

# On Some Strong and $\Delta$ - Convergence Results for SKC Mappings in Hyperbolic Spaces

Preety Malik<sup>1\*</sup>, Madhu Aggarwal<sup>2</sup> and Renu Chugh<sup>3</sup>

<sup>1</sup>Department of Mathematics, Government College for Women, Rohtak-124001 (India)

<sup>2</sup>Department of Mathematics, Vaish College, Rohtak-124001(INDIA)

<sup>3</sup>Department of Mathematics, Maharshi Dayanand University, Rohtak-124001(INDIA)

**Abstract:** The aim of this paper is to prove some results on strong and  $\Delta$ -convergence of  $S$ -iterative scheme for SKC mappings in hyperbolic spaces. The results presented here extend and improve the results of Nanjaras et. al. [15], Karapinar and Tas [16] and Khan and Abbas [17].

**Keywords:** Hyperbolic spaces;  $\Delta$ -convergence; strong convergence; total asymptotically quasi nonexpansive mappings; common fixed point; Iterative procedures.

## 1. Introduction and Preliminaries

In 2008, Suzuki [12] introduced a condition on mappings and named it condition (C).

**Definition 1.1 [12, 13]** Let  $T$  be a mapping on a subset  $K$  of a Banach space  $E$ . Then  $T$  is said to satisfy (C)-condition if

$$\frac{1}{2} \|x - Tx\| \leq \|x - y\| \text{ implies that } \|Tx - Ty\| \leq \|x - y\|$$

for all  $x, y \in K$ .

**Definition 1.2.** Let  $K$  be nonempty subset of a Banach space  $X$ . Then  $T : K \rightarrow K$  is called

- a) **nonexpansive** if  $\|Tx - Ty\| \leq \|x - y\|$ , for all  $x, y \in K$ .
- b) **quasi-nonexpansive**[14] if  $\|Tx - p\| \leq \|x - p\|$ , for all  $x \in K, p \in F(T)$ .

where  $F(T)$  is the set of fixed points of  $T$ .

This condition is weaker than nonexpansiveness and stronger than quasi- nonexpansiveness. For such mappings he proved some fixed point and convergence theorems.

**Theorem 1.3 [12]** If  $K$  is a weakly compact convex subset of a uniformly convex in every direction Banach space and if  $T : K \rightarrow K$  is a mapping satisfying condition (C), then  $T$  has a fixed point.

Then Karapinar and Tas [16] modified Suzuki (C) condition as:

**Definition 1.4 [16]** Let  $T$  be a mapping on a subset  $K$  of a Banach space  $E$ . Then  $T$  is said to satisfy Suzuki-Ciric (C)-condition (SCC) if

$$\frac{1}{2} \|x - Tx\| \leq \|x - y\| \text{ implies that } \|Tx - Ty\| \leq M(x, y)$$

where

$$M(x, y) = \max \{ \|x - y\|, \|x - Tx\|, \|Ty - y\|, \|Tx - y\|, \|x - Ty\| \}$$

for all  $x, y \in K$ .

Moreover,  $T$  is said to satisfy Suzuki- (KC)-condition [(SKC)- condition] if

$$\frac{1}{2} \|x - Tx\| \leq \|x - y\| \text{ implies that } \|Tx - Ty\| \leq N(x, y)$$

where

$$N(x, y) = \max \left\{ \|x - y\|, \frac{1}{2} [\|x - Tx\| + \|Ty - y\|], \frac{1}{2} [\|Tx - y\| + \|x - Ty\|] \right\} \text{ for}$$

all  $x, y \in K$ .

**Proposition 1.5 [16]** Let  $T$  be a mapping on a subset  $K$  of a Banach space  $E$  and satisfy (SKC)- condition. Then

$$\|x - Ty\| \leq 5\|Tx - x\| + \|x - y\| \text{ holds for all } x, y \in K.$$

Takahashi [38] introduced the concept of convexity in metric space  $(X, d)$  as follows:

**Definition 1.6 [11]** A map  $W : X^2 \times [0, 1] \rightarrow X$  is a convex structure in  $X$  if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all  $x, y, u \in X$  and  $\lambda \in [0, 1]$ . A metric space  $(X, d)$  together with a convex structure  $W$  is known as convex metric space and is denoted by  $(X, d, W)$ .

A nonempty subset  $C$  of a convex metric space is

convex if  $W(x, y, \lambda) \in C$  for all  $x, y \in C$  and  $\lambda \in [0, 1]$ .

All normed spaces and their subsets are the examples of convex metric spaces. But there are many examples of convex metric spaces which are not imbedded in any normed space, (see Takahashi [11]). After that several authors extended this concept in many ways. One such convex structure is hyperbolic space which was introduced by Kohlenbach [9] as follows:

**Definition 1.7 [9]** A hyperbolic space  $(X, d, W)$  is a metric space  $(X, d)$  together with a convexity mapping  $W : X^2 \times [0, 1] \rightarrow X$  satisfying

$$(W1) \quad d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$$

$$(W2) \quad d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| d(x, y)$$

$$(W3) \quad W(x, y, \lambda) = W(y, x, 1 - \lambda)$$

$$(W4) \quad d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$$

for all  $x, y, z, w \in X$  and  $\lambda, \lambda_1, \lambda_2 \in [0, 1]$ .

Clearly every hyperbolic space is convex metric space but converse need not true. For example, if  $X = R$  (the set of reals),  $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$

and define  $d(x, y) = \frac{|x - y|}{1 + |x - y|}$  for  $x, y \in R$ , then

$(X, d, W)$  is a convex metric space but not a hyperbolic space.

A hyperbolic space  $(X, d, W)$

is said to be uniformly convex [8] if for all

$$u, x, y \in X, r > 0 \text{ and } \varepsilon \in (0, 2]$$

, there exists a  $\delta \in (0, 1]$  such that

$$d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r, \text{ whenever}$$

$$d(x, u) \leq r, d(y, u) \leq r \text{ and } d(x, y) \geq \varepsilon r.$$

A map  $\eta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$  which provides such a  $\delta = \eta(r, \varepsilon)$  for  $u, x, y \in X, r > 0$  and  $\varepsilon \in (0, 2]$  is called modulus of uniform convexity of  $X$ . We call  $\eta$  to be monotone if it decreases with  $r$  (for a fixed  $\varepsilon$ ).

A sequence  $\{x_n\}$

in  $(X, d)$

is Fejer monotone with respect to a subset  $K$  of  $X$  if  $d(x_n, x) \leq d(x_{n-1}, x)$  for all  $x \in K$ .

Let  $\{x_n\}$  be a bounded sequence in a metric space  $X$ . We define a functional  $r(\cdot, \{x_n\}) : X \rightarrow R^+$  by  $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n)$  for all  $x \in K$ . The asymptotic radius of  $\{x_n\}$  with respect to  $K \subseteq X$  is defined as,  $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}$ .

A point  $y \in K$  is called the asymptotic centre of  $\{x_n\}$  with respect to  $K \subseteq X$  if

$r(y, \{x_n\}) \leq r(x, \{x_n\})$  for all  $x \in K$ . The set of all asymptotic centres of  $\{x_n\}$  is denoted by  $A(\{x_n\})$ .

A sequence  $\{x_n\}$  in  $X$  is said to  $\Delta$ -converge to  $x \in X$  if  $x$  is the unique asymptotic centre of  $\{u_n\}$  for every subsequence  $\{u_n\}$  of  $\{x_n\}$  [11]. In this case, we write  $x$  as  $\Delta$ -limit of  $\{x_n\}$ , i.e.,  $\Delta\text{-}\lim_n x_n = x$ .

Also  $\Delta$ -convergence coincides with weak convergence in Banach spaces with opial's property [7].

**Lemma 1.8 [6]** Let  $(X, d, W)$  be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity  $\eta$ . Then every bounded sequence  $\{x_n\}$  in  $X$  has a unique asymptotic centre with respect to any nonempty closed convex subset  $K$  of  $X$ .

**Lemma 1.9 ([1])** Let  $K$  be a nonempty closed convex subset of a uniformly convex hyperbolic space and  $\{x_n\}$  be a bounded sequence in  $K$  such that  $A(\{x_n\}) = \{y\}$ . If  $\{y_m\}$  is another sequence in  $K$  such that  $\lim_{m \rightarrow \infty} r(y_m, \{x_n\}) = \rho$  (a real number), then  $\lim_{m \rightarrow \infty} y_m = y$ .

**Lemma 1.10 [1]** Let  $(X, d, W)$

be a uniformly convex hyperbolic space with monotone modulus of uniform convexity  $\eta$ . Let  $x \in X$

and  $\{a_n\}$  be a sequence in  $[b,c]$  for some  $b,c \in [0,1]$ .

If  $\{w_n\}$  and  $\{z_n\}$

are sequences in  $X$  such that

$\limsup_{n \rightarrow \infty} d(w_n, x) \leq r$ ,  $\limsup_{n \rightarrow \infty} d(z_n, x) \leq r$  and

$\lim_{n \rightarrow \infty} d(W(w_n, z_n, a_n), x) = r$  for some  $r \geq 0$ , then

$\lim_{n \rightarrow \infty} d(w_n, z_n) = 0$

Now the iterative schemes in terms of convex structure are as follows:

Let  $(X, d, W)$  be a hyperbolic space and  $T : X \rightarrow X$  be a selfmap of  $X$ . For  $x_0 \in X$ ,

(1.1.1) Picard iterative scheme:

$$x_{n+1} = Tx_n, n = 0, 1, 2, \dots$$

(1.1.2) Mann iterative scheme [18]:

$$x_{n+1} = W(x_n, Tx_n, \alpha_n), n = 0, 1, 2, \dots$$

where  $\{\alpha_n\}_{n=0}^{\infty}$  is a real sequence in  $[0,1]$ .

(1.1.3) Ishikawa iterative scheme [19]:

$$x_{n+1} = W(x_n, Ty_n, \alpha_n)$$

$$y_n = W(x_n, Tx_n, \beta_n), n = 0, 1, 2, \dots$$

where  $\{\alpha_n\}_{n=0}^{\infty}$  and  $\{\beta_n\}_{n=0}^{\infty}$  are real sequences in  $[0,1]$ .

(1.1.4) S-iterative scheme [5]:

$$x_{n+1} = W(Tx_n, Ty_n, \alpha_n)$$

$$y_n = W(x_n, Tx_n, \beta_n), n = 0, 1, 2, \dots$$

where  $\{\alpha_n\}_{n=0}^{\infty}$  and  $\{\beta_n\}_{n=0}^{\infty}$  are sequences of positive numbers in  $[0,1]$ .

Many authors have studied the strong and  $\Delta$ -convergence of various iterative schemes in hyperbolic spaces (see [1], [6], [20], [21], [22], [23], [24]). In the next section, we establish strong and  $\Delta$ -convergence of S-iterative scheme in hyperbolic spaces for SKC mappings. The obtained results extend and improve the results of Nanjaras et. al.

[15], Karapinar and Tas [16] and Khan and Abbas [17].

## 2. CONVERGENCE RESULTS

**Lemma 2.1.** Let  $C$  be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space  $X$  with modulus of uniform convexity  $\eta$ . Let  $T : C \rightarrow C$  be a SKC mapping with  $F(T) \neq \emptyset$ . Then for the iterative procedure  $\{x_n\}$  defined by (1.1.4) with  $0 < a \leq a_n, b_n \leq b < 1$  for all  $n \in N$  and for same  $a, b$ , we have

$$(i) \quad \lim_{n \rightarrow \infty} d(x_n, q) \text{ exists for all } q \in F.$$

$$(ii) \quad \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

**Proof.** Let  $q \in F$ . Then

$$\begin{aligned} d(x_{n+1}, q) &= d(W(Tx_n, Ty_n, a_n), q) \\ &\leq (1 - a_n)d(Tx_n, q) + a_n d(Ty_n, q) \\ &\leq (1 - a_n)d(x_n, q) + a_n d(y_n, q). \end{aligned} \tag{2.1.1}$$

$$\begin{aligned} \text{But } d(y_n, q) &= d(W(x_n, Tx_n, b_n), q) \\ &\leq (1 - b_n)d(x_n, q) + b_n d(Tx_n, q) \\ &\leq (1 - b_n)d(x_n, q) + b_n d(x_n, q) \\ &= d(x_n, q). \end{aligned} \tag{2.1.2}$$

Combining (2.1.1) and (2.1.2), we have

$$d(x_{n+1}, q) \leq d(x_n, q). \tag{2.1.3}$$

Thus  $\{d(x_n, q)\}$  is decreasing and hence  $\lim_{n \rightarrow \infty} d(x_n, q)$  exists for all  $q \in F$ . This proves part (i).

$$\text{Let } \lim_{n \rightarrow \infty} d(x_n, q) = c. \tag{2.1.4}$$

By (2.1.1), we have

$$d(x_{n+1}, q) \leq (1 - a_n)d(x_n, q) + a_n d(y_n, q).$$

Thus,

$$a_n d(x_n, q) \leq d(x_n, q) + a_n d(y_n, q) - d(x_{n+1}, q).$$

that is,

$$\begin{aligned} d(x_n, q) &\leq d(y_n, q) + \frac{1}{a_n} [d(x_n, q) - d(x_{n+1}, q)] \\ &\leq d(y_n, q) + \frac{1}{a} [d(x_n, q) - d(x_{n+1}, q)]. \end{aligned}$$

This gives

$$\liminf_{n \rightarrow \infty} d(x_n, q) \leq \liminf_{n \rightarrow \infty} d(y_n, q) + \liminf_{n \rightarrow \infty} \frac{1}{d} [d(x_n, q) - d(x_{n+1}, q)]$$

so that,  $c \leq \liminf_{n \rightarrow \infty} d(y_n, q)$ . (2.1.5)

By (2.1.2) and (2.1.4), we get

$$\limsup_{n \rightarrow \infty} d(y_n, q) \leq c. \tag{2.1.6}$$

Combining it with (2.1.5), we have

$$\lim_{n \rightarrow \infty} d(y_n, q) = c. \tag{2.1.7}$$

Also,

$$d(Tx_n, q) \leq d(x_n, q).$$

Thus,

$$\limsup_{n \rightarrow \infty} d(Tx_n, q) \leq c. \tag{2.1.8}$$

Now,

$$c = \lim_{n \rightarrow \infty} d(y_n, q) = \lim_{n \rightarrow \infty} d(W(x_n, Tx_n, b_n), q).$$

With the help of (2.1.4), (2.1.8) and Lemma 1.3, we have

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \tag{2.1.9}$$

**Theorem 2.2.** Let  $X, C, T, F, \{a_n\}, \{b_n\}$  and  $\{x_n\}$  be as in Lemma 2.1. Then  $\{x_n\}$   $\Delta$ -converges to a point of  $F$ .

**Proof:** Let  $q \in F$ . Then by Lemma 2.1,  $\lim_{n \rightarrow \infty} d(x_n, q)$  exists for all  $q \in F$ . Thus  $\{x_n\}$  is bounded. As proved in Lemma 2.1, we have  $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ .

Firstly, we show that  $\omega_\Delta(x_n) \subseteq F$ . Let  $u \in \omega_\Delta(x_n)$ , then there exists a subsequence  $\{u_n\}$  of  $\{x_n\}$ , such that,  $A(\{u_n\}) = \{u\}$ . Then using (2.1.9), we have

$$\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0. \tag{2.2.1}$$

We claim that  $u \in F(T)$ . Now, we define a sequence  $\{z_m\}$  in  $C$  by  $z_m = T^m u$ . Then,

$$d(z_m, u_n) = d(T^m u, u_n) \leq 5d(u_n, Tu_n) + d(u, u_n).$$

Taking lim sup on both sides of above inequality and using (2.2.1), we have

$$r(z_m, \{u_n\}) = \limsup_{n \rightarrow \infty} d(z_m, u_n) \leq \limsup_{n \rightarrow \infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that  $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \rightarrow 0$  as  $m \rightarrow \infty$ .

It follows from Lemma 1.2 that  $\lim_{m \rightarrow \infty} T^m u = u$ .

Using uniform continuity of  $T$ , we have

$$Tu = T\left(\lim_{m \rightarrow \infty} T^m u\right) = \lim_{m \rightarrow \infty} T^{m+1} u = u.$$

which implies,  $u \in F(T)$ .

Moreover by Lemma 2.1,  $\lim_{n \rightarrow \infty} d(x_n, u)$  exists.

Suppose  $x \neq u$ . By the uniqueness of asymptotic centers, we have

$$\limsup_{n \rightarrow \infty} d(u_n, u) < \limsup_{n \rightarrow \infty} d(u_n, x) \leq \limsup_{n \rightarrow \infty} d(x_n, x)$$

$$< \limsup_{n \rightarrow \infty} d(x_n, u) = \limsup_{n \rightarrow \infty} d(u_n, u),$$

a contradiction. Hence  $x = u$ . Since  $\{u_n\}$  is an arbitrary sequence of  $\{x_n\}$ , therefore  $A(\{u_n\}) = \{u\}$  for all subsequences  $\{u_n\}$  of  $\{x_n\}$ , that is,  $\{x_n\}$   $\Delta$ -converges to  $x \in F(T)$ .

**Theorem 2.3** Let  $X, C, T$  and  $\{x_n\}$  be as in Lemma 2.1. Then  $\{x_n\}$  converges strongly to a point of  $F$  if and only if  $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$ ,

$$\text{where } d(x, F) = \inf\{d(x, p); p \in F\}.$$

**Proof:** Necessity is obvious. Conversely, suppose that  $\liminf_{n \rightarrow \infty} d(x_n, F)$  exists. But by

$$\text{hypothesis, } \liminf_{n \rightarrow \infty} d(x_n, F) = 0,$$

$$\text{therefore we have } \lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

Next, we show that  $\{x_n\}$  is a Cauchy sequence in  $C$ . Let  $\varepsilon > 0$  be arbitrary chosen. Since  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ ,

there exists a positive integer  $n_0$  such that

$$d(x_n, F) < \frac{\varepsilon}{4}, \text{ for all } n \geq n_0.$$

$$\text{In particular, } \inf\{d(x_{n_0}, p) : p \in F\} < \frac{\varepsilon}{4}.$$

Thus, there must exist  $p^* \in F$  such

$$\text{that } d(x_{n_0}, p^*) < \frac{\varepsilon}{2}.$$

Now, for all  $m, n \geq n_0$ , we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(x_n, p^*) \leq 2d(x_{n_0}, p^*)$$

$$< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Hence,  $\{x_n\}$  is a Cauchy sequence in  $C$ . Since,  $C$  is a closed subset of a complete Hyperbolic space  $X$ , so it is also complete. Thus,  $\{x_n\}$  must converge to a point  $q$  in  $C$ . Also,  $\lim_{n \rightarrow \infty} d(x_n, F) = 0$  gives that  $d(q, F) = 0$ . Since  $F$  is closed, so we have  $q \in F$ .

**Theorem 2.4** Let  $X, C, T$  and  $\{x_n\}$  be as in Lemma 2.1. Let  $T$  satisfy the condition (A), then  $\{x_n\}$  converges strongly to a point of  $F$ .

**Proof:** We have proved in Lemma 2.1 that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0 \quad (2.4.1)$$

From the condition (A) and (2.4.1), we have

$$\lim_{n \rightarrow \infty} f(d(x_n, F(T))) \leq \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0,$$

Hence  $\lim_{n \rightarrow \infty} f(d(x_n, F(T))) = 0$ .

Since  $f : [0, \infty) \rightarrow [0, \infty)$

is a nondecreasing function satisfying  $f(0) = 0$ ,

$f(r) > 0$  for all  $r \in (0, \infty)$ , therefore we

have  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ .

Now all conditions of Theorem 2.3 are satisfied, therefore by its conclusion  $\{x_n\}$  converges strongly to a point of  $F$ .

## REFERENCES

- [1] A. R. Khan, H. Fukhar-ud-din and M. A. Khan: An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, *Fixed Point Theory Appl.*, vol.54, (2012), 12 pages.
- [2] H. F. Senter and W. G. Dotson: Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.*, vol. 44, (1974), 375–380.
- [3] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Austral. Math. Soc.*, vol. 43, (1991), 153-159.
- [4] K. K. Tan and H. K. Xu: Approximating fixed points of nonexpansive mappings by Ishikawa iteration process, *J. Math. Anal.*, vol. 178, (1993), 301-308.
- [5] R. P. Agarwal, D. O'Regan, and D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, vol. 8, no. 1, (2007), 61-79.
- [6] L. Leustean: Nonexpansive iterations in uniformly convex W-hyperbolic spaces, *Contemp. Math.*, vol. 513, (2010), 193-210.
- [7] T. Kuczumow: An almost convergence and its applications, *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.*, vol. 32, (1978), 79-88.
- [8] T. Shimizu and W. Takahashi: Fixed points of multivalued mappings in certain convex metric spaces, *Topol. Methods Nonlinear Anal.*, vol. 8, (1996), 197-203.
- [9] U. Kohlenbach: Some logical metatheorems with applications in functional analysis, *Trans. Amer. Math. Soc.*, vol. 357(2005), 89-128.
- [10] W. Kirk and B. Panyanak: A concept of convergence in geodesic spaces, *Nonlinear Anal.*, vol. 68, (2008), 3689-3696.
- [11] W. Takahashi: A convexity in metric spaces and nonexpansive mapping, *Kodai Math. Sem. Rep.*, vol. 22, (1970), 142-149.
- [12] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mapping, *J. Math. Anal. Appl.*, vol. 340, (2008), 1088-1095.
- [13] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer. Math. Soc.*, vol. 136, (2008), 1861-1869.
- [14] S. B. Diaz and F. B. Metcalf, On the structure of the set of sequential limit points of successive approximations, *Bull. Amer. Math. Soc.*, vol. 73, (1967), 516-519.
- [15] B. Nanjaras, B. Panyanak, W. Phuengrattana, Fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mappings in CAT(0) spaces, *Nonlinear Analysis: Hybrid Systems*, Vol. 4(2010), 25-31.
- [16] E. Karapinar and Kenan Tas, Generalized (C)-conditions and related fixed point theorems, *Computers and Mathematics with Applications*, Vol. 61(2011), 3370-3380.
- [17] Saifeer Hussain Khan and Mujahid Abbas, Strong and  $\Delta$ -convergence of some iterative schemes in CAT(0) spaces, *Computers and Mathematics with Applications*, Vol. 61(2011), 109-116.
- [18] W. R. Mann: Mean value methods in iteration, *Proc. Amer. Math. Soc.*, vol.44, (1953), 506-510.
- [19] S. Ishikawa: Fixed point by a new iteration method, *Proc. Amer. Math. Soc.*, vol.44, no.1, (1974), 147-150.
- [20] R. Chugh, Preety, M. Aggarwal, Some convergence results for modified S-iterative scheme in hyperbolic spaces, *International Journal of Computer Applications*, vol. 80, no.6, (2013), 20-23.
- [21] J. K. Kim, R.P. Pathak, S. Dashputre, S. D. Diwan and R. Gupta, *International Journal of Mathematics and Mathematical Sciences*, vol. 2015, (2015), 1-6.
- [22] H. Fukhar-ud-din, A. Kalsoom, Fixed point approximation of asymptotically nonexpansive mappings in hyperbolic spaces, *Fixed Point Theory and Applications*, 2014: 33, (2014), 1-15.
- [23] A. Sahin, M. Basarir, Some convergence results for modified SP-iteration scheme in hyperbolic spaces, *Fixed Point Theory and Applications*, 2014:133, (2014), 1-11.
- [24] S. M. Kang, S. Dashputre, B. L. Nalagar, A. Rafiq, On the convergence of fixed points for Lipschitz type mappings in hyperbolic spaces, *Fixed Point Theory and Applications*, 2014:229, (2014), 1-15.