Cone Metric Spaces, Cone Rectangular Metric Spaces and Common Fixed Point Theorems

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ABSTRACT: *A common fixed point theorem for a pair of weakly compatible mappings is proved in a cone metric pace. We also proved a fixed pint theorem in cone rectangular metric space by using rational type contractive condition.*

Introduction: The study of common fixed pint of mapping satisfying certain contractive conditions has been taken a important role in resent research activity. In 1976, Jungek [5] proved a common fixed point theorem for commuting mappings, generalized the famous Banach contraction principle. Sessa [10] introduced the notion of weakly commuting mappings. Also, Jungek [6] introduced the notion of compatible mappings in order to generalize the concept of weak commutativity. Again, Pant [9] Defined R-Weakly commuting maps and established some common fixed pint theorem, assuming the continuity of at least one of the mappings.

Jungek and Rhoades [7] defined a pair of self mappings to be weakly compatible if they commute at their coincidence pints. Then applying these concepts, several authors have obtained coincidence point results for various classes of mappings in a metric space. On the other hand Huang and Zhang [4] have introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space, and they have established some fixed point theorems for contractive type mappings in a normal cone metric space. On the other hand Huang and Zhang [4] have introduced the concept of cone metric spaces, where the set of real numbers is replaced by an ordered Banach space, and they have established some fixed point theorems for contractive type mappings in complete cone metric spaces. The study of fixed point theorems in such spaces is followed by some other mathematicians.

Following the idea of Branciari [3], Agam, Arshad and Beg [2] extended the notion of cone metric spaces by replacing the triangular inequality by a rectangular inequality.

The aim of this paper is to establish a common fixed point theorem for a pair of weakly compatible mappings in a cone metric space without exploiting the notion of the continuity and we also proved a fixed point theorem in cone rectangular metric space by using rational type contraction mapping.

Let E be a real Banach space. A subset P of E is called a cone if, and only if

(a) P is called, non empty and $p \neq \{0\}$

(b) $a,b \in$ R and $a,b \geq 0$ and $x,y \in$ P implies $ax + by \in$ P

(c) $P \cap (-P) = \{0\}$

Given a cone $P \in E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in R$, A cone is normal if there is a number $K > 0$ such that for all $x, y \in E$, the inequality $0 \leq x \leq y$ implies. $\|x\| \leq K\|y\|$

The least positive number satisfying the above inequality is called the normal constant of P, while $x << y$ stands for $y - x \in$ int P (interior of P).

DEFINITION 1.1. Let X be a non empty. Suppose that the mapping $d : X \times X \to E$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, yz \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space.

DEFINITION 1.2. Let (X, d) be a cone metric space we say that $\{x_n\}$ is a Cauchy sequence if for energy C in E with $c >> 0$, there is N such that for all $n > N$, $d(x_n, x) << C$ for some fixed x in X.

A cone metric space X is said to be complete if energy Cauchy sequence in X is convergent in X.

It is known that $\{x_n\}$ converges to $x \in X$ if, and only if $d(x_n, x) \to 0$ as $n \to \infty$. Also the limit of a convergent sequence is unique provided P is a normal cone with normal constant k.

DEFINITION 1.3. Let f and g be self mappings of a set X. If $w = fx = gx$ for some x in X. then x is called a coincidence point f and g, and w is called a point of coincidence of f and g.

DEFINITION 1.4. Two self mappings f and g of a set X are said to be weakly compatible if they commute at their coincidence points, that is if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

DEFINITION 2.1. Let X be a non empty set, suppose the mapping $d : X \times X \to E$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$, $x \neq y$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and for all distinct points $u, v \in X / \{x, y\}$ (rectangular property)

Then d is called a cone rectangular metric on X, and (X, d) is called a cone rectangular metric space.

Note that any cone metric space is a cone rectangular metric space but the converse is not true is general.

DEFINITION 2.2. Let (X,d) be a cone rectangular metric space let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $C \in E$, $c >> 0$ there is N such that for all $n > N$, $d(x_n, x) << C$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. We denote this by $x_n \to +\infty$.

LEMMA 2.3. Let (X,d) be a cone rectangular metric space, P be a normal cone. Let $\{x_n\}$ be a sequence in X. Then $x_n \to x$ as $n \to +\infty \Leftrightarrow \| d(x_n, x) \| \to 0$ as $n \to +\infty$.

Note that if (X,d) is a cone metric space and $\{x_n\}$ is convergent sequence in X. Then the limit of $\{x_n\}$ is unique.

DEFINITION 2.4. Let (x,d) be a cone rectangular metric space, $\{x_n\}$ be a sequence in X. If for any $C \in E$ with $0 << C$, therer is N such that for all $n, m > N$, $(x_n, x_m) << C$, then $\{x_n\}$ is called a Cauchy sequence in X.

DEFINITION 2.5. Let (X,d) be a cone rectangular metric space and P be a normal cone. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence of and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

DEFINITION 2.6. Let (X,d) be a cone rectangular metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone rectangular metric space.

THEOREM 1:- Let (X,d) be a cone metric space and P be a normal cone with normal constant k. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition

$$d(fx, fy) \le r[d(gx, gy) + d(fx, fy) + d(fx, gy) + d(fy, gx) + d(fy, gy) + d(fx, gx)]$$

Where $r \in [0, \frac{1}{6})$ is a constant. If the range of g contains the range of f and $g(X)$ is complete subspace of X. Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Then since $f(X) \subset g(X)$, we choose a point x_1 in X such that $f(x_0) = g(x_1)$ continuing this process, having chosen x_n in X we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$
\begin{aligned}
d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\
&\leq r[d(gx_n, gx_{n-1}) + d(fx_n, fx_{n-1}) + d(fx_n, gx_{n-1}) + d(fx_{n-1}, gx_n) \\
&\qquad + d(fx_n, gx_n) + d(fx_{n-1}, gx_{n-1})] \\
&\leq r[d(gx_n, gx_{n-1}) + d(gx_{n+1}, gx_n) + 2d(gx_{n+1}, gx_n) + 2d(gx_n, gx_{n-1})] \\
&\leq r[3d(gx_n, gx_{n-1}) + 3d(gx_{n+1}, gx_n)] \\
&\leq \frac{3r}{1-3r} d(gx_n, gx_{n-1})
\end{aligned}
$$

$$d(gx_{n+1}, gx_n) \leq hd(gx_n, gx_{n-1})$$

$$h = \frac{3r}{1-3r}$$

Now for $n > m$ we get

$$
\begin{aligned}
d(gx_n, gx_m) &\leq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots\cdots + d(gx_{m+1}, gx_m) \\
&\leq (h^{n-1} + h^{n-2} + \cdots\cdots h^m) d(gx_{m+1}, gx_m) \\
&\leq \frac{h^m}{1-h} d(gx_1, gx_0)
\end{aligned}
$$

Using the normality of cone P implies that

$$\| d(gx_n, gx_m) \| \leq \frac{h^m}{1-h} K \| d(gx_1, gx_0) \|$$

$d(gx_n, gx_m) \to 0$ as $n, m \to \infty$

$\{gx_n\}$ is a Cauchy sequence in X since $g(X)$ is complete subspace of X so there exists q in $g(X)$ such that

$$g(p) = q$$

$$d(gx_n, f_p) \le r[d(gx_{n-1}, g_p) + d(gx_n, f_p) + d(gx_{n-1}, f_p) + d(f_p, g_p)$$
$$+ d(fx_{n-1}, g_p) + d(fx_{n-1}, gx_{n-1})]$$

$$\le r[d(gx_{n-1}, gp) + d(gx_n, f_p) + d(gx_{n-1}, g_p) + d(gx_{n-1}, g_p)]$$

$$\le \frac{3r}{1-r}[d(gx_{n-1}, g_p)]$$

Using normality of cone

$$\| d(gx_n, f_p) \| \le K\frac{3r}{1-r} \| d(gx_{n-1}, g_p \| = 0 \text{ as } n \to \infty.$$

$$d(gx_n, f_p) \to 0 \text{ as } n \to \infty$$

The uniqueness of a limit in a cone metric space implies that

$$f(pO = g(p)$$

Again we show f and g have a unique point of coincidence, if possible assume that there exists another point t in X such that $f(t) = g(t)$

$$d(gt, gp) = d(ft, fp)$$

$$\le r[d(gt, gp) + d(ft, fp) + d(ft, gp) + d(fp, gt) + d(ft, gt) + d(fp, gp)]$$

$$\le r[d(gt, gp) + d(ft, fp) + d(gp, gt) + d(gp, gt)]$$

$$d(gt, gp) \le \frac{r}{1-2r}[d(gp, fp)] \to 0$$

This show f and g both have same fixed point.

Theorem 2. Let (X, d) be a complete cone rectangular metric space. P be a normal cone with normal constant K. Suppose a mapping $f : X \to X$ satisfying contractive condition

$$d(fx, fy) \le \alpha\left[\frac{d(x, fx)d(y, fy)}{d(x, y)} + d(x, fx) + d(y, fy) + d(x, y)\right] \forall xy \in X,$$

where $\alpha \in [0, \frac{1}{5})$. Then,

(i) f has a unique fixed point in X

(ii) For any $x \in X$, the iterative sequence $T^n_{(x)}$ converges to the fixed point.

Now for $x \in X$ we have

$$d(Tx, T^2x) = d(Tx, TTx)$$

$$\leq \alpha \left[\frac{d(x,Tx)d(Tx,T^2x)}{d(x,Tx)} + d(x,Tx) + d(Tx,T^2x) + d(x,Tx) \right]$$

$$\leq 2\alpha \left[d(Tx,T^2x) + d(x,Tx) \right]$$

$$d(Tx,T^2x) \leq \frac{2\alpha}{1-2\alpha} d(x,Tx)$$

$$d(T^2x,T^3x) = d(TTx,TT^2x)$$

$$\leq \alpha \left[\frac{d(Tx,T_x^2)d(T_x^2,T_x^3)}{d(Tx,T_x^2)} + d(Tx,T^2x)dd(T^2x,T^3x) + d(Tx,T^2x) \right]$$

$$\leq \alpha \left[d(T^2x,T^3x) + d(Tx,T^2x) + d(T^2x,T^3x) + d(Tx,T^2x) \right]$$

$$d(T^2x,T^3x) \leq \frac{2\alpha}{1-2\alpha} d(Tx,T^2x)$$

$$\leq \left(\frac{2\alpha}{1-2\alpha} \right)^2 d(x,Tx)$$

Thus in general, if n is a positive integer, then

$$\alpha(T^nx, T^{n+1}x) \leq \left(\frac{2\alpha}{1-2\alpha} \right)^n d(x,Tx)$$

$$\leq k^n d(x,Tx)$$

where $k = \dfrac{2\alpha}{1-\alpha} \in [0, \frac{1}{5})$.

We divide the proof into two case.

FIRST CASE: Let $T^m x = T^n x$ for some $m,n \in N, m \neq n$. Let $m > n$. Then $T^{m-n}(T^n x) = T^n x$, i.e. $T^p y = y$ where $p = m-n$, $y = T^n x$. Now since $p > 1$, we have

$$d(y,Ty) = d(T^p y, T^{p+1} y)$$

$$\leq r^p d(y,Ty).$$

Since $r \in [0,1)$ we obtain $-d(y,Ty) \in P$ and $d(y,Ty) \in P$ which implies that $\| d(y,Ty) \| = 0$, i.e., $Ty = y$.

SECOND CASE: Assume that $T^m x \neq T^n x$ for all $m,n \in N$, $m \neq n$. Clearly, we have

$$d(T^n x, T^{n+1} x) \leq r^n d(x,Tx) \leq \frac{r^n}{1-r} d(x,Tx)$$

And

$$d(T^n x, t^{n+2} x) \leq \alpha(d(T^{n-1} x, T^n x) + d(T^{n+1} x, T^{n+2} x))$$

$$\leq \alpha(r^{n-1} d)x,Tx) + r^{n+1} d(x,Tx))$$

$$\leq r^n d(x,Tx) + r^{n+1} d(x,Tx)$$

$$\leq \frac{r^n}{1-r} d(x,Tx).$$

Now if $m > 2$ is odd then writing $m = 2\ell + 1, \ell \geq 1$ and using the fact that $T^p x \neq T^r x$ for $p,r \in N$, $p \neq r$, we can easily show that

$$d(T^n x, T^{n+m} x) \leq (d(T^{n-1} x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{n+2\ell-1} x, T^{n+2\ell} x)$$

$$\leq r^n d(x,Tx) + r^{n+1} d(x,Tx) + \cdots + r^{n+2\ell} d(x,Tx)$$

$$\leq \frac{r^n}{1-r} d(x,Tx).$$

Again if $m > 2$ is even then writing $m = 2\ell \geq 2$ and using the same arguments as before, we can get

$$d(T^n x, T^{n+m} x) \le d(T^n x, T^{n+2} x) + d(T^{n+2} x, T^{n+3} x) + d(T^{n+3} x, T^{n+4} x) + \cdots + d(T^{n+2\ell-1} x, T^{n+2\ell} x)$$

$$\le r^n d(x, Tx) + r^{n+2} d(x, Tx) + r^{n+3} d(x, Tx) + \cdots + r^{n+2\ell} d(x, Tx)$$

$$\le \frac{r^n}{1-r} d(x, Tx).$$

Thus combining all the cases we have

$$d(T^n x T^{n+m} x) \le \frac{r^n}{1-r} d(x, Tx), \forall m, n \in N.$$

Hence, we get

$$\| d(T^n x, T^{n+m} x) \| \le \frac{r^n}{1-r} \| d(x, Tx) \|, \forall m, n \in N.$$

Since $k \frac{r^n}{1-r} \| d(x, Tx) \| \to 0$ as $n \to +\infty, (T^n x)$ is a Cauchy sequence. By the completeness of X , there is $x^* \in X$ such that $T^n x \to x^*$ as $n \to +\infty$.

We shall now show that $Tx^* x^*$. Without any loss of generality, we can assume that $T^* x \neq x^*$, Tx^* for any $r \in N$. We have

$$d(x^*, Tx^*) \le d(x^*, T^{n+1} x) + d(T^{n+1} x, Tx^*)$$

$$\le d(x^*, T^n x) + d(T^n x, T^{n+1}) + \alpha (d(T^n x, T^{n+1} x) + d(x^*, Tx^*))$$

This implies that

$$d(x^*, Tx^*) \le \frac{1}{1-\alpha} (d(x^*, T^n x) + (1+\alpha) d(T^n x, T^{n+1} x)).$$

Hence,

$$\| d(x^*, Tx^*) \| \le \frac{k}{1-\alpha} (\| d(x^*, T^n x) \| + (1+\alpha) \| d(T^n x, T^{n+1} x) \|) \to 0$$ as $n \to +\infty$.

So we obtain $d(Tx^*, x^*) = 0$, i.e., $x^* = Tx^*$.

Now, if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \le \alpha(d(x^*, Tx^*) + d(y^* Ty^*)) = 0$$

which implies that $\|d(x^*, y^*)\| = 0$ i.e., $x^* = y^*$.

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