Matrices Associated with Class of Unitary Arithmetical Functions

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Abstract:

If f is an arithmetical function and [f(i,j)] is the matrix then we evaluated

det [f(i,j)] as product of f, has Dirichlet convolution of f and Other theorem and corollary are evaluated.

KEY WORDS:

Arithmetical function, determinants, Ramanujan fraction, even functions

Introduction:

The determinant of the nxn matrix $[(i,j)]$ which has greatest common divisor (i,j) of I and j as its i,j –entry is the product $\emptyset^*(1)\ \emptyset^*(2)\emptyset^*(3)$ ------$\emptyset^*(n)$, where $\emptyset^*$ is unitary Euler's totient function. In this paper we also proved that if f is an arithmetical function and [f(i,j)] is the nxn matrix having f evaluated at the greatest common divisor of I and j as its I,j entry, then

det [f(i,j)] $= (f.\mu)(1)$= $(f.\mu)(1)\ (f.\mu)(2)\ (f.\mu)(3)$---- $(f.\mu)(n)$,where μ is the Mobius function and f. μ is the Dirichlet convolution of f and μ.

Tom Apostol [1] extended smith's result by showing that if f and g are arithmetical functions and β^* is defiend for positive integers m and r by

$$\beta^*(m,r) = \sum_{d//(m,r)^*} f^*(d) g^*\left({}^{r}/_{d}\right),$$

Then $det[\beta^*(i,j)] = [g^*(1)]^n f^*(1)f^*(1)f^*(2)$------$f^*(n)$. He noted that , as a consequence of this , $det[c^*(i,j)] = n!$, where $c^*(m,r)$ is Unitary Ramanujan sum. It is defined as

$$C^*(m,r) = \sum_{k(mod\ \ r)} exp\left(\frac{2\pi im}{k}\right) = \sum_{d//(m,r)^*} d\mu^*\left(\frac{r}{d}\right)$$

$$(k,r)^* = 1$$

Where the first sum is over a unitary reduced residue system (mod r). Since $C^*(r,r) = \emptyset^*(r)$, this function is a unitary generalization of Euler's totient function.

Paul McCarthy generalized smith's and Apostol's results to the class of even functions

(mod r).he evaluated the determents of nxn matrices of the form$[\beta^*(i,j)]$. Where $\beta^*(m,r)$ is a even function of m (mod r). Acomplex-valued function $\beta^*(m,r)$ of the positive integral variables m and r is said to be a even function of m (mod r) ,if $\beta^*(m,r) = \beta^*((m,r),r)$for all values of m . the functions considered by Smith and Apostol are even functions of m (mod r) for every r.

Ecford Cohen showed that if $\beta^*(m,r)$ is an even function of m (mod r), then β^* can expressed in the form

$$\beta^*(m,r) = \sum_{d//r} C^*(m,d)\,\alpha^*(d,r), \qquad (1)$$

where the coefficients $\alpha^*(d,r)$ are uniquely determined by

$$\alpha^*(d,r) = \frac{1}{r}\sum_{e//r} C^*\left({}^{r}/_{d}, e\right)\beta^*\left({}^{r}/_{e}, r\right)$$

Another characterization, also obtained in this paper is that $\beta^*(m,r)$ is an even function of m(mod r) if and only if there is a function F^* of two positive integral variables such that

$$\beta^*(m,r) = \sum_{d//(m,r)} F^*\left(d, {}^{r}/_{d}\right) \qquad (2)$$

If for each r, $\beta^*(m,r)$ is an even function of m(mod r) given (1) &(2),then Mc carthy showed that det$[\beta^*(i,j)] = n!\ \alpha^*(1,1)\ \alpha^*(2,2)\ \alpha^*(3,3)$----- $\alpha^*(n,n) = F^*(1,1)F^*(2,1)F^*(3,1)$--$F^*(n,1)$.for the functions considered by Apostol, $F^*(m,r) = f^*(m)g^*(r)$.

A set S= $\{x_1 x_2 x_3 \ldots . x_n\}$ of distinct positive integers is factor-closed if it contains every divisor of x for any $x \in S$.let f^* be an arithmetical function and $[f^*(x_{i,} x_j)]$ denote n x n matrix having f^* evaluated at the greatest common divisor of $x_{i,}$and x_j as its i,j entry .Smith also stated that the following result is true. If S is factor –closed ,then the determent of the matrix $[f^*(x_{i,} x_j)]$ is the product $(f^*.\mu^*)(x_1)(f^*.\mu^*)(x_2)\ldots..(f^*.\mu^*)(x_n)$.The purpose of this paper is to extend the results of Smith ,Apostol, and Mc Carthy to matrixes of the form $[\beta^*(x_{i,} x_j)]$,where $\beta^*(m,r)$ is an even function of m (mod r) and to obtain some new results concerning the structure and inverses of the matrices. We use some of the results to study matrices of the form$f^*[x_{i,} x_j\ ]$ which have f^* evaluated at the product of $x_{i,}$and x_j as their i,j-entry ,where is quadratic.

1. MATRICES ASSOCIATED WITH EVEN FUNCTION (mod r)

Throughout this paper ,let f(m),g(m),and h(m) be arithmetical functions (i,e complex-valued functions of a real variable that vanish when the argument is not positive integer).we assume that f,g,h have Dirichlet invers, which are denoted by $f' g' h'$, respectively.

Our fist results are on n x n matrices of the form $\Psi^*(m,r)$ is defined for all positive integers m and r as

$$\Psi^*(m,r) = \sum_{d//(m,r)^*} f^*(d) g^*\left({}^{m}/_{d}\right) h^*\left({}^{r}/_{d}\right), \qquad (3)$$

And S= $\{x_1 x_2 x_3 \ldots . x_n\}$ is a set of distinct positive integers.

Let $\zeta(d)$ be the function defined by $\zeta(d) = 1$ for all d. if g = ζ, $\Psi$is called a generalized Ramanujan's sum. These sums generalize Dirichlet convolution $\Psi^*(m,m) = ((f^*.h^*)(m))$ and were studied by Apostol .A well-known example is Ramanujan's sum $C^*(m,r)$,which is obtained from (3) by setting $f^*(d) = d$ for all d, g= ζ ,and $h^* = \mu^*$. We use our result on matrix $[\Psi^*(x_{i,} x_j)]$ to study the matrix $[\beta^*(x_{i,} x_j)]$ when β^* is even function of m(mod r).

LEMMA 1:

If T = $\{y_1 y_2 y_3 \ldots . y_m\}$ is a factor –closed set containing S, then $[\Psi^*(x_{i,} x_j)]$ = GAH^T where A = diag $(f^*(y_1), f^*(y_2)\ldots.. f^*(y_m))$ and the n x m matrices G and H are defined by $G = \left[g^*\left({}^{x_i}/_{x_j}\right)\right]$ and $H = \left[h^*\left({}^{x_i}/_{x_j}\right)\right]$.

Proof: Calculating the i-j entry of the product GAH^T gives

$$\sum_{k=1}^{m} f^*(y_k) g^*\left({}^{x_i}/_{y_k}\right) h^*\left({}^{x_j}/_{y_k}\right) = \sum_{\substack{d//x_j \\ d//x_j}} f^*(d) g^*\left({}^{x_i}/_{d}\right) h^*\left({}^{x_j}/_{d}\right) = [\Psi^*(x_{i,} x_j)]$$

LEMMA 2 :

If S= $\{x_1 x_2 x_3 \ldots . x_n\}$ is factor –closed , then $\left[f^*\left({}^{x_i}/_{x_j}\right)\right]^{-1} = \left[f^{*\prime}\left({}^{x_i}/_{x_j}\right)\right]$.

Proof. Calculating the i-j entry of the product $[f^*\left({}^{x_i}/_{x_j}\right)]\ [f^{*\prime}\left({}^{x_i}/_{x_j}\right)]$ gives

$$\sum_{k=1}^{m} f^*\left({}^{x_i}/_{x_k}\right) f^{*'}\left({}^{x_k}/_{x_j}\right) = \sum_{d//{}^{x_i}/_{x_j}} [f^{*'}(d)\ [f^*\left({}^{x_i}/_{x_j\, d}\right) = \begin{cases} 1 & if\ x_j = x_i \\ 0 & other\ wise \end{cases}$$

THEOREM 1 :

If S= $\{x_1 x_2 x_3 \dots . x_n\}$ is factor – closed, then each of the following is true.

(i) $\det[\Psi^*(x_{i,} x_j)] = [g^*(1)h^*(1)]^n f^*(x_1)f^*(x_2)f^*(x_3) \dots . f^*(x_n)$;

(ii) if $\det[\Psi^*(x_{i,} x_j)] \neq 0$, $[\Psi^*(x_{i,} x_j)]^{-1} = (a_{ij})$, where

$$a_{ij} = \sum_{\substack{x_{i,}//x_k \\ x_{j,}//x_k}} \left[\frac{1}{(f^*(x_k))}\right] h'(x_{k,}/x_i) g'(x_{k,}/x_j)$$

Proof: By lemma 1, $[\Psi^*(x_{i,} x_j)] = GAH^T$, A = diag $(f^*(x_1), f^*(x_2) \dots . . f^*(x_n))$ and n x n matrices G and H are defined by $G = \left[g^*\left({x_i}/{x_j}\right)\right]$ and $H = \left[h^*\left({x_i}/{x_j}\right)\right]$. Since any permutation of the elements in S yields a similar matrix we may assume that $x_1 < x_2 < x_3 \dots . x_n$

Thus G and H are triangular with diagonal elements$g^*(1)h^*(1)$, respectively. this proves (i). if $\det[\Psi^*(x_{i,} x_j)] \neq 0$, $[\Psi^*(x_{i,} x_j)]^{-1} = (GAH^T)^{-1}$. Therefore using lemma (2), we obtain (ii).

COROLLARY 1.

Let S= $\{x_1 x_2 x_3 \dots . x_n\}$ be factor – closed. if β^* is defined for positive integers m and r by $\beta^*(m, r) = \sum_{d//(m,r)^*} f^*(d) g^*\left({r}/{d}\right)$ then each of the following is true:

(i) $\det[\beta^*(x_{i,} x_j)] = [g^*(1)]^n f^*(x_1)f^*(x_2)f^*(x_3) \dots . f^*(x_n)$

(ii) if $\det[\beta^*(x_{i,} x_j)] \neq 0$, $[\beta^*(x_{i,} x_j)]^{-1} = (a_{ij})$, where

$$a_{ij} = \sum_{\substack{x_{i,}//x_k \\ x_{j,}//x_k}} \left[\frac{1}{f^*(x_k)}\right] h^{*'}(x_{k,}/x_i) g^{*'}(x_{k,}/x_j)$$

COROLLARY 2.

If β^* is defined for positive integers m and r by $\beta^*(m, r) = \sum_{d//(m,r)^*} f^*(d) g^*\left({r}/{d}\right)$ then $\det[\beta^*(i, j)] = [g^*(1)]^n f^*(1)f^*(2)f^*(3) \dots . f^*(n)$.

COROLLARY 3.

Let S= $\{x_1 x_2 x_3 \dots . x_n\}$ be factor – closed, then each of the following is true:

(i) $\det[C^*(x_{i,} x_j)] = x_1 x_2 x_3 \dots . x_n$,

(ii) $[C^*(x_{i,} x_j)]^{-1} = (a_{ij})$, where

$$a_{ij} = \sum_{\substack{x_{i,}//x_k \\ x_{j,}//x_k}} \left[\frac{1}{(x_k)}\right] \mu^*(x_{k,}/x_j).$$

Proof: if we set $f^*(d) = d$ for all d, $g^* = \zeta$,then $\Psi^* = C^*$. Hence applying thermo 2,obtain (i)&(ii).

THEOREM 2.

Let S= $\{x_1 x_2 x_3 \dots . x_n\}$ be factor – closed. if β^* is defined for positive integers m and r by $\beta^*(m, r) = \sum_{d//r} C^*(m, d)\alpha^*(d, r)$ then $\det[\beta^*(x_{i,} x_j) = \prod_{i=1}^n x_i \alpha^*(x_{i,} x_j)$.

Proof : we have $[\beta^*(x_{i,} x_j)] = [C^*(x_{i,} x_j)]\ [\alpha^*(x_{i,} x_j) e_{ij}]$, where $e_{ij} = 1$. if ${x_i}/{x_j}$ and 0 otherwise . Since we may assume that $x_1 < x_2 < x_3 \dots . x_n$, $\det[\beta^*(x_{i,} x_j) = \prod_{i=1}^n x_i \alpha^*(x_{i,} x_j)$

COROLLARY 4.

if β^* is defined for positive integers m and r by $\beta^*(m,r) = \sum_{d//r} C^*(m,d)\alpha^*(d,r)$ then $\det[\beta^*(i,j) =$ n! $\alpha^*(1,1)\, \alpha^*(2,2)\, \alpha^*(3,3)$----- $\alpha^*(n,n)$.

2. MATRICES ASSOCIATED WITH COMLETELY EVEN FUNCTIONS (mod r)

Given any unitary arithmetical function $f^*(m)$,we denote by the $f^*(m,r)$ the function f^* evaluated at the greatest common divisor of m and r, Cohen called the function $f^*(m,r)$ a completely even function (mod r).
Let S= $\{x_1 x_2 x_3 \ldots . x_n\}$ be a set of distinct positive integers . if I(M) = m is the identity function , the n x n matrix (S) = $[I^*(x_{i,}\, x_j)]$ having the greatest common divisor $(x_{i,}\, x_j)$ of x_i and x_j as i,j entry is called the greatest common divisor (GCD) matrix on S.In this section we extend some of the results to matrices of the form $[f^*(x_{i,}\, x_j)]$

COROLLARY 5.

If S= $\{x_1 x_2 x_3 \ldots . x_n\}$ be factor – closed, then each of the following is true:

(i) (Smith) $\det[f^*(x_{i,}\, x_j)] = (f^*.\mu^*)(x_1)(f^*.\mu^*)(x_2) \ldots .. (f^*.\mu^*)(x_n)$,

(ii) (Bourque and Ligh) if $\det[f^*(x_{i,}\, x_j)] \neq 0$, $\left[f^*(x_{i,}\, x_j)\right]^{-1} = (a_{ij})$, where

$$a_{ij} = \sum_{\substack{x_{i,}//x_k \\ x_{j,}//x_k}} \left[\frac{1}{(f^*.\mu^*)(x_k)}\right] \mu^*(x_{k,}/x_i)\mu^*(x_{k,}/x_j)$$

Proof: Set $g^* = h^* = \zeta$ and substitute $(f^*.\mu^*)$ for f^* in theorem 1.

Example 1.

For any real number ε, let the functions $\zeta_{\,\varepsilon}$ and $\varphi_\varepsilon{}^*$ be defined by $\zeta_{\,\varepsilon}(m) = m^\varepsilon$ and $\varphi_\varepsilon{}^*(m) = (\zeta_{\,\varepsilon}\,.\mu^*)(m) = \sum_{d//m} d^\varepsilon \mu^*\left(\frac{m}{d}\right)$. Since $\varphi_\varepsilon{}^*$ is the Dirichlet convolution of two multiplicative functions, it is multiplicative and for prime power P^r $(r \geq 1)$,
$\varphi_\varepsilon{}^*(P^r) = P^{\varepsilon r} - P^{\varepsilon(r-1)}$. thus $\varepsilon > 0$ we see that $\varphi_\varepsilon{}^*(m) > 0$ for all m. If S= $\{x_1 x_2 x_3 \ldots . x_n\}$ be factor – closed and $\varepsilon > 0$,by Corollary 5 each of the following is true.

(i) $\left[(x_{i,}\, x_j)^\varepsilon\right]^{-1} = (a_{ij})$ where

$$a_{ij} = \sum_{\substack{x_{i,}//x_k \\ x_{j,}//x_k}} \left[\frac{1}{\varphi_\varepsilon{}^*(x_k)}\right] \mu^*(x_{k,}/x_i)\mu^*(x_{k,}/x_j)$$

(ii) $\det[(x_{i,}\, x_j)^\varepsilon] = \varphi_\varepsilon{}^*(x_1)\varphi_\varepsilon{}^*(x_2) \ldots .. \varphi_\varepsilon{}^*(x_n)$,

Example 2.

Foe a real number ε, let σ_ε be defined by $\sigma_\varepsilon(m) = (\zeta_{\,\varepsilon}\,.\zeta_{\,0})(m) = \sum_{d//m} d^\varepsilon$. The functions $\tau(m) = \sigma_0(m)$ and σ(m) = $\sigma_1(m)$ give the number of divisors of m and the sum of the divisors of m , respectively. we have $\sigma_\varepsilon.\,\mu^* = \zeta_{\,\varepsilon}.\zeta.\mu^* = \zeta_{\,\varepsilon}$. thus $(\sigma_\varepsilon.\,\mu^*)(m) > 0$ for all m. . If S= $\{x_1 x_2 x_3 \ldots . x_n\}$ be factor – closed and $\varepsilon > 0$,by Corollary 5 each of the following is true.

(i) $\left[\sigma_\varepsilon(x_{i,}\, x_j)\right]^{-1} = (a_{ij})$ where

$$a_{ij} = \sum_{\substack{x_{i,}//x_k \\ x_{j,}//x_k}} \left[\frac{1}{(x_k)}\right]^{\varepsilon} \mu^*(x_{k,}/x_i)\mu^*(x_{k,}/x_j)$$

(ii) Det $[\sigma_\varepsilon(x_{i,} x_j)] = (x_1 x_2 x_3 \ldots . x_n)^\varepsilon$.

If A is n x m matrix with $m \geq n$, we denote by $A(d_1 d_2 d_3 \ldots . d_n)$ the n x n submatrix of A which contains the columns 1≤ $d_1 < d_2 < d_3 \ldots . < d_n \leq m$.

THEOREM 3

Let S= $\{x_1 x_2 x_3 \ldots . x_n\}$ be a set of distinct positive integers. If $h^*(d) = g^*(d)$ ∈R and f(d) > 0 whenever d//x for any x ∈ S, then each of the following is true .

(i) $[\Psi^*(x_{i,} x_j)]$ is positive definite:

(ii) det$[\Psi^*(x_{i,} x_j)]$ = $[g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n)$ if and only if S is factor closed :

(iii) $[g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n) \leq$ det$[\Psi^*(x_{i,} x_j)] \leq \Psi^*(x_{1,} x_1)\, \Psi^*(x_{2,} x_2) \ldots .. \Psi^*(x_{n,} x_n)$.

Proof: Let $m \geq x$ for all x ∈ S. by lemma 1, $[\Psi^*(x_{i,} x_j)]$ = GAH^T, where $A = diag\left(f^*(1) f^*(2) \ldots \ldots f^*(m)\right)$ and n x m matrix G is defind by $G = \left[g^*\left({}^{x_i}/_j\right)\right]$.Thus $[\Psi^*(x_{i,} x_j)]$ = AA^T, where A = G$A^{1/2}$. Hence by the Binet –Cauchy formula ,

Det AA^T $= \sum_{1 \leq k_1 < k_2 < \cdots \ldots k_n \leq m} [detA(k_1 k_2 k_3 \ldots . k_n)]^2$

$$= \sum_{1 \leq k_1 < k_2 < \cdots \ldots k_n \leq m} f^*(k_1) f^*(k_2) \ldots \ldots f^*(k_n) \left(det\left[g^*\left({}^{x_i}/_j\right)\right]\right)^2. \qquad (4)$$

One term in $f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n) \left(det\left[g^*\left({}^{x_i}/_j\right)\right]\right)^2 = [g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n)$.Snice each term is nonnegative, det$[(x_{i,} x_j)] \geq [g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n) > 0$. This proves (i)and (ii) . by the theorem 1, if S is factored-closed , det$[\Psi^*(x_{i,} x_j)] = [g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n)$. to prove converse we may assume $x_1 < x_2 < x_3 \ldots . x_n$. if S is not factor –closed ,let r be the smallest integer such that d//x_r for some d $\notin S$. Define $k_i$for $1 \leq i \leq n$ by $k_i = x_{i,}$ if $i \neq r$ and $k_i = d$ if i= r

Since $d \nmid x_i\ for\ i < r$, $[g^*\left({}^{x_i}/_j\right)]$ is upper triangular with $g^*(1)$ as the diagonal elements. Consequently , from (4), we have det$[\Psi^*(x_{i,} x_j)] \geq [g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n) + [g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n) > [g^*(1)]^{2n}\, f^*(x_1) f^*(x_2) \ldots \ldots f^*(x_n)$.

3.MATRICES ASSOCIATED WITH QUADRATIC FUNCTIONS

In this section we use the following theorem of Vaidyanathaswamy, concerning quadratic functions and our results on matrix $[\Psi^*(x_{i,} x_j)]$ is to investigate matrices of the form $[f^*(x_{i,} x_j)]$
Where $f^*(m)$a quadratic function .An arithmetical function is $f^*(m)$ is said to be quadratic if it is the Dirichlet product of two completely multiplicative functions .

THEOREM 4

If $f^* = g^*.h^*$, where $g^*$and h^* are completely multiplicative functions, then f^* satisfies the identity

$$f^*(mr) = \sum_{d//(m,r)^*} f^*(m/d) f^*(r/d) g^*(d) h^*(d) \mu^*(d)$$

COROLLARY. 6

Let $f^* = g^*.h^*$, where $g^*$and h^* are completely multiplicative. If S is factor –closed ,then each of the following is true:

i) det $[f^*(x_{i,} x_j)] = \prod_{i=1}^{n} g^*(x_{i,})h^*(x_{i,})\mu^*(x_{i,})$;

ii) if det $[f^*(x_{i,} x_j)] \neq 0, [f^*(x_{i,} x_j)]^{-1} = (a_{ij})$, where a_{ij} defined as

$$a_{ij} = \sum_{\substack{x_i/x_k \\ x_j/x_k}} \left(\frac{\mu^*(x_k)}{g^*(x_k)h^*(x_k)}\right) f'^*({x_k}/{x_i}) f'^*({x_k}/{x_j}).$$

Proof : this corollary follows from theorems 1 & 4 .

COROLLARY. 7

Let $f^* = g^*.h^*$, where $g^*$and h^* are completely multiplicative. If $f^*(d) \in R$ and $g^*(d)h^*(d)\mu^*(d) > 0$ whenever d//x for any x $\in s$, then each of the following is true : i)$[f^*(x_{i,} x_j)]$ is positive definite

ii)det $[f^*(x_{i,} x_j)] = \prod_{i=1}^{n} g^*(x_i)h^*(x_i)\mu^*(x_i)$ if and only if S factor-closed

(iii) $\prod_{i=1}^{n} g^*(x_{i,})h^*(x_{i,})\mu^*(x_{i,}) \leq$ det $[f^*(x_{i,} x_j)] \leq f^*(x_1{}^2)f^*(x_2{}^2) \ldots f^*(x_n{}^2)$.

Proof. This corollary follows from theorem 3 & 4.

EXAMPLE .3

For S= $\{x_1 x_2 x_3 \ldots . x_n\}$ be a set of distinct positive integers each of the following is true :

i).det $[\sigma_\varepsilon(x_{i,} x_j)] = \prod_{i=1}^{n} x_i{}^\varepsilon \mu^*(x_i)$

ii).det $[\sigma_\varepsilon(x_{i,} x_j)] \neq 0, [\sigma_\varepsilon(x_{i,} x_j)]^{-1} = (a_{ij})$, where

$$a_{ij} = \sum_{\substack{x_i/x_k \\ x_j/x_k}} (x_k)^{-\varepsilon}\mu^*(x_k)\sigma_\varepsilon{}'^*({x_k}/{x_i})\sigma_\varepsilon{}'^*({x_k}/{x_j})$$

EXAMPLE.4

Let $\beta^*(m)$ be the number of integers d such that $1 \leq d \leq m$ and (d, m) = r^2 for some r.This function is called Square –totient. if μ^*(x) $\neq 0$ for all x $\in s = \{x_1 x_2 x_3 \ldots . x_n\}$,then each of the following is true:

(i) $[\beta^*(x_{i,} x_j)$ is positive definite,

(ii) det $[\beta^*(x_{i,} x_j)] = x_1 x_2 x_3 \ldots . x_n$ if and only if S is factor-closed,

(iii) $x_1 x_2 x_3 \ldots . x_n \leq$ det $[\beta^*(x_{i,} x_j)] \leq \beta^*(x_1{}^2)\beta^*(x_2{}^2) \ldots \beta^*(x_n{}^2)$.

We have $\beta^* = \zeta_1.\lambda$, where Lioville's function λ is defined by $\lambda(1) = 1$ and if

m = $p_1{}^{a_1}.p_2{}^{a_2} \ldots p_r{}^{a_r}$ as a product of distinct primes p_i, then $\lambda(m) = (-1)^k$, where

k = $a_1 + a_{2.} \ldots . a_r$. this applying Corollary 7, we obtain (i) – (iii) . if S is factor closed , then

det $[\beta^*(x_{i,} x_j)] = \prod_{i=1}^{n} x_i |\mu^*(x_i)|$. Moreover, if det $[\beta^*(x_{i,} x_j)] \neq 0$,then

$[\beta^*(x_{i,} x_j)]^{-1} = (a_{ij})$, where a_{ij} is defined by

$$a_{ij} = \sum_{\substack{x_i/x_k \\ x_j/x_k}} (x_k)^{-1} \beta'^* \left({x_k}/{x_j} \right) \beta'^* \left({x_k}/{x_i} \right)$$

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