Relative p-th Order of Entire Functions of Several Complex Variables

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Abstract**—***In this paper we introduce the idea of relative p-th order of entire functions of several complex variables. After proving some basic results, we observe that the relative p-th order of a transcendental entire function with respect to an entire function is the same as that of its partial derivatives. Further we study the equality of relative p-th order of two entire functions when they are asymptotically equivalent.*

Keywords —*Entire functions, polydisc, relative order, relative p-th order, several complex variables.*

I. Introduction

Let f and g be two non-constant entire functions and

$$F(r) = \max\{|f(z)| : |z| = r\},$$
$$G(r) = \max\{|g(z)| : |z| = r\},$$

be the maximum modulus functions of f and g respectively. Then $F(r)$ is strictly increasing and continuous function of r and its inverse

$F^{-1}: (|f(0)|, \infty) \to (0, \infty)$ exists and $\lim_{R\to\infty} F^{-1}(R) = \infty$.

Bernal [3] introduced the definition of relative order of f with respect to g as

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^{\mu}) \text{for all } r > r_0(\mu) > 0\}.$$

Notation 1.1.[11] $log^{[0]}x = x$, $exp^{[0]}x = x$and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

In [10] Lahiri and Banerjee considered a more general definition of order as follows:

Definition 1.2. [10] If$p \geq 1$ is a positive integer, then the p-th generalized relative order of f with respect to g, denoted by $\rho_g^p(f)$ is defined by

$$\rho_g^p(f) = \inf\{\mu > 0 : F(r) < G(exp^{[p-1]}r^{\mu}) \text{for all } r > r_0(\mu) > 0\}.$$

Note 1.3. If $p = 1$ then $\rho_g^p(f) = \rho_g(f)$. If $p = 1$, $g(z) = \exp z$, then $\rho_g^p(f) = \rho(f)$, the classical order of f.

To investigate the basic properties of relative order of entire function of several complex variables we first need the following definition of order of entire functions.

Let$f(z_1, z_2)$ be a non-constant entire function of two complex variables $z_1$and z_2, holomorphic in the closed poly disc

$$\{(z_1, z_2): |z_j| \leq r_j, j = 1,2 \text{for all } r_1 \geq 0, r_2 \geq 0\}.$$

Let $F(r_1, r_2) = \max\{|f(z_1, z_2)|: |z_j| \leq r_j, j = 1,2\}$.

Then by the Hartogs theorem and maximum principle {[6], p-2, p-51} $F(r_1, r_2)$ is increasing function of r_1, r_2. The order $\rho = \rho(f)$ of $f(z_1, z_2)$ is defined {[6], p-338} as the infimum of all positive numbers μ for which

$$F(r_1, r_2) < \exp[(r_1 r_2)^{\mu}] \quad \ldots (1.1)$$

holds for all sufficiently large values of r_1 and r_2.
In other words
$\rho(f) = \inf\{\mu > 0 : F(r_1, r_2) < \exp[(r_1 r_2)^{\mu}]$ for all $r_1 \geq R(\mu), r_2 \geq R(\mu)\}$.

Equivalent formula for $\rho(f)$ is $\{[6], p - 339$ $(see\ also$ $[1])\}$

$$\rho(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log \log F(r_1, r_2)}{\log(r_1 r_2)}.$$

A more general approach to the problem of relative order of entire functions has been demonstrated by Kiselman [9].

Let h and k be two functions defined on $\mathbb{R}$ such that h, k: $\mathbb{R} \to [-\infty, \infty]$. The order of h relative to k is
$order(h{:}k) = \inf\ [a > 0{:}\ \exists\ c_a \in \mathbb{R}, \forall\ x \in \mathbb{R},$
$f(x) \leq a^{-1}g(ax) + c_a]$.

If H be an entire function then the growth function of H is defined by

$$h(t) = \sup[log|H(z)|, |z| \leq e^t],\ t \in \mathbb{R}.$$

If H and K are two entire functions then the order of H relative to K is now defined by

$$order\ (H{:}\,K) = order(h{:}\,k).$$

As observed by Kiselman [8], the expression $a^{-1}g(ax) + c_a$ may be replaced by $g(ax) + c_a$ if $g(t) = e^t$ because then the infimum in the cases coincide. Taking $c_a = 0$ in the above definition, one may easily verify that

$$order(H : K) = \rho_K(H)$$

i.e., the order $(H : K)$ coincides with the Bernal's definition of relative order.

Further if $K = exp\, z$ then order $(H : K)$ coincides with the classical order of H.

The papers [7], [8] and [9] made detailed investigations on entire functions and relative order $(H : K)$ but our analysis of relative order, generated from Bernal's relative order, made in the present paper have little relevance to the studies made in the above papers by Kiselman and others.

In 2007 Banerjee and Dutta [2] introduced the definition of relative order of an entire function $f(z_1, z_2)$ with respect to an entire function $g(z_1, z_2)$ as follows:

Definition 1.4. [2] Let $g(z_1, z_2)$ be an entire function holomorphic in the closed polydisc $\{(z_1, z_2) : |z_j| \le r_j ; j = 1, 2\}$ and let

$$G(r_1, r_2) = \max\{|g(z_1, z_2)| : |z_j| \le r_j ; j = 1; 2\}.$$

The relative order of f with respect to g, denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f) = \inf\{\mu > 0 : F(r_1, r_2) < G(r_1^{\mu}, r_2^{\mu}); \quad \text{for } r_1 \ge R(\mu), r_2 \ge R(\mu)\}.$$

The definition coincides with that of classical (1.1) if $g(z_1, z_1) = e^{z_1 z_2}$.

In [5] Dutta and Mandal considered a more general definition of order after Banerjee and Dutta as follows:

Definition 1.5. [5] Let $f(z_1, z_2)$ and $g(z_1, z_2)$ be two entire functions of two complex variables z_1, z_2 with maximum modulus functions $F(r_1, r_2)$ and $G(r_1, r_2)$ respectively then relative p-th order of f with respect to g, denoted by $\rho_g^p(f)$ and is defined by

$$\rho_g^p(f) = \inf\{\mu > 0 : F(r_1, r_2) < G\left(exp^{[p-1]} r_1^{\mu},\ exp^{[p-1]} r_2^{\mu}\right); for\ r_i \ge R(\mu); i = 1,2\},$$

where $p \ge 1$ is a positive integer.

Note 1.6. If we consider p = 1 then Definition 1.5 coincide with Definition 1.4.

Recently Dutta [4] introduced the idea of relative order of entire functions of several complex variables.

Definition 1.7. [4] Let $f(z_1, z_2, \dots, z_n)$ and $g(z_1, z_2 \dots, z_n)$ be two entire functions of n complex variables $z_1, z_2 \dots, z_n$ with maximum modulus functions $F(r_1, r_2, \dots, r_n)$ and $G(r_1, r_2, \dots, r_n)$ respectively then relative order of f with respect to g, denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f) = \inf\{\mu > 0 : F(r_1, r_2, \dots, r_n) < G\left(r_1^{\mu}, r_2^{\mu}, \dots, r_n^{\mu}\right); for\ r_i \ge R(\mu); i = 1,2, \dots, n\}.$$

Note 1.8. If we consider $n = 2$ then Definition 1.7 coincide with Definition 1.4.

In this paper we introduce the idea of relative p –th order of entire functions of several complex variables.

Definition 1.9. Let $f(z_1, z_2, \dots, z_n)$ and $g(z_1, z_2 \dots, z_n)$ be two entire functions of n complex variables $z_1, z_2 \dots, z_n$ with maximum modulus functions $F(r_1, r_2, \dots, r_n)$ and $G(r_1, r_2, \dots, r_n)$ respectively then relative p-th order of f with respect to g, denoted by $\rho_g^p(f)$ and is defined by

$$\rho_g^p(f) = \inf\{\mu > 0 : F(r_1, r_2, \dots, r_n) < G\left(exp^{[p-1]} r_1^{\mu}, exp^{[p-1]} r_2^{\mu}, \dots, exp^{[p-1]} r_n^{\mu}\right); for\ r_i \ge R(\mu); i = 1,2, \dots, n\},$$

where $p \ge 1$ is a positive integer.

Note 1.10. If we consider p = 1 then Definition 1.9 coincide with Definition 1.7 also when $n = 2, p = 1$ then it coincides with Definition 1.4.

Definition 1.11. [4] The function $g(z_1, z_2, \dots, z_n)$ is said to have the property (R) if for any $\sigma > 1$ and for all large $r_1, r_2, \dots, r_n$,

$$[G(r_1, r_2, \dots, r_n)]^2 < G(r_1^{\sigma}, r_2^{\sigma}, \dots, r_n^{\sigma}).$$

The function $g(z_1, z_2, \dots, z_n) = e^{z_1 z_2 \dots z_n}$ has the property (R) but the function $g(z_1, z_2, \dots, z_n) = z_1 z_2 \dots z_n$ does not have the property (R).

Throughout we shall assume f, g, h etc. are non-constant entire functions of several complex variables and $F(r_1, r_2, \dots, r_n), G(r_1, r_2, \dots, r_n), H(r_1, r_2, \dots, r_n)$ etc. denotes respectively their maximum modulus in the polydisc $\{(z_1, z_2, \ldots, z_n) : |z_j| \leq r_j, j = 1, 2, \ldots, n\}$. Also we shall consider only non-constant polynomials.

II. LEMMAS

The following lemmas will be required.

Lemma 2.1. [4] Let g has the property (R). Then for any positive integer n and for all $\sigma > 1$,

$$[G(r_1, r_2, \ldots, r_n)]^p < G(r_1^{\sigma}, r_2^{\sigma}, \ldots r_n^{\sigma})$$

holds for all large $r_1, r_2, \ldots, r_n$.

Lemma 2.2. [4] Let $f(z_1, z_2, \ldots, z_n)$ be non-constant entire and $\alpha > 1$; $0 < \beta < \alpha$. Then $F(\alpha r_1, \alpha r_2, \ldots, \alpha r_n) > \beta F(r_1, r_2, \ldots, r_n)$ for all large $r_1, r_2, \ldots, r_n$.

Lemma 2.3. [4] Let $f(z_1, z_2, \ldots, z_n)$ be non-constant entire function, $s > 1, 0 < \mu < \lambda$ and n is a positive integer. Then

(a) $\exists\, K = K(s, f) > 0$ such that $[F(r_1, r_2, \ldots, r_n)]^s \leq KF(r_1^s, r_2^s, \ldots r_n^s)$ for $r_1, r_2, \ldots, r_n > 0$;

(b) $\mathrm{Lim}_{r_1, r_2, \ldots, r_n \to \infty} \frac{F(r_1^s, r_2^s, \ldots, r_n^s)}{F(r_1, r_2, \ldots, r_n)} = \infty = \lim_{r_1, r_2, \ldots, r_n \to \infty} \frac{F\left(r_1^{\lambda}, r_2^{\lambda}, \ldots, r_n^{\lambda}\right)}{F\left(r_1^{\mu}, r_2^{\mu}, \ldots, r_n^{\mu}\right)}$.

Lemma 2.4. Let $f(z_1, z_2, \ldots, z_n)$ be a transcendental entire function then

$$\frac{F(r_1, r_2, \ldots, r_n)}{r_1} \leq \bar{\mathrm{F}}(r_1, r_2, \ldots, r_n) \leq \frac{F(2r_1, r_2, \ldots, r_n)}{r_1} \leq F(2r_1, r_2, \ldots, r_n) \text{ for } r_1, r_2, \ldots, r_n \geq 1,$$

where $\bar{\mathrm{F}}(r_1, r_2, \ldots, r_n) = \max_{|z_j| = r_j, j = 1, 2, \ldots, n} \left|\frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_1}\right|$.

Lemma 2.4 follows from Theorem 5.1 in [4].

III. PRELIMINARY THEOREM

Theorem 3.1. Let f, g, h be entire functions of several complex variables. Then

(a) if f is a polynomial and g is transcendental entire, then $\rho_g^p(f) = 0$;

(b) if $F(r_1, r_2, \ldots, r_n) \leq H(r_1, r_2, \ldots, r_n)$ for all large $r_1, r_2, \ldots, r_n$ then $\rho_g^p(f) \leq \rho_g^p(h)$.

Proof.

(a) If f is a polynomial and g is transcendental entire, then there exists a positive integer p such that

$$F(r_1, r_2, \ldots, r_n) \leq Mr_1^p r_2^p \ldots r_n^p$$

and

$$G(r_1, r_2, \ldots, r_n) \leq Kr_1^m r_2^m \ldots r_n^m$$

for all large $r_1, r_2, \ldots, r_n$ where M and K are constant and $m > 0$ may be any real number. We have then for all large $r_1, r_2, \ldots, r_n$ and $\mu > 0$,

$$\begin{aligned} &G\left(exp^{[p-1]} r_1^{\mu}, exp^{[p-1]} r_2^{\mu}, \ldots, exp^{[p-1]} r_n^{\mu}\right) \\ &> K(exp^{[p-1]} r_1^{\mu} . exp^{[p-1]} r_2^{\mu}, \ldots, exp^{[p-1]} r_n^{\mu})^m \\ &> Mr_1^p r_2^p \ldots r_n^p, \text{by choosing } m \text{ suitably} \\ &\geq F(r_1, r_2, \ldots, r_n). \end{aligned}$$

Thus for all large $r_1, r_2, \ldots, r_n$ and $\mu > 0$,

$$\begin{aligned} &F(r_1, r_2, \ldots, r_n) \\ &< G\left(exp^{[p-1]} r_1^{\mu}, exp^{[p-1]} r_2^{\mu}, \ldots, exp^{[p-1]} r_n^{\mu}\right). \end{aligned}$$

Since $\mu > 0$ is arbitrary, we must have

$$\rho_g^p(f) \leq 0 \quad i.e., \quad \rho_g^p(f) = 0.$$

(b) Let $\epsilon > 0$ be arbitrary then from the definition of relative p −th order, we have

$$\begin{aligned} &H(r_1, r_2, \ldots, r_n) \\ &< G\left(exp^{[p-1]} r_1^{\rho_g^p(h)+\epsilon}, exp^{[p-1]} r_2^{\rho_g^p(h)+\epsilon}, \ldots, \right. \\ &\left. exp^{[p-1]} r_n^{\rho_g^p(h)+\epsilon}\right). \end{aligned}$$

So for all large $r_1, r_2, \ldots, r_n$,

$$\begin{aligned} &F(r_1, r_2, \ldots, r_n) \\ &\leq H(r_1, r_2, \ldots, r_n) \\ &\quad < G(exp^{[p-1]} r_1^{\rho_g^p(h)+\epsilon}, \\ &\quad exp^{[p-1]} r_2^{\rho_g^p(h)+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_g^p(h)+\epsilon}). \end{aligned}$$

So,

$$\rho_g^p(f) \leq \rho_g^p(h) + \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$\rho_g^p(f) \leq \rho_g^p(h).$$

This completes the proof.

IV. SUM AND PRODUCT THEOREMS

Theorem 4.1. Let f_1 and f_2 be entire functions of several complex variables having relative $p - $th orders $\rho_g^p(f_1)$ and $\rho_g^p(f_2)$ respectively. Then

(i) $\rho_g^p(f_1 \pm f_2) \leq \max\{\rho_g^p(f_1),\ \rho_g^p(f_2)\}$

and

(ii) $\rho_g^p(f_1 . f_2) \leq \max\{\rho_g^p(f_1),\ \rho_g^p(f_2)\}$

provided g has the property (R).

The equality holds in (i) if $\rho_g^p(f_1) \neq \rho_g^p(f_2)$.

Proof. First suppose that relative p −th order of f_1 and f_2 are finite, if one of them or both are infinite then the results are trivial. Let $f = f_1 + f_2$, $\rho = \rho_g^p(f)$, $\rho_i = \rho_g^p(f_i), i = 1,2$ and $\rho_1 \leq \rho_2$. Therefor for any $\epsilon >0$ and for all large $r_1, r_2, \ldots, r_n$

$$F_1(r_1, r_2, \ldots, r_n) < G(exp^{[p-1]} r_1^{\rho_1+\epsilon}, exp^{[p-1]} r_2^{\rho_1+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_1+\epsilon}) \leq G(exp^{[p-1]} r_1^{\rho_2+\epsilon}, exp^{[p-1]} r_2^{\rho_2+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_2+\epsilon})$$

and

$$F_2(r_1, r_2, \ldots, r_n) \leq G(exp^{[p-1]} r_1^{\rho_2+\epsilon}, exp^{[p-1]} r_2^{\rho_2+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_2+\epsilon}$$

hold.

So for all large $r_1, r_2, \ldots, r_n$ using Lemma 2.2 we have

$$F(r_1, r_2, \ldots, r_n) \leq F_1(r_1, r_2, \ldots, r_n) + F_2(r_1, r_2, \ldots, r_n) < 2G(exp^{[p-1]} r_1^{\rho_2+\epsilon}, exp^{[p-1]} r_2^{\rho_2+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_2+\epsilon}) < G(3exp^{[p-1]} r_1^{\rho_2+\epsilon}, 3exp^{[p-1]} r_2^{\rho_2+\epsilon}, \ldots, 3exp^{[p-1]} r_n^{\rho_2+\epsilon}) < G(exp^{[p-1]} r_1^{\rho_2+3\epsilon}, exp^{[p-1]} r_2^{\rho_2+3\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_2+3\epsilon})$$

$$\therefore \quad \rho \leq \rho_2 + 3\epsilon$$

Since $\epsilon > 0$ arbitrary,

$$\rho \leq \rho_2 \qquad \ldots (4.1)$$

Next let $\rho_1 < \rho_2$ and suppose $\rho_1 < \mu < \lambda < \rho_2$.

Then for all large $r_1, r_2, \ldots, r_n$

$$F_1(r_1, r_2, \ldots, r_n) < G\left(exp^{[p-1]} r_1^{\mu}, exp^{[p-1]} r_2^{\mu}, \ldots, exp^{[p-1]} r_n^{\mu}\right) \ldots (4.2)$$

and there exists non-decreasing sequence $\{r_{ik}\}, r_{ik} \to \infty; i = 1,2, \ldots, n$ as $k \to \infty$ such that

$$F_2(r_{1k}, r_{2k}, \ldots, r_{nk}) > G\left(exp^{[p-1]} r_{1k}^{\lambda}, exp^{[p-1]} r_{2k}^{\lambda}, \ldots, exp^{[p-1]} r_{nk}^{\lambda}\right), \ldots \ (4.3)$$

for $k = 1,2, \ldots$

Using Lemma 2.3(b), we see that for all large $r_1, r_2, \ldots, r_n$

$$G(r_1^{\lambda}, r_2^{\lambda}, \ldots r_n^{\lambda}) > 2G(r_1^{\mu}, r_2^{\mu}, \ldots r_n^{\mu}) \qquad \ldots (4.4)$$

So from (4.2), (4.3) and (4.4),
$F_2(r_{1k}, r_{2k}, \ldots, r_{nk}) > 2F_1(r_{1k}, r_{2k}, \ldots, r_{nk})$ for $k = 1,2, \ldots \ldots$

Therefore for all large k using Lemma 2.2 and (4.3) we have

$$F(r_{1k}, r_{2k}, \ldots, r_{nk}) \geq F_2(r_{1k}, r_{2k}, \ldots, r_{nk}) - F_1(r_{1k}, r_{2k}, \ldots, r_{nk}) > \frac{1}{2} F_2(r_{1k}, r_{2k}, \ldots, r_{nk}) > \frac{1}{2} G\left(exp^{[p-1]} r_{1k}^{\lambda}, exp^{[p-1]} r_{2k}^{\lambda}, \ldots, exp^{[p-1]} r_{nk}^{\lambda}\right) > G\left(\frac{1}{3} exp^{[p-1]} r_{1k}^{\lambda}, \frac{1}{3} exp^{[p-1]} r_{2k}^{\lambda}, \ldots, \frac{1}{3} exp^{[p-1]} r_{nk}^{\lambda}\right) > G\left(exp^{[p-1]} r_{1k}^{\lambda-\epsilon}, exp^{[p-1]} r_{2k}^{\lambda-\epsilon}, \ldots, exp^{[p-1]} r_{nk}^{\lambda-\epsilon}\right),$$

where $\epsilon > 0$ is arbitrary.

This gives $\rho \geq \lambda - \epsilon$ and since $\lambda \in (\rho_1, \rho_2)$ and $\epsilon > 0$ is arbitrary,
we have

$$\rho \geq \rho_2 \qquad \ldots (4.5)$$

From (4.1) and (4.5) we have

$$\rho_g^p(f_1 + f_2) = \rho_g^p(f_2) = \max\{\rho_g^p(f_1), \rho_g^p(f_2)\}.$$

For the second part, we let $f = f_1 . f_2$, $\rho = \rho_g^p(f)$ and $\rho_g^p(f_1) \leq \rho_g^p(f_2)$.
Then for arbitrary $\epsilon > 0$ we have

$$F(r_1, r_2, \ldots, r_n) \leq F_1(r_1, r_2, \ldots, r_n) . F_2(r_1, r_2, \ldots, r_n) < G(exp^{[p-1]} r_1^{\rho_1+\epsilon}, exp^{[p-1]} r_2^{\rho_1+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_1+\epsilon}) . G(exp^{[p-1]} r_1^{\rho_2+\epsilon}, exp^{[p-1]} r_2^{\rho_2+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_2+\epsilon} \leq [G(exp^{[p-1]} r_1^{\rho_2+\epsilon}, exp^{[p-1]} r_2^{\rho_2+\epsilon}, \ldots, exp^{[p-1]} r_n^{\rho_2+\epsilon})]^2 < G(exp^{[p-1]} r_1^{\sigma(\rho_2+\epsilon)}, exp^{[p-1]} r_2^{\sigma(\rho_2+\epsilon)}, \ldots, exp^{[p-1]} r_n^{\sigma(\rho_2+\epsilon)}),$$

for any $\sigma > 1$, since g has the property (R).
So

$$\rho \leq \sigma(\rho_2 + \epsilon).$$

Now letting $\epsilon \to 0$ and $\sigma \to 1_+$, we have

$$\rho \leq \rho_2.$$

Therefore

$$\rho_g^p(f_1 . f_2) \leq \rho_g^p(f_2) = \max\{\rho_g^p(f_1), \rho_g^p(f_2)\}.$$

This completes the proof.

V. Relative p-th Order of the Partial Derivatives

Regarding the relative p −th order of f and its partial derivatives $\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_n}$ with respect to g and $\frac{\partial g}{\partial z_1}, \frac{\partial g}{\partial z_2}, \ldots, \frac{\partial g}{\partial z_n}$ we prove the following theorem.

Theorem 5.1. If f and g are transcendental entire functions of several complex variables and g has the property (R) then

$$\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) = \rho_g^p(f) = \rho^p_{\frac{\partial g}{\partial z_1}}(f).$$

Proof. From the definition of $\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)$, we have for any $\epsilon > 0$,

$$\bar{F}(r_1, r_2, \ldots, r_n) < G(exp^{[p-1]}r_1^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}) \text{for} r_1, r_2, \ldots, r_n \geq r_0(\epsilon).$$

Hence from Lemma 2.4 and Lemma 2.1 we have

$$F(r_1, r_2, \ldots, r_n) \leq r_1\bar{F}(r_1, r_2, \ldots, r_n)$$
$$< r_1 G(exp^{[p-1]}r_1^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon})$$
$$\leq [G(exp^{[p-1]}r_1^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}]^2$$
$$\leq G(exp^{[p-1]}r_1^{\sigma[\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon]}, exp^{[p-1]}r_2^{\sigma[\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon]}, \ldots, exp^{[p-1]}r_n^{\sigma[\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon]})$$

for every $\sigma > 1$ since g has the property (R).

So,

$$\rho_g^p(f) \leq [\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) + \epsilon]\sigma.$$

Letting$\sigma \to 1_+$, since $\epsilon > 0$ is arbitrary, we have

$$\rho_g^p(f) \leq \rho_g^p\left(\frac{\partial f}{\partial z_1}\right) \qquad \ldots \quad (5.1)$$

Similarly using $\bar{F}(r_1, r_2, \ldots, r_n) \leq F(2r_1, r_2, \ldots, r_n)$ of Lemma 2.4 we have

$$\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) \leq \rho_g^p(f). \qquad \ldots (5.2)$$

So from (5.1) and (5.2) we have

$$\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) = \rho_g^p(f). \qquad \ldots (5.3)$$

Since g is a transcendental entire function therefore using Lemma 2.4, we obtain

$$\frac{G(r_1, r_2, \ldots, r_n)}{r_1} \leq \bar{G}(r_1, r_2, \ldots, r_n)$$
$$\leq G(2r_1, r_2, \ldots, r_n). \qquad \ldots (5.4)$$

Now by the definition of $\rho^p_{\frac{\partial g}{\partial z_1}}(f)$ and (5.4) for given $\epsilon > 0$ we have

$$F(r_1, r_2, \ldots, r_n) < \bar{G}(exp^{[p-1]}r_1^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+\epsilon}, exp^{[p-1]}r_2^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+\epsilon})$$
$$\leq G(2exp^{[p-1]}r_1^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+\epsilon}, exp^{[p-1]}r_2^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+\epsilon})$$
$$< G(exp^{[p-1]}r_1^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+2\epsilon}, exp^{[p-1]}r_2^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+2\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho^p_{\frac{\partial g}{\partial z_1}}(f)+2\epsilon}).$$

So

$$\rho_g^p(f) \leq \rho^p_{\frac{\partial g}{\partial z_1}}(f) + 2\epsilon.$$

Since $\epsilon >$ 0 be arbitrary, this gives

$$\rho_g^p(f) \leq \rho^p_{\frac{\partial g}{\partial z_1}}(f).$$

Again from (5.4)

$$F(r_1, r_2, \ldots, r_n) \leq G(exp^{[p-1]}r_1^{\rho_g^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(f)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho_g^p(f)+\epsilon})$$

$$\leq r_1.\bar{G}(exp^{[p-1]}r_1^{\rho_g^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(f)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho_g^p(f)+\epsilon})$$

$$\leq [\bar{G}(exp^{[p-1]}r_1^{\rho_g^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(f)+\epsilon}, \ldots, exp^{[p-1]}r_n^{\rho_g^p(f)+\epsilon})]^2$$

$$\leq \bar{G}(exp^{[p-1]}r_1^{\sigma[\rho_g^p(f)+\epsilon]}, exp^{[p-1]}r_2^{\sigma[\rho_g^p(f)+\epsilon]}, \ldots, exp^{[p-1]}r_n^{\sigma[\rho_g^p(f)+\epsilon]}),$$

for any $\sigma > 1$.
So

$$\rho^p_{\frac{\partial g}{\partial z_1}}(f) \leq \sigma[\rho_g^p(f) + \epsilon].$$

Now letting $\sigma \to 1_+$, since $\epsilon > 0$ is arbitrary

$$\rho^p_{\frac{\partial g}{\partial z_1}}(f) \leq \rho_g^p(f)$$

and so

$$\rho^p_{\frac{\partial g}{\partial z_1}}(f) = \rho^p_g(f). \quad \dots(5.5)$$

From (5.3) and (5.5) we have

$$\rho^p_g\left(\frac{\partial f}{\partial z_1}\right) = \rho^p_g(f) = \rho^p_{\frac{\partial g}{\partial z_1}}(f).$$

This proves the theorem.

Note 5.2. Similar results holds for other partial derivatives.

VI. Asymptotic behaviour

Definition 6.1. [4] Two entire functions g_1 and g_2 are said to be asymptotically equivalent ifthere exists $l, 0 < l < \infty$ such that

$$\frac{G_1(r_1,r_2,\dots,r_n)}{G_2(r_1,r_2,\dots,r_n)} \to l \text{ as } r_1,r_2,\dots,r_n \to \infty,$$

and in this case we write $g_1 \sim g_2$.

If $g_1 \sim g_2$ then clearly $g_2 \sim g_1$.

Theorem 6.2. If $g_1 \sim g_2$ and if f is an entire function of several complex variables then

$$\rho^p_{g_1}(f) = \rho^p_{g_2}(f).$$

Proof.Let $\epsilon > 0$, then from Lemma 2.2 and for all large $r_1, r_2, \dots, r_n$

$$G_1(r_1,r_2,\dots,r_n) < (l+\epsilon)G_2(r_1,r_2,\dots,r_n) < G_2(\alpha r_1, \alpha r_2, \dots, \alpha r_n) \quad \dots(6.1)$$

where$\alpha > 1$ is such that $l + \epsilon < \alpha$.

Hence

$$F(r_1,r_2,\dots,r_n) < G_1(exp^{[p-1]} r_1^{\rho^p_{g_1}(f)+\epsilon}, exp^{[p-1]} r_2^{\rho^p_{g_1}(f)+\epsilon}, \dots, exp^{[p-1]} r_n^{\rho^p_{g_1}(f)+\epsilon})$$

$$< G_2(exp^{[p-1]} r_1^{\rho^p_{g_1}(f)+2\epsilon}, exp^{[p-1]} r_2^{\rho^p_{g_1}(f)+2\epsilon}, \dots, exp^{[p-1]} r_n^{\rho^p_{g_1}(f)+2\epsilon})$$

Since $\epsilon > 0$ is arbitrary, we have for all large $r_1, r_2, \dots, r_n$

$$\rho^p_{g_2}(f) \le \rho^p_{g_1}(f).$$

The reverse inequality is clear because $g_2 \sim g_1$ and so

$$\rho^p_{g_1}(f) = \rho^p_{g_2}(f).$$

This proves the theorem.

Note 6.3. Converse of the Theorem 6.2 is not always true and the condition $g_1 \sim g_2$ is not necessary, which are shown by the following examples.

Example 6.4. Consider the functions

$f(z_1,z_2,\dots,z_n) = z_1 z_2 \dots z_n$,
$g_1(z_1,z_2,\dots,z_n)$
$= \log^{[p-1]} z_1 \log^{[p-1]} z_2 \dots \log^{[p-1]} z_n$
and
$g_2(z_1,z_2,\dots,z_n)$
$= (\log^{[p-1]} z_1 \log^{[p-1]} z_2 \dots \log^{[p-1]} z_n)^2$.

Then we have

$$g_1 \nsim g_2, \rho^p_{g_1}(f) = 1 \text{ and } \rho^p_{g_2}(f) = \frac{1}{2}.$$

Example 6.5. Consider the functions

$f(z_1,z_2,\dots,z_n) = e^{z_1 z_2 \dots z_n}$,
$g_1(z_1,z_2,\dots,z_n)$
$= \exp(\log^{[p-1]} z_1 \log^{[p-1]} z_2 \dots \log^{[p-1]} z_n)$
and
$g_2(z_1,z_2,\dots,z_n)$
$= \exp\{2(\log^{[p-1]} z_1 \log^{[p-1]} z_2 \dots \log^{[p-1]} z_n)\}$.

Then we have

$$g_1 \nsim g_2 \text{ but } \rho^p_{g_1}(f) = \rho^p_{g_2}(f).$$

Theorem 6.6. Let f_1, f_2, g be entire functions of several complex variables and $f_1 \sim f_2$. Then

$$\rho^p_g(f_1) = \rho^p_g(f_2).$$

The proof is similar to the proof of theTheorem 6.2.

Note 6.7. Converse of the Theorem 6.6 is not always true and the condition $f_1 \sim f_2$ is not necessary, which are shown by the following examples.

Example 6.8. Consider the functions

$f_1(z_1,z_2,\dots,z_n) = z_1 z_2 \dots z_n$,
$f_2(z_1,z_2,\dots,z_n) = (z_1 z_2 \dots z_n)^2$
and
$g(z_1,z_2,\dots,z_n)$
$= \log^{[p-1]} z_1 \log^{[p-1]} z_2 \dots \log^{[p-1]} z_n$.

Then we have

$$f_1 \nsim f_2 \text{ and } \rho^p_g(f_1) \neq \rho^p_g(f_2).$$

Example 6.9. Consider the functions

$f_1(z_1,z_2,\dots,z_n) = e^{z_1 z_2 \dots z_n}$,
$f_2(z_1,z_2,\dots,z_n) = e^{2 z_1 z_2 \dots z_n}$

and

$g(z_1, z_2, \ldots, z_n)$
$= \exp\{\log^{[p-1]} z_1 \log^{[p-1]} z_2 \ldots \log^{[p-1]} z_n\}$.

Then we have

$f_1 \nsim f_2$ but $\rho_g^p(f_1) = \rho_g^p(f_2)$.

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