Existence and uniqueness for Volterra nonlinear integral equation

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Abstract : We study the existence and uniqueness theorem of a functional Volterra integral equation in the space of Lebesgue integrable on unbounded interval by using the Banach fixed point theorem.

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1. Introduction

The subject of nonlinear integral equation considered as an important branch of mathematics because it is used for solving of many problems such as physics, engineering and economics [1,2].

In this paper, we will prove the existence and uniqueness theorem of a functional Volterra integral equation in the space of Lebesgue integrable $L_1(R^+)$ on unbounded interval of the kind :

$$x(t) = g(t)\, f\big(t, x(t)\big) + h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big)ds, \qquad (1.1)$$

where $t > 0$ in the unbounded interval $R^+ = [\,0, \infty\,)$.

2. Preliminaries

Let R be the field of real number, R^+ be the interval $[0,\infty)$. If A is a Lebesgue measurable subset of R **,** then the symbol meas(A) stands for the Lebesgue measure of A.

Further, denote by $L_1(A)$ the space of all real functions defined and Lebesgue mesurable on the set A. The norm of a function $x \in L_1(A)$ is defined in the standard way by the formula,

$$\|x\| = \left\|x_{L_1(A)}\right\| = \int_A |x(t)|dt \qquad (2.1)$$

Obviously $L_1(A)$ forms a Banach space under this norm. The space $L_1(A)$ will be called the lebesgue space. In the case when $A = R^+$ we will write L_1 instead of $L_1(R^+)$.

One of the most important operator studied in nonlinear functional analysis is the so called the superposition operator [3]. Now, let us assume that $I \subset R$ is a given interval, bounded or not.

Definition 2.1 [4] Assume that a function $f(t,x) = f: I \times R \to R$ satisfies the so-called Carathéodory condition, i. e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in I$. Then to every function $x = x(t)$ which is measurable on I we may assign the function $(Fx)(t) = f(t, x(t))$, $t \in I$. The operator F defined in such a way is said to be the **superposition operator** generated by the function f.

Theorem 2.1 [5] The superposition operator F generated by a function f maps continuously the space $L^1(I)$ into itself if and only if $|f(t,x)| \leq a(t) + b|x|$ for all $t \in I$ and $x \in R$, where $a(t)$ is a function the from $L^1(I)$ and b is a nonnegative constant.

This theorem was proved by Krasnoselskii [2] in the case when I is bounded interval. The generalization to the case of an unbounded interval I was given by Appell and Zabrejko [6].

Definition 2.2 [7] A function $f : A \to R^m$, $A \subset R^n$, is said to be Lipschitz condition if there exists a constant L, $L > 0$ (is called the Lipschitz constant of f on A) such that

$|f(x) - f(y)| \leq L\,|x - y|$, for all $x, y \in A$.

Definition 2.3 [8] Let (X, d) be a metric space and $T : X \to X$ is called contraction mapping, if there exist a number $\gamma < 1$, such that :
$d(Tx, Ty) \leq \gamma\, d(x, y), \quad \forall\, x, y \in X.$

Theorem 2.2 [9] (Banach fixed point theorem).

Let X be a closed subset of a Banach space E and $T : X \to X$ be a contraction, then T has a unique fixed point.

3. Existence Theorem

Define the operator H associated with integral Equation (1.1) take the following form.

$$Hx = Ax + Bx. \tag{3.1}$$

Where

$$(Ax)(t) = g(t)\, f(t, x(t)),$$
$$(Bx)(t) = h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big)ds$$
$$= h(t) + KFx(t),$$

where $(Kx)(t) = \int_0^t k(t,s)\, x(s)\, ds$,

$Fx = f(t, x)$, are linear operator at superposition respectively.

We shall treat the equation (3.1) under the following assumptions listed below.

i) $g : R^+ \to R$ is bounded function such that

$$M = \sup_{t \in R^+} |g(t)|,$$

and $h : R^+ \to R$, such that $h \in L_1(R^+)$.

ii) $k : R^+ \times R^+ \to R$ such that :

$$\int_0^t |k(t,s)| dt \leq C$$

iii) $f : R^+ \times R \to R$ satisfies Lipschitz condition with positive constant L such that :

$$|f(t, x(t)) - f(t, y(t))| \leq L|x(t) - y(t)|, \quad \text{for all } t \in R^+.$$

iv) $LM + LC < 1$.

Now, for the existence of a unique solution of our equation, we can see the following theorem .

Theorem 3.1 If the assumptions (i)-(iv) are satisfied, then the equation (1.1) has a unique solution, where $x \in L_1(R^+)$.

Proof : first we will prove that $H : L_1(R^+) \to L_1(R^+)$,

second will prove that H is contraction .

Consider the operator H as :

$$Hx(t) = g(t)\, f\big(t, x(t)\big) + h(t) + \int_0^t k(t,s)\, f\big(s, x(s)\big) ds$$

Then our equation (3.1) becomes

$$x(t) = Hx(t).$$

We notice that by assumption (iii), we have

$$\begin{aligned} |f(t,x)| &= |f(t,x) - f(t,0) + f(t,0)| \\ &\le |f(t,x) - f(t,0)| + |f(t,0)| \\ &\le L\,|x-0| + |f(t,0)| \\ &\le L\,|x| + a(t) \end{aligned}$$

Where

$$|f(t,0)| = a(t)$$

To prove that $H : L_1(R^+) \to L_1(R^+)$

Let $x \in L_1(R^+)$,

then we have

$$\begin{aligned} \|(Hx)(t)\| &= \int_0^\infty |(Hx)(t)| dt \\ &= \int_0^\infty \left| g(t)\, f\big(t,x(t)\big) + h(t) + \int_0^t k(t,s)\, f\big(s,x(s)\big) ds \right| dt \\ &\le \int_0^\infty \left| g(t)\, f\big(t,x(t)\big) \right| dt \\ &\quad + \int_0^\infty \left| h(t) + \int_0^t k(t,s)\, f\big(s,x(s)\big) ds \right| dt \\ &\le \int_0^\infty |g(t)|\,[a(t) + L|x(t)|] dt \\ &\quad + \int_0^\infty |h(t)| dt + \int_0^\infty \left| \int_0^t k(t,s)\, f\big(s,x(s)\big) ds \right| dt \\ &\le M\|a\| + LM \int_0^\infty |x(t)| dt + \|h\| \end{aligned}$$

$$+ \int_0^t |k(t,s)| dt \int_0^\infty \left|f\big(t,x(t)\big)\right| ds$$

$$\leq M\|a\| + LM \int_0^\infty |x(t)| dt + \|h\|$$

$$+ C \int_0^\infty [a(s) + L|x(s)|] ds$$

$$\leq M\|a\| + LM\|x\| + \|h\|$$

$$+ C \left[\left|\int_0^\infty a(s) ds\right| + L \int_0^\infty |x(s)|\, ds\right]$$

$$\leq M\|a\| + LM\|x\| + \|h\| + C\|a\| + LC\|x\|$$

$$\leq [\,M + C]\|a\| + \|h\| + [LM + LC]\|x\| < \infty.$$

Then

$$H : L_1\,(R^+) \;\rightarrow\; L_1\,(R^+)$$

Now, to prove that H is contraction,

let $x, y \in L_1\,(R^+)$,

then

$$\int_0^\infty |(Hx)(t) - (Hy)(t)| dt = \int_0^\infty \Big| g(t) f\big(t,x(t)\big) + h(t) + \int_0^t k(t,s) f\big(s,x(s)\big) ds - g(t) f\big(t,y(t)\big) - h(t) - \int_0^t k(t,s) f\big(s,y(s)\big) ds \Big| dt$$

$$\leq \int_0^\infty |g(t)| \left|f\big(t,x(t)\big) - f\big(t,y(t)\big)\right| dt$$

$$+ \int_0^\infty \left| \int_0^t k(t,s) f\big(s,x(s)\big) ds - \int_0^t k(t,s) f\big(s,y(s)\big) ds \right| dt$$

$$\leq M \int_0^\infty L|x(t) - y(t)| dt$$

$$+ \int_0^\infty \int_0^t k(t,s) \left|f\big(s,x(s)\big) - f\big(s,y(s)\big)\right| ds\, dt$$

$$\leq LM \int_0^\infty |x(t) - y(t)| dt + C \int_0^\infty L|x(s) - y(s)| dt$$

$$\leq \mathrm{L}M\|x - y\| + LC\|x - y\|$$

$$\leq [\mathrm{L}M + LC]\|x - y\|$$

Hence, by using Banach fixed point theorem,

H has a unique point, which is the solution of the equation (1.1), where $x \in L_1\,(R^+)$. ■

References

[1] M. M. El-Borai, Wagdy G. El-Sayed and Faez N. Ghaffoori, On The Solvability of Nonlinear Integral Functional Equation, (IJMTT) Vo 34, No. 1, June 2016.

[2] P. P. Zarejko, A.I. Koshlev, M. A. Krasnoselskii, S. G. Mikhlin, L. S. Rakovshchik, V. J. Stecenko, Integral Equations, Noordhoff, Leyden, 1975.

[3] J. Appell and P. P. Zabroejko, Continuity properties of the superposition operator, No. 131, Univ. Augsburg, (1986).

[4] M. M. A. Metwali, On solutions of quadratic integral equations, Adam Mickiewicz University (2013).

[5] J. Bana's and W. G. El-Sayed, Solvability of functional and Integral Equations in some classes of integrable functions, 1993.

[6] J. Appell and P. P. Zabroejko, Continuity properties of the superposition operator, No. 131, Univ. Augsburg, (1986).

[7] R. R. Van Hassel, Functional Analysis, December 16, (2004).

[8] I. G. Petrovski and R. A .Silverman, Ordinary Differential Equations, 1973.

[9] Ravi P. Agarwal, Maria Meehan and Donal O'regan, Fixed Point Theory and Applications Cambridge University Press, 2004.