

CARTESIAN PRODUCT OF HYPERBOLIC (F, g, r, η, ξ) STRUCTURE

By

Dr. Shankar Lal,

Assistant Professor, Department of Mathematics, HNB Garhwal University S.R.T. Campus,
Badshahithaul, Tehri Garhwal Uttarakhand, India

ABSTRACT. In the present paper we have studied the Cartesian product of hyperbolic (F, g, r, η, ξ) structure. Cartesian product of two manifolds has been defined and studied by Pandey. In this paper we have taken Cartesian product of (F, g, r, η, ξ) structure manifolds, where r is some finite integer and studied some properties of curvature and Ricci tensor of such a product manifold. In section one; introductory part of hyperbolic (F, g, r, η, ξ) structure is defined. In section two, we prove that the some theorems of product of hyperbolic (F, g, r, η, ξ) structure as well as others important structure. In section three, we have studied some properties of Curvature and Ricci tensor and some theorems. In the end we have discussed the Cartesian product of hyperbolic (F, g, r, η, ξ) structure.

KEY WORDS: Cartesian product, hyperbolic (F, g, r, η, ξ) structure, Curvature and Ricci tensor, Tachibana manifolds, KH-structure, Einstein space etc.

2010 SUBJECT CLASSIFICATION: 83C15, 83C57, 53B50.

1. INTRODUCTION. Let M_1, M_2, \dots, M_r be (F, g, r, η, ξ) -structure manifolds each of class C^∞ and of dimension n_1, n_2, \dots, n_r respectively. Suppose $(M_1)m_1, (M_2)m_2, \dots, (M_r)m_r$ be their tangent spaces at $m_1 \in M_1, m_2 \in M_2, \dots, m_r \in M_r$, then the product space $(M_1)m_1 \times (M_2)m_2 \times \dots \times (M_r)m_r$ contains vector fields of the form (X_1, X_2, \dots, X_r) ,

where $X_1 \in (M_1)m_1, X_2 \in (M_2)m_2, \dots, X_r \in (M_r)m_r$. Vector addition and scalar multiplication on above product space are defined as follows.

$$(1.1) \quad (X_1, X_2, \dots, X_r) + (Y_1, Y_2, \dots, Y_r) = (X_1 + Y_1, X_2 + Y_2, \dots, X_r + Y_r),$$

$$(1.2) \quad \lambda(X_1, X_2, \dots, X_r) = (\lambda X_1, \lambda X_2, \dots, \lambda X_r), \text{ where } X_i, Y_i \in (M_i)m_i,$$

$i = 1, 2, \dots, r$ and λ is a scalar.

Under these conditions the product space $(M_1)m_1 \times (M_2)m_2 \times \dots \times (M_r)m_r$ forms a vector space.

Define a linear transformation F on the product space

$$(1.3) \quad F(X_1, X_2, \dots, X_r) = (F_1 X_1, F_2 X_2, \dots, F_r X_r),$$

where F_1, F_2, \dots, F_r are linear transformations on $(M_1)m_1, (M_2)m_2, \dots, (M_r)m_r$ respectively.

If f_1, f_2, \dots, f_r be C^∞ functions over the spaces $(M_1)m_1, (M_2)m_2, \dots, (M_r)m_r$ respectively. We define the C^∞ function (f_1, f_2, \dots, f_r) on the product space as

$$(1.4) \quad (X_1, X_2, \dots, X_r)(f_1, f_2, \dots, f_r) \\ = (X_1 f_1, X_2 f_2, \dots, X_r f_r).$$

Let D_1, D_2, \dots, D_r be the connection on the manifolds M_1, M_2, \dots, M_r respectively. We define the operator D on the product space as

$$(1.5) \quad D_{(X_1, X_2, \dots, X_r)}(Y_1, Y_2, \dots, Y_r) = (D_{1_{X_1}} Y_1, D_{2_{X_2}} Y_2, \dots, D_{r_{X_r}} Y_r).$$

Then D satisfies all four properties of a connection and thus it is a connection on the product manifold.

2. SOME RESULTS

THEOREM 2.1. The product manifold $M_1 \times M_2 \times \dots \times M_r$ admits a (F, g, r, η, ξ) structure if and only if the manifolds M_1, M_2, \dots, M_r are (F, g, r, η, ξ) structure manifolds.

PROOF. Suppose M_1, M_2, \dots, M_r are (F, g, r, η, ξ) structure manifolds. Thus there exist tensor fields F_1, F_2, \dots, F_r each of type $(1, 1)$ on M_1, M_2, \dots, M_r respectively satisfying

$$(2.1) \quad F_i^2(X_i) = X_i + \xi(X_i)\eta, \quad i = 1, 2, \dots, r \text{ and } a \text{ is any complex number and arbitrary vector field } X, \text{ not equal to zero.}$$

$$(2.1a) \quad \bar{X} \stackrel{Def}{=} FX$$

$$(2.1b) \quad \xi(\eta) = -1 \quad \text{and} \quad \bar{\eta} = \xi(\bar{X}) = 0.$$

In view of equation (1.3) it follows that there exists a linear transformation F on $M_1 \times M_2 \times \dots \times M_r$ satisfying

$$(2.2) \quad F^2(X_1, X_2, \dots, X_r) = (F^2_1 X_1, F^2_2 X_2, \dots, F^2_r X_r) \\ = (X_1 + \xi(X_1)\eta, X_2 + \xi(X_2)\eta, \dots, X_r + \xi(X_r)\eta).$$

Thus the product manifold admits a structure.

Let us define a Riemannian metric g on the product manifold $M_1 \times M_2 \times \dots \times M_r$ as

$$(2.3) \quad g(\bar{X}, \bar{Y}) \stackrel{Def}{=} -g(X, Y) - \xi(X)\xi(Y)$$

$$(2.3) \quad g((\bar{X}_1, \bar{X}_2, \dots, \bar{X}_r), (\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_r)) \\ = -g_1(X_1, Y_1) - \xi(X_1)\eta - g_2(X_2, Y_2) - \xi(X_2)\eta \dots - g_r(X_r, Y_r) - \xi(X_r)\eta,$$

where g_1, g_2, \dots, g_r are the Riemannian metrics over the manifolds $M_1 \times M_2 \times \dots \times M_r$ respectively.

If $\xi_1, \xi_2, \dots, \xi_r$ be vector fields and $\eta_1, \eta_2, \dots, \eta_r$ be 1-forms on the GF-structure manifolds M_1, M_2, \dots, M_r respectively, then a vector field ξ and a 1-form η on the product manifold is defined as

$$(2.4) \quad \eta(X)\xi = (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \dots, \eta_r(X_r)\xi_r)$$

We now prove the following results.

THEOREM 2.2. The product manifold $M_1 \times M_2 \times \dots \times M_r$ admits a generalized almost contact structure if and only if the manifold M_1, M_2, \dots, M_r possess the same structure.

PROOF. Let M_1, M_2, \dots, M_r are generalized almost contact manifolds of hyperbolic (F, g, r, η, ξ) structure. Thus there exists tensor fields F_i of type (1,1), vector fields ξ_i and 1-forms η_i , $i = 1, 2, \dots, r$ satisfying.

$$(2.5) \quad F_i^2(X_i) = X_i + \eta_i(X_i)\xi_i,$$

for product manifold $M_1 \times M_2 \times \dots \times M_r$.

$$F^2(X_1, X_2, \dots, X_r) = (F^2_1 X_1, F^2_2 X_2, \dots, F^2_r X_r),$$

which by the help of equations (2.4) and (2.5) takes the form

$$F^2(X_1, X_2, \dots, X_r) = (X_1, X_2, \dots, X_r) + (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \dots, \eta_r(X_r)\xi_r)$$

or $F^2(X) = X + \eta(X)\xi$.

Hence the product manifold admits a generalized almost contact metric hyperbolic (F, g, r, η, ξ) structure.

THEOREM 2.3. The product manifold $M_1 \times M_2 \times \dots \times M_r$ admits a KH-structure if and only if the manifolds M_1, M_2, \dots, M_r are KH-structure manifolds.

PROOF. Suppose M_1, M_2, \dots, M_r are KH-structure manifolds. Thus

$$\begin{aligned}
 (2.6) \quad (D_{1_{x_1}} F_1)(Y_1) &= (D_{2_{x_2}} F_2)(Y_2) \\
 &= \dots \\
 &= (D_{r_{x_r}} F_r)(Y_r) \\
 &= 0.
 \end{aligned}$$

As D is a connection on the product manifold of hyperbolic (F, g, r, η, ξ) structure. Hence

$$\begin{aligned}
 (2.7) \quad (D_{(x_1, x_2, \dots, x_r)} F)(Y_1, Y_2, \dots, Y_r) &= D_{(x_1, x_2, \dots, x_r)} \{F(Y_1, Y_2, \dots, Y_r) \\
 &\quad - F\{D_{(x_1, x_2, \dots, x_r)}(Y_1, Y_2, \dots, Y_r)\}
 \end{aligned}$$

This in view of equation (1.3) and equation (1.5) takes the form

$$\begin{aligned}
 (D_{(x_1, x_2, \dots, x_r)} F)(Y_1, Y_2, \dots, Y_r) &= D_{(x_1, x_2, \dots, x_r)} (F_1 Y_1, F_2 Y_2, \dots, F_r Y_r) \\
 &\quad - F(D_{1_{x_1}} Y_1, D_{2_{x_2}} Y_2, \dots, D_{r_{x_r}} Y_r) \\
 &= -(D_{1_{x_1}} F_1 Y_1, D_{2_{x_2}} F_2 Y_2, \dots, D_{r_{x_r}} F_r Y_r) \\
 &\quad - (F_1 D_{1_{x_1}} Y_1, F_2 D_{2_{x_2}} Y_2, \dots, F_r D_{r_{x_r}} Y_r) \\
 &= ((D_{1_{x_1}} F_1)(Y_1), (D_{2_{x_2}} F_2)(Y_2), \dots, (D_{r_{x_r}} F_r)(Y_r)) \\
 &= 0.
 \end{aligned}$$

Thus the product manifold is KH-structure manifold of hyperbolic (F, g, r, η, ξ) structure.

THEOREM 2.4 The product manifold $M_1 \times M_2 \times \dots \times M_r$ of (F, g, r, η, ξ) structure manifolds M_1, M_2, \dots, M_r is almost Tachibana if and only if the manifolds M_1, M_2, \dots, M_r are separately Tachibana manifolds.

PROOF. Let a GF-structure manifolds M_1, M_2, \dots, M_r are almost Tachibana manifolds.

Then

$$(2.8) \quad (D_{i_{x_i}} F_i)(Y_i) + (D_{i_{y_i}} F_i)(Y_i) = 0, \quad i = 1, 2, \dots, r.$$

3. CURVATURE AND RICCI TENSOR. Let $X = (X_1, X_2, \dots, X_r)$ and

$Y = (Y_1, Y_2, \dots, Y_r)$ be C^∞ vector fields on the product manifold $M_1 \times M_2 \times \dots \times M_r$

and $F = (f_1, f_2, \dots, f_r)$ be a C^∞ function. Then

$$\begin{aligned} (3.1) \quad & [(X_1, X_2, \dots, X_r), (Y_1, Y_2, \dots, Y_r)](f_1, f_2, \dots, f_r) \\ &= (X_1, X_2, \dots, X_r)\{(Y_1, Y_2, \dots, Y_r) \\ & \quad (f_1, f_2, \dots, f_r)\} - (Y_1, Y_2, \dots, Y_r) \\ & \quad \{(X_1, X_2, \dots, X_r)(f_1, f_2, \dots, f_r)\} \\ &= ([X_1, Y_1]f_1, [X_2, Y_2]f_2, \dots, [X_r, Y_r]f_r). \end{aligned}$$

Suppose $K_i(X_i, Y_i, Z_i), i = 1, 2, \dots, r$ be the curvature tensors of the

(F, g, r, η, ξ) structure of manifolds M_1, M_2, \dots, M_r respectively. If $K(X, Y, Z)$ be the

Curvature tensor of the product manifold $M_1 \times M_2 \times \dots \times M_r$. Then we have

$$(3.2) \quad K(X, Y, Z) = [K_1(X_1, Y_1, Z_1), K_2(X_2, Y_2, Z_2), \dots, K_r(X_r, Y_r, Z_r)].$$

If $W = (W_1, W_2, \dots, W_r)$ be a vector field on the product manifold. Then

$$(3.3) \quad K'(X, Y, Z, W) = g(K(X, Y, Z, W)),$$

$$(3.4) \quad K'(X, Y, Z, W) = K'_1(X_1, Y_1, Z_1, W_1) + K'_2(X_2, Y_2, Z_2, W_2) + \dots + K'_r(X_r, Y_r, Z_r, W_r).$$

Thus we have

THEOREM 3.1. The product manifold $M_1 \times M_2 \times \dots \times M_r$ is of constant curvature if and only if (F, g, r, η, ξ) structure manifolds M_1, M_2, \dots, M_r are separately of constant curvature.

THEOREM 3.2. The Ricci tensor of the product manifold $M_1 \times M_2 \times \dots \times M_r$ is the sum of the Ricci tensor of the (F, g, r, η, ξ) structure manifolds M_1, M_2, \dots, M_r .

THEOREM 3.3 The product manifold $M_1 \times M_2 \times \dots \times M_r$ is an Einstein space if and only if the (F, g, r, η, ξ) structure manifolds M_1, M_2, \dots, M_r are separately Einstein spaces.

PROOF. Let the product manifold $M_1 \times M_2 \times \dots \times M_r$ be an Einstein space. Thus

$$(3.5) \quad Ric(X, Y) = Cg(X, Y)$$

Where $C = \frac{K}{n}$, K being the scalar curvature and n being the dimension of the product manifold. Then

$$Ric(X_i, Y_i) = Cg_i(X_i, Y_i) \quad , \quad i = 1, 2, \dots, r.$$

Therefore the manifolds M_1, M_2, \dots, M_r are also Einstein spaces.

Also

$$(3.7) \quad C = \frac{K}{n} = \frac{K_1}{n_1} = \frac{K_2}{n_2} = \dots = \frac{K_r}{n_r}.$$

Conversely if M_1, M_2, \dots, M_r be Einstein spaces so is the product manifold $M_1 \times M_2 \times \dots \times M_r$ and the scalar curvatures of $M_1 \times M_2 \times \dots \times M_r$ and of

M_1, M_2, \dots, M_r are the ratio $n : n_1 : n_2 : \dots : n_r$ where $n, n_1, n_2 : \dots : n_r$ are their respective dimensions. It can be easily checked that

$K = K_1 + K_2 + K_3 + \dots + K_r$ provided the product space is an Einstein space.

DISCUSSION. The Cartesian product of hyperbolic (F, g, r, η, ξ) structure are important role of dealing the extended the product of n-dimensional space of modeling heavenly body with cover because it is construct the higher dimensional space and they allow more complicated structures. We can easily calculate all structures and spaces of manifold of hyperbolic (F, g, r, η, ξ) structure.

REFERENCES

1. Adati, T. and T. Miyazawa, (1979): On P-Sasakian manifolds satisfying certain condition, Tensor, (N.S), vol. 33 pp. 173-178.
2. Bagewadi, C.S. and Venkatesha, (2006): Torsion-forming vector field in a 3-dimensional trans- Sasakian manifold Differential Geometry- Dynamical Systems, vol. 8 pp.23-28.
3. De, U.C. and A. Sarkar, (2009). On a type of P-Sasakian manifold. Math. Reports, vol. 11 (61) pp.139-144.
4. De, U.C. and A.A. Shaikh, (2007). Differential Geometry of Manifolds. Alpha Science International Ltd. Oxford, UK, pp: 259-272.
5. Dube, K.K. (1973): On almost hyperbolic Hermite manifolds, An. Univ. Din Timisoara, siri mat. Vol. XI, face I pp. 49-54.
6. Hicks, N.J. (1965): Notes on Differential Geometry, D. Van Nostrand company inc., New York.
7. Pandey, H.B. (1981): Cartesian product of two manifolds, Indian Journal of Pure & Applied Mathematics Vol. 12, No. 1, pp 55-60.
8. Pant, J. (1986): Hypersurface immersed in a GF-structure manifold, Demonstratio Mathematica, Warszawa Poland, Vol. XIX No. 3, pp 693-697.
9. Upreti, J. (1999): Manifold with generalized r-contact product structure, J. of T.S.I. vol. 17 pp. 82-88.
10. Yano, K. and Kon, M. (1984): Structures on manifolds, word scientific publishing co. pvt. Ltd. Singapore.