindrical**	**Spherical**
0.05000	2.56E-06	6.38E-06	0.00455	2.00E-10	4.71E-10
0.02500	1.72E-07	4.15E-07	0.00417	1.42E-10	3.33E-10
0.01667	3.48E-08	8.32E-08	0.00385	1.03E-10	2.42E-10
0.01250	1.12E-08	2.65E-08	0.00357	7.66E-11	1.80E-10
0.01000	4.61E-09	1.09E-08	0.00333	5.85E-11	1.37E-10
0.00833	2.23E-09	5.28E-09	0.00313	4.51E-11	1.06E-10
0.00714	1.21E-09	2.86E-09	0.00294	3.52E-11	8.26E-11
0.00625	7.12E-10	1.68E-09	0.00278	2.79E-11	6.56E-11
0.00556	4.45E-10	1.05E-09	0.00263	2.25E-11	5.30E-11
0.00500	2.93E-10	6.88E-10	0.00250	1.86E-11	4.34E-11

4. Conclusion

The expectation of using fourth order discretization and obtaining better results, in a very expressive way was achieved. It is important to note that a more detailed study of the cost benefit, for example, the increase of the computational cost versus the numerical efficiency when choosing between a second or fourth order discretization should be evaluated in more complex applications.

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