A Comparative Study on Gauss Elimination Method and Simplex Method of Linear Optimization Problem

Rahidas Kumar[1] and Sahdeo Mahto[2]
[1]Research Scholar, Department of Mathematics, Ranchi University, Ranchi, Jharkhand, india
[2]Asso, Prof, Department of Mathematics, University, Ranchi, Jharkhand, india

Abstract: *This paper deals a new approach for the solution of linear optimization problem with the help of Gauss Elimination Method of matrix. This method is based on the square matrix which converted into upper triangular matrix by elementary row transformation. This method we get direct solution without any iteration. Also we show that this method is better than Simplex Method.*

Keywords: *Gauss Elimination Method, linear programming problem, system of linear equation, upper triangular matrix, augmented matrix, Simplex Method*

Introduction: Generally all linear programming problem are related with maximization (or minimization) of linear function subject to set of constraints. The Simplex method of the linear programming was developed by Prof. G.B. Dantzig in 1947[1],[2]. He demonstrated how to use an objective function to find the optimal solution from amongst the several feasible solutions to the linear programming problem. Further, development of computers last three decades has made it easy for the Simplex Method to solve large scale linear programming problems very quickly [3],[4]. However in 1984, Narendra Karmakar of AT and T Bell laboratories was developed a new algorithm for solving very large scale linear programming problem. Further in 1998 , this method was modified by P. Kanniappan and K. Thangavel[5],[6].

This article, we use Gauss Elimination Method of matrix for the system of linear inequalities in linear programming problem and solve it[7]. By applying Gauss Elimination Method we get the solution after initial iteration.

Here we use some basic terminology [8]-[10] which are given below.

Optimal (optimum) solution: A feasible solution to a lpp is said to be optimal solution if it is also optimize the objective function Z of the problem.

Coefficient Matrix : All the coefficient of the set of linear system of equations in matrix form is Coefficient Matrix.

Constant Matrix: All the constant of the set of linear system of equation in matrix form is Constant Matrix.

Augmented Matrix : Combination matrix of coefficient matrix and constant matrix is Augmented matrix. If A is coefficient matrix and B is constant matrix then [A|B] is augmented matrix.

Elementary row transformation: If a matrix convert into another matrix by only row operation (addition and subtraction of any two rows, multiplication and division by any scalar of any rows) is Elementary row transformation.

Upper Triangular Matrix : A square matrix is upper triangular matrix if all element below principal diagonal are zero.

This paper has six sections. The first section gives knowledge about Gauss Elimination Method to solve linear programming problem, the second section introduce Simplex Method, the third section illustrates real life linear optimization problem, the fourth section, the numerical example solved by Gauss elimination method , the fifth section highlights the Simplex Method and the last sixth section gives comparison of Gauss Elimination Method with Simplex Method.

I. GAUSS ELIMINATION METHOD TO SOLVE LINEAR OPTIMIZATION PROBLEM

Gauss Elimination Method is direct method which consists of transforming the given system of simultaneous equations to an equivalent upper triangular system. From this transformed system, the required solution can be obtained by the back substitution.

Steps to solve a system of linear inequalities using Gauss Elimination Method.

- Transform the Linear Optimization Problem in to canonical form i.e., if the objective function is maximization then all inequalities sing must be ' $\leq$ ' types and if objective function is minimization then all inequalities sing must be '$\geq$' types.
- Convert all inequalities in to equalities then system of n linear equations in n variables x_1 , x_2 ,x_3 ,..., x_n as a matrix equation AX = B , where A = $[a_{ij}]$ is n×n matrix of real coefficients, X = $[x_i]$ is n×1 matrix of variables and B = $[b_i]$ is n×1 matrix of constants.
- Construct augmented matrix [A | B] and convert A matrix into upper triangular matrix. i.e., [U | B'] where U is the upper triangular matrix and B' is the transform form of B.
- Now solve UX = B', we will get the solution by back substitution.

II. SIMPLEX METHOD TO SOLVE LINEAR OPTIMIZATION PROBLEM

All Optimization problems solve by Simplex Method. But here we focus linear optimization problems because it's comparing Gauss Elimination Method. Given problem always optimize to maximization if it is not optimize to maximization then it's convert to maximization, objective function. All inequalities convert into equalities by introducing non negative slack and surplus variables. If we cannot get the initial basic matrix (B) = I_n (Identity matrix) then we introduce one more variable which is Artificial variables. And last we solve the optimization problem by general Simplex method, Two-phase method or Big-M method according to situation.

- This is the iteration method. We construct the Simplex tables step by step.
- When all $\Delta_j = Z_j - C_j \geq 0$ in the Simplex table then stop the iteration and we get the optimum values and optimal solution.

III. NUMERICAL ILLUSTRUCTION

A firm can produce three types of cloth say A, B and C. Three kinds of wool are required for it, say red wool, green wool and blue wool. One unit length of type A cloth needs 2 yards of red wool and 3 yards of blue wool; one unit length of type B cloth needs 3 yards of red wool, 2 yards of green wool and 2 yards of blue wool, one unit of type C cloth needs 5 yards of green wool and 4 yards of blue wool. The firm has only a stock of 8 yards of red wool, 10 yards of green wool and 15 yards of blue wool. It is assumed that the income obtained from one unit length of type A cloth is 3.00 of type B cloth is 5.00 and of type cloth is 4.00[11].

<u>Formulation of the problem</u>:

For clear understanding of the problem, first we construct a table for the given data's.

Kinds of wool	Types of cloth $A(x_1)$	 $B(x_2)$	 $C(x_3)$	Stock of wool available with the firm in(yards)
Red Green Blue	2 0 3	3 2 2	0 5 4	8 10 15
Income from one unit length of cloth in Rs.	3.00	5.00	4.00	

Let the firm produce x_1, x_2, and x_3 yards of three types of A,B and C respectively. Total profit in Rs. of the firm is given by $Z = 3x_1 + 5x_2 + 4x_3$

Total quantity of red wool require to prepare x_1, x_2, and x_3 yards of three cloth of types A, B and C is $2x_1 + 3x_2 + 0x_3$ yards.

Since the stock of red wool available is only 8 yards.

So $2x_1 + 3x_2 + 0x_3 \leq 8$

Similarly total quantity of green wool required is

$0x_1 + 2x_2 + 5x_3 \leq 10$

And total quantity of blue wool required is

$3x_1 + 2x_2 + 4x_3 \leq 15$

Hence the problem of the firm formulated as linear programming problem is as follows:

Optimize (Maximize) $Z = 3x_1 + 5x_2 + 4x_3$

Subject to the constraints

- $2x_1 + 3x_2 + 0x_3 \leq 8$
- $0x_1 + 2x_2 + 5x_3 \leq 10$
- $3x_1 + 2x_2 + 4x_3 \leq 15$, $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$

IV. SOLVE THE ABOVE LINEAR OPIMIZATION PROBLEM BY GAUSS ELIMINATION METHOD:

The above system of inequalities written as,

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 5 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 15 \end{bmatrix}$$

Now write the augmented matrix

$$\text{i.e,} \begin{bmatrix} 2 & 3 & 0 & \vdots & 8 \\ 0 & 2 & 5 & \vdots & 10 \\ 3 & 2 & 5 & \vdots & 15 \end{bmatrix}$$

Convert the coefficient matrix into upper triangular by elementary row transformation.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & \vdots & 4 \\ 0 & 2 & 5 & \vdots & 10 \\ 0 & 0 & -\frac{41}{5} & \vdots & -\frac{62}{5} \end{bmatrix}$$

Also it is written as

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -\frac{41}{5} \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -\frac{62}{5} \end{bmatrix}$$

Now the back substitution we get the solution, $x_1 = 89/41$, $x_2 = 50/41$ and $x_3 = 62/41$

And hence the optimize (maximize) Z = 765/41

V. SOLVE THE ABOVE PROBLEM BY SIMPLEX METHOD

Since the above linear optimization problem, optimize to maximize then convert all inequalities into equalities by introducing slack variables s_1 , s_2 and s_3 respectively. Now we write the linear optimization problem in standard form.

Maximize $Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$
Subject to constraints:

- $2x_1 + 3x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 8$
- $0x_1 + 2x_2 + 5x_3\ 0s_1 + s_2 + 0s_3 = 10$
- $3x_1 + 2x_2 + 4x_3\ 0s_1 + 0s_2 + s_3 = 15$; x_1 , x_2 , x_3, s_1, s_2 and $s_3 \geq 0$

Initial basic feasible solution $X_B = \begin{bmatrix} s1 \\ s2 \\ s3 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 15 \end{bmatrix}$

The iterative simplex table are:

TABLE 1, INITIAL ITARATION

		C_j	3	5	4	0	0	0	Minimum Ratio.
B	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B/X_2
S_1	0	8	2	**3**	0	1	0	0	8/3 →
S_2	0	10	0	2	5	0	1	0	10/2
S_3	0	15	3	2	4	0	0	1	15/3
Z= 0		Δ_j	-3	-5 ↑	-4	0 ↓	0	0	

Here introduce X_2 and drop S_1

		C_j	3	5	4	0	0	0	Minimum Ratio.
B	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B/X_3
X_2	5	8/3	2/3	1	0	1/3	0	0	____
S_2	0	14/3	-4/3	0	**5**	-2/3	1	0	14/15 →
S_3	0	29/3	5/3	0	4	-2/3	0	1	29/12
Z= 40/3		Δ_j	1/3	0	-4↑	-5/3	0 ↓	0	

TABLE 2, SECOND ITERATION

Here introduce X_3 and drop S_2

TABLE 3, THIRD ITERATION

		C_j	3	5	4	0	0	0	Minimum Ratio.
B	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B/X_1
X_2	5	8/3	2/3	1	0	1/3	0	0	4
X_3	4	14/15	-4/15	0	1	-2/15	1/5	0	_____
S_3	0	89/15	**41/15**	0	0	-2/15	-4/5	1	89/41 →
Z= 256/15		Δ_j	-11/15 ↑	0	0	17/15	4/5 ↓	0	

Here introduce X_1 and drop S_3

TABLE 4, FOURTH ITERATION

		C_j	3	5	4	0	0	0
B	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
X_2	5	50/ 41	0	1	0	44/41	8/41	-10/ 41
X_3	4	62/ 41	0	0	1	-6/41	5/41	4/41
X_1	3	89/41	1	0	0	-2/41	-12/ 41	15/ 41
Z= 765/ 41		Δ_j	0	0	0	190/ 41	24/ 41	11 /41

Here all $\Delta_j = Z_j - C_j \geq 0$, so simplex table is optimal. And hence the optimal basic feasible solution is x_1 = 89/41 x_2 = 62/41 and x_3= 50/41 ; Maximize Z = 765/41

VI. COMPARISON OF GAUSS ELIMINATION METHOD AND SIMPLEX METHOD

A. Similarity

- Both methods are doing step by step
- Both methods give us an actual solution.

B. DIFFERENCE

- In Simplex Method we introduce slack variables but in Gauss Elimination Method we cannot use any extra variables.
- In Simplex Method, we have to take at least two iterations but Gauss Elimination Method, we have to take exactly two iterations.

CONCLUSIONS AND REMARKS

Gauss Elimination Method has minimum calculations as compare to Simplex Method. Gauss Elimination Method has simple calculation as compare to the Simplex Method. More and more useful of Gauss Elimination Method becomes very mechanical. So this method is a faster than of the Simplex Method.

REFERENCES:

[1] Dantzig G. B, *Linear Programming and Extension*. Princenton, University Press, Princenton, NJ

[2] Dantzig G. B., 1951, *Maximization of linear function subject to linear inequalities,* in T. C. Koopmans(ed.), *Activity Analysis OF Production and Allocation,* John Wiley & Sons, New York, 339-347.

[3] Bland R. G., 1977. *New finite pivoting rules for the simplex method, Mathematics of Operations Research 2,* 103- 107

[4] Teukolsky, WH., SA., Vetterling, WT. And Flannery, BP, 2007. *NumericalRecipes: The Art of Science Computing(3rd ed.),*New York: Cambridge University Press.

[5] Karmakar, N. 1984. *A new polynomial time algorithm for linear programming , Combinatorica, 4,* 141- 158.

[6] P. Kanniappan and k. Thangavel, 1998 *Modified Fourier's method of solving linear programming problems, Opsearch,* 35, 45- 56.

[7] Householder, Alston S., 1975 . *The Theory of Matrices in Numerical Analysis,* New York; Dover Publications.

[8] Shenoy, G. V. 1998. *Linear Programming (Method and Applications),* Second Edition, New Age International Publishers.

[9]Horn , Roger A. And Johnson, Charles R., 1985. *Matrix Analysis,* Cambridge university Press. Section 3.5

[10] Poole, David, 2006, *Linear Algebra "A Modern Introduction"* (2nd ed.), Canada; Thomson Books/ Cole.

[11] Gupta,R. K., *Operations Research,* Krishna Prakashan Media (P) Ltd.2013