A Model for Optimal Reserve Inventory between Two Machines With Reference to the Distribution of Repair Time

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Abstract- *In Inventory control theory, many suitable models for real life systems are constructed with the objective of determining the optimal inventory level. In a system where the machines are in series for producing the finished products. The reserve of semi-finished products between two machines becomes unavoidable to minimize the idle time of machines in series. In this model the repair time of machines is assumed to be a random variable and it follows exponential distribution which satisfies the so-called change of distribution property. Also, the truncation point of the repair time distribution is itself a random variable and it follows mixed exponential distribution. Under this assumption an optimal reserve inventory is obtained.*

Keywords: *Reserve Inventory, Change of distribution property, Repair time, Truncation point and Optimal reserve.*

I. INTRODUCTION

Determining the behavior of the inventory system in terms of the mathematical model brings out a determination of optimal inventory under different real-life circumstances. When working systems are considered, avoiding the breakdown of the system if quiet possible by keeping reserve inventory between the machines are in series. A system in which there are two machines M_1 and M_2 are in series. The output of the machine M_1 is the input of the machine M_2. The breakdown of M_1 causes the idle time of M_2, since there is no input to M_2 from M_1. The idle time of M_2 is very costly and hence to avoid it, a reserve inventory is maintained in between M_1 and M_2. If the reserve inventory is surplus in quantity, there is an inventory holding cost. If the reserve inventory slacks in its quantity, then it assumes the high idle time cost. Hence, in order to balance these costs, an optimal reserve inventory must be maintained between these two machines. Therefore, the problem is to determine the optimal reserve between M_1 and M_2. Here the repair time of M_1 is considered as a random variable. The very basic model has been discussed by Hanssmann(1962)[(1)]. The extension over this model is discussed by Ramachandran and Sathyamoorthi (1981)[(2)].

This type of models has been discussed by many authors under the assumptions that the repair time is a random variable. Sachithananthamet.al(2006)[(3)] discussed this model with the assumption that the probability function of the repair time undergoes a parametric change after the truncation point. The very basic concept of change of distribution property was discussed by Stangl(1995)[(4)], Suresh Kumar(2006)[(5)] has used their concept in the stock model approach. The model for optimal reserve inventory between two machines under the assumption that the repair time of machine M_1 satisfies the change of distribution property with the truncation point being a random variable is derived by Sachithanantham et.al(2007)[(6)]. And the same model with a modification of the probability function of truncation point is discussed by Ramathilagam et. al(2014)[(7)]. The improvement over this model is being discussed in this paper, in which the truncation point is a random variable, which follows the exponential distribution with parameter α.

II. NOTATIONS

h : Inventory holding cost/ unit/ unit-time.
d : Idle time cost due to M_2 / unit-time.

μ : Mean time interval between successive breakdown of machine M_1, assuming exponential distribution of inter-arrival times.

t : Continuous random variable denoting the repair time of M_1 with pdf g(.) and c.d.f G(.)

r : Constant consumption rate of M_2 per unit of time.

S : Reserve inventory between M_1 and M_2.

T : Random variable denoting the idle time of M_2

θ : Parameter of exponential distribution before the truncation point.

λ : Parameter of exponential distribution after the truncation point.

α : Parameter of exponential distribution.

III. MODEL I

If T is a random variable denoting idle time of M_2, it can be seen that

$$T = \begin{cases} 0 & if\ \frac{s}{r} > t \\ t - \frac{s}{r} & if\ \frac{s}{r} \leq t \end{cases} \quad (1)$$

The expected total cost of inventory holding and idle time of M_2 per unit of time is given by

$$E(c) = hs + \frac{d}{\mu}\int_{\frac{s}{r}}^{\infty}\left(t - \frac{s}{r}\right)g(t)dt \quad (2)$$

The optimal reserve $\hat{S}$ can be obtained by solving the equation $\frac{dE(c)}{ds} = 0$.

The expression for optimal reserve inventory is given by

$$G\left(\frac{\hat{S}}{r}\right) = 1 - \frac{r\mu h}{d}$$

This result is discussed in Hanssmann.F (1962). It may be observed that the above expression for optimal value of S has a constraint that $\frac{h\mu r}{d}$ is always less than unity.Otherwise the solution is not a feasible one. Hence, a slight modification in the expression for the expected total cost can be incorporated as below.

IV. MODEL II

In this model it is assumed that the repair time distribution satisfies the change of distribution property is basically discussed by Stangl(1995), suresh Kumar(2006). Under this assumption

$$g(t) = \begin{cases} g_1(t,\theta), & t \leq X_0 \\ g_2(t,\lambda), & t > X_0 \end{cases}$$

where X_0 is a random variable denoting the truncation point of repair time and it is distributed as exponential with parameter α.

$$g(t,\theta) = \begin{cases} \theta e^{-\theta t} & , if\ t \leq X_0 \\ e^{-\theta x_0}\, e^{-(t-x_0)}\left({(t-x_0)^{\lambda-1}}/{\Gamma\lambda} \right) & , if\ t > X_0 \end{cases}$$

$$f(t) = g_1(t,\theta)P(t \leq X_0) + g_2(t,\theta)P(t > X_0)$$

$$f(x_0) = \alpha e^{-\alpha x_0}, 0 \leq x_0 < \infty$$

$$\text{Here } P(t \leq X_0) = P(X_0 \geq t) = \int_t^{\infty} f(x_0)\, dx_0$$

$$= \int_t^{\infty} \alpha e^{-\alpha x_0}\, dx_0$$

$$= \alpha\left(\frac{e^{-\alpha x_0}}{-\alpha}\right)_t^{\infty}$$

$$\therefore P(t \leq x_0) = e^{-\alpha t}$$

$$f(t) = g_1(t,\theta)\, e^{-\alpha t} + \int_0^t \alpha e^{-\alpha x_0}\, g_2(t,\theta) dx_0$$

It may be observed that the random variable 'T' defined in equation (1) also undergoes a parametric change and the average idle time of M_2 is

$$E(T) = \int_{\frac{s}{r}}^{\infty} (t - {}^{s}/_{r})\, f(t) dt$$

$$= \int_{\frac{s}{r}}^{\infty} (t - {}^{s}/_{r}) \left\{ g(t,\theta) e^{-\alpha t} + \int_0^t \alpha e^{-\alpha x_0}\, h(t,\theta) dx_0 \right\} dt$$

$$= \int_{\frac{s}{r}}^{\infty} (t - {}^{s}/_{r}) \{\theta e^{-\theta t}\, e^{-\alpha t} \mathrm{dt}\} + \alpha \int_{\frac{s}{r}}^{\infty} (t - {}^{s}/_{r})\, e^{-\alpha x_0} e^{-\theta x_0} e^{-(t-x_0)} \left({}^{(t-x_0)^{\lambda-1}}/_{\Gamma\lambda} \right) \mathrm{d}x_0 dt$$

$$= \theta \int_{\frac{s}{r}}^{\infty} (t - {}^{s}/_{r})\, e^{-(\alpha+\theta)t} dt + {}^{\alpha}/_{\Gamma\lambda} \int_{\frac{s}{r}}^{\infty} \int_0^t (t - {}^{s}/_{r})\; e^{-(\alpha+\theta)x_0}\; e^{-(t-x_0)} (t-x_0)^{\lambda-1} \mathrm{d}x_0 dt$$

$$= \theta I_1 + {}^{\alpha}/_{\Gamma\lambda} (t - {}^{s}/_{r}) e^{-t} \int_{\frac{s}{r}}^{\infty} \left\{ \int_0^t e^{-(\alpha+\theta)x_0}\, e^{x_0} (t-x_0)^{\lambda-1}\, \mathrm{d}x_0 \right\} dt$$

$$= \theta I_1 + {}^{\alpha}/_{\Gamma\lambda} \int_{s/r}^{\infty} (t - {}^{s}/_{r})\, e^{-t} \left[\int_0^t e^{-(\alpha+\theta-1)x_0}\, (t-x_0)^{\lambda-1}\, \mathrm{d}x_0 \right] dt$$

$$Let \quad I_\lambda = {}^{1}/_{\Gamma\lambda} \int_0^t e^{-(\alpha+\theta-1)x_0} (t-x_0)^{\lambda-1}\, \mathrm{d}x_0$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ (t-x_0)^{\lambda-1} \left[\frac{e^{-(\alpha+\theta-1)x_0}}{-(\alpha+\theta-1)} \right]_0^t - \int_0^t \frac{e^{-(\alpha+\theta-1)x_0}}{-(\alpha+\theta-1)} (\lambda-1)(t-x_0)^{\lambda-2} (-\mathrm{d}x_0) \right\}$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)} - \frac{(\lambda-1)}{(\alpha+\theta-1)} \int_0^t e^{-(\alpha+\theta-1)x_0} (t-x_0)^{\lambda-2} \mathrm{d}x_0 \right\}$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)} - \frac{(\lambda-1)}{(\alpha+\theta-1)} \int_0^t e^{-(\alpha+\theta-1)x_0} (t-x_0)^{\lambda-2} \mathrm{d}x_0 \right\}$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)} - \frac{(\lambda-1)}{(\alpha+\theta-1)} \left[\frac{t^{\lambda-2}}{(\alpha+\theta-1)} - \frac{(\lambda-2)}{(\alpha+\theta-1)} I_{\lambda-3} \right] \right\}$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)} - \frac{(\lambda-1)}{(\alpha+\theta-1)^2}\, t^{\lambda-2} + \frac{(\lambda-1)(\lambda-2)}{(\alpha+\theta-1)^2} I_{\lambda-3} \right\}$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)} - \frac{(\lambda-1)}{(\alpha+\theta-1)^2}\, t^{\lambda-2} + \frac{(\lambda-1)(\lambda-2)}{(\alpha+\theta-1)^2} \left[\frac{t^{\lambda-3}}{(\alpha+\theta-1)} - \frac{(\lambda-3)}{(\alpha+\theta-1)} I_{\lambda-4} \right] \right\}$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)} - \frac{(\lambda-1)}{(\alpha+\theta-1)^2}\, t^{\lambda-2} + \frac{(\lambda-1)(\lambda-2)}{(\alpha+\theta-1)^3}\, t^{\lambda-3} - \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{(\alpha+\theta-1)^3} I_{\lambda-4} \right\}$$

$$= {}^{1}/_{\Gamma\lambda} \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)} - \frac{(\lambda-1)}{(\alpha+\theta-1)^2}\, t^{\lambda-2} + \frac{(\lambda-1)(\lambda-2)}{(\alpha+\theta-1)^3}\, t^{\lambda-3} - \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{(\alpha+\theta-1)^4}\, t^{\lambda-4} \right.$$
$$\left. + \frac{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{(\alpha+\theta-1)^5}\, t^{\lambda-5} - \cdots + \frac{(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)\ldots 3.2.1}{(\alpha+\theta-1)^{\lambda}}\, t^0 \right\}$$

$$= \left\{ \frac{t^{\lambda-1}}{(\alpha+\theta-1)(\lambda-1)!} - \frac{t^{\lambda-2}}{(\alpha+\theta-1)^2(\lambda-2)!} + \frac{t^{\lambda-3}}{(\alpha+\theta-1)^3(\lambda-3)!} - \cdots + \frac{t^3}{(\alpha+\theta-1)^{\lambda-3}3!} \right.$$

$$\left. - \frac{t^2}{(\alpha+\theta-1)^{\lambda-2}2!} + \frac{t^1}{(\alpha+\theta-1)^{\lambda-1}1!} - \frac{t^0}{(\alpha+\theta-1)^{\lambda}0!} \right\}$$

$$= \sum_{i=1}^{\lambda} \frac{(-1)^{i+1}\, t^{\lambda-i}}{(\alpha+\theta-1)^i(\lambda-i)!}$$

$$\mathrm{E(T)} = \theta \mathrm{I}_1 + \alpha \int_{\frac{s}{r}}^{\infty} (t - {}^{s}\!/_{r})\, e^{-t} \left[\sum_{i=1}^{\lambda} \frac{(-1)^{i+1}\, t^{\lambda-i}}{(\alpha+\theta-1)^i(\lambda-i)!} \right] dt$$

$$\mathrm{E(T)} = \theta \int_{s/r}^{\infty} (t - {}^{s}\!/_{r})\, e^{-(\alpha+\theta)t} dt + \alpha \int_{s/r}^{\infty} (t - {}^{s}\!/_{r})\, e^{-t} \left[\sum_{i=1}^{\lambda} \frac{(-1)^{i+1}\, t^{\lambda-i}}{(\alpha+\theta-1)^i(\lambda-i)!} \right] dt$$

Thus, the expected total cost is

$$E(c) = hs + \frac{d}{\mu}\ \mathrm{E(T)}$$

$$E(c) = hs + \frac{d}{\mu} \left\{ \theta \int_{\frac{s}{r}}^{\infty} (t - {}^{s}\!/_{r})\, e^{-(\alpha+\theta)t} dt + \alpha \int_{\frac{s}{r}}^{\infty} (t - {}^{s}\!/_{r})\, e^{-t} \left[\sum_{i=1}^{\lambda} \frac{(-1)^{i+1}\, t^{\lambda-i}}{(\alpha+\theta-1)^i(\lambda-i)!} \right] dt \right\}$$

$$E(c) = \mathrm{hs} + \frac{\mathrm{d}}{\mu}\{\theta \mathrm{T}_1 + \alpha \mathrm{T}_2\}$$

$$\frac{dE(c)}{ds} = 0 \quad \Rightarrow\ h + \frac{\theta d}{\mu}\frac{d\mathrm{T}_1}{ds} + \frac{\alpha d}{\mu}\frac{d\mathrm{T}_2}{ds} = 0$$

$$\frac{d\mathrm{T}_1}{ds} = \frac{d}{ds}\int_{\frac{s}{r}}^{\infty} (t - {}^{s}\!/_{r})\, e^{-(\alpha+\theta)t} dt$$

$$= \left(-\frac{1}{r}\right) \int_{\frac{s}{r}}^{\infty} e^{-(\alpha+\theta)t} dt$$

$$= \left(-\frac{1}{r}\right) \left[\frac{e^{-(\alpha+\theta)t}}{-(\alpha+\theta-1)} \right]_{\frac{s}{r}}^{\infty}$$

$$= \left(\frac{-1}{r(\alpha+\theta)}\right) \left[e^{-\frac{(\alpha+\theta)s}{r}} \right]$$

$$\frac{d\mathrm{T}_1}{ds} = \left[\frac{-\, e^{-(\alpha+\theta)s/r}}{r(\alpha+\theta)}\right]$$

$$\frac{d\mathrm{T}_2}{ds} = \frac{d}{ds}\int_{\frac{s}{r}}^{\infty} (t - {}^{s}\!/_{r})\, e^{-t}\left[\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}\, t^{\lambda-i}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\right]dt$$

$$= \left(-\frac{1}{r}\right)\int_{\frac{s}{r}}^{\infty} e^{-t}\left[\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}\, t^{\lambda-i}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\right]dt$$

$$= \left(-\frac{1}{r}\right)\sum_{i=1}^{\lambda}(-1)^{i+1}\int_{\frac{s}{r}}^{\infty} e^{-t}\,\frac{t^{\lambda-i}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\,dt$$

$$= \left(-\frac{1}{r}\right)\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\int_{\frac{s}{r}}^{\infty} t^{\lambda-i}e^{-t}\,dt$$

$$= \left(-\frac{1}{r}\right)\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\, e^{-\frac{s}{r}}\sum_{j=0}^{\lambda-\mathrm{i}}\frac{\left(\frac{s}{r}\right)^{j}}{j!}$$

$$= \left(-\frac{e^{-s/r}}{r}\right)\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\sum_{j=0}^{\lambda-\mathrm{i}}\frac{(s/r)^{j}}{j!}$$

$$\frac{dE(c)}{ds} = 0 \;\Rightarrow\; h + \frac{\theta d}{\mu}\left[\frac{-\, e^{-(\alpha+\theta)s/r}}{r(\alpha+\theta)}\right] + \frac{\alpha d}{\mu}\left[\left(-\frac{e^{-s/r}}{r}\right)\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\sum_{j=0}^{\lambda-\mathrm{i}}\frac{(s/r)^{j}}{j!}\right] = 0$$

$$h = \frac{\theta d}{\mu r(\alpha+\theta)}\, e^{-\frac{(\alpha+\theta)s}{r}} + \frac{\alpha d}{\mu r}\, e^{-\frac{s}{r}}\left[\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\sum_{j=0}^{\lambda-\mathrm{i}}\frac{\left(\frac{s}{r}\right)^{j}}{j!}\right]$$

$$\frac{h\mu r}{d} = \frac{\theta}{(\alpha+\theta)}\, e^{-(\alpha+\theta)s/r} + \alpha e^{-s/r}\left[\sum_{i=1}^{\lambda}\frac{(-1)^{i+1}}{(\alpha+\theta-1)^{i}(\lambda-i)!}\sum_{j=0}^{\lambda-\mathrm{i}}\frac{(s/r)^{j}}{j!}\right]$$

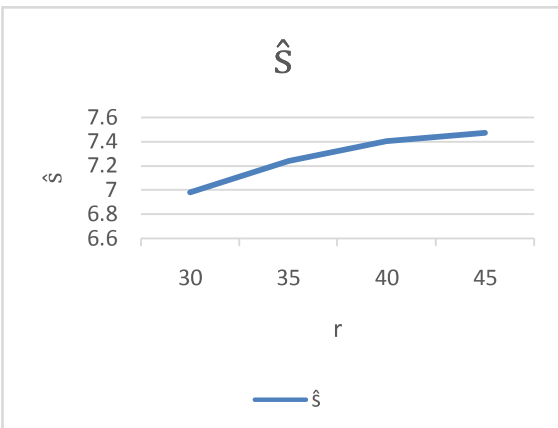
V. NUMERICAL ILLUSTRATIONS

The variations in the values of optimal reserve inventory " $\hat{S}$ " ,consequent on the changes in the parametersr, h, d,μ and αhave been studied by taking numerical illustrations. The tables and the corresponding graphs are given.

Case (i)

For h=5,d=3000,μ=2, λ =5,α =4, θ=3, the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of r.

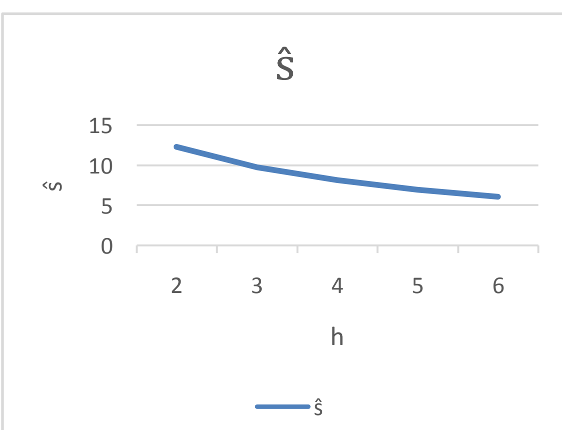
R	**30**	**35**	**40**	**45**
$\hat{S}$	6.98	7.24	7.4	7.47



Case (ii)

For r=30, d=3000, μ=2, λ =5,α =4,θ=3, the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of h.

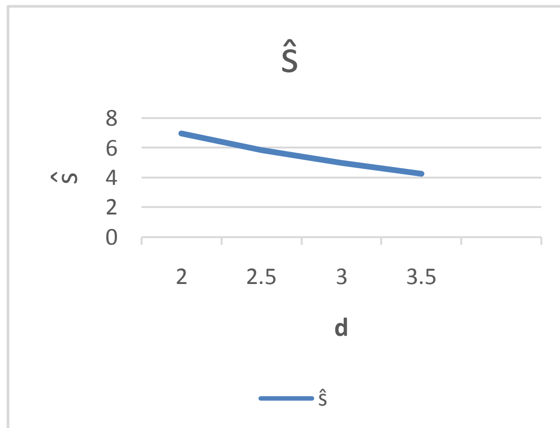
H	**2**	**3**	**4**	**5**	**6**
$\hat{S}$	12.34	9.75	8.14	6.98	6.06



Case (iii)

For r=30, h=5, μ=2,λ =5, α =4, θ=3, the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of d.

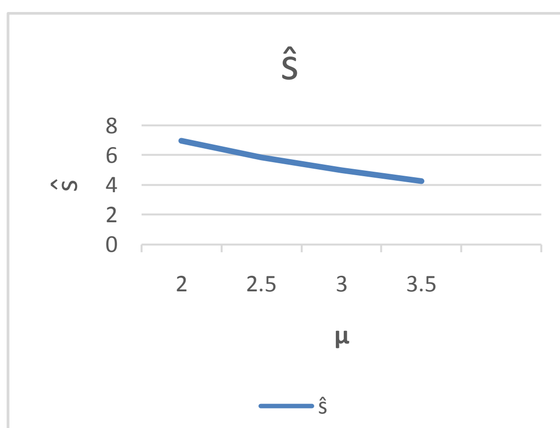
d	**2500**	**3000**	**3500**	**4000**
$\hat{S}$	6.06	6.98	7.78	8.49



Case (iv)

For r=30, h=5, d=3000, λ =5, α =4, θ=3, the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of μ.

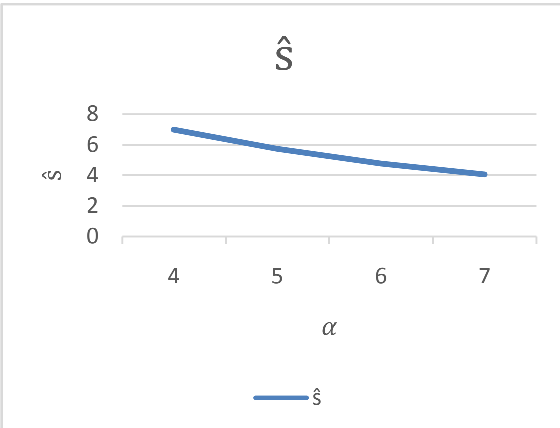
μ	**2**	**2.5**	**3**	**3.5**
$\hat{S}$	6.98	5.86	4.98	4.25



Case (v)

For r=30, h=5, d=3000, μ=2,λ =5, θ=3, the optimal value of S is obtained and the variations in $\hat{S}$ for the changes in the value of α.

α	4	5	6	7
$\hat{S}$	6.98	5.71	4.76	4.04



VI. CONCLUSIONS

From the tables and graphs, it is observed that,

As r, the consumption rate of M_2 increases, the optimal reserve inventory $\hat{S}$ increases.

As h, the parameter of the holding cost increases, the optimal reserve inventory $\hat{S}$ decreases. It is understood that, the model suggests small size inventory, when the inventory holding cost increases.

As d, the parameter of the idle time cost increases, the optimal reserve inventory $\hat{S}$ decreases.

As μ, the parameter of the Mean time interval between successive breakdown of machine M_1increases, the optimal reserve inventory $\hat{S}$ decreases.

Asα, the parameter of exponential distribution increases, the optimal reserve inventory $\hat{S}$ decreases.

REFERENCES

[1] F.Hanssmann, *Operations Research in Production and Inventory Control*.John Wiley and sons, inc.New York.1962

[2] V. Ramachandran and R. Sathiyamoorthi, "Optimal Reserves for Two Machines."*IEEE Transactions. On Reliability*. Vol.R.30.No.4. p. 397. 1981

[3] S. Sachithanantham, V. Ganesan and R. Sathiyamoorthi, "Optimal Reserves for Two Machines with Repair time having SCBZ Property."*Bulletin of Pure and Applied Sciences.* Vol.25E(No.2) PP.487-297.2006

[4] D.K. Stangl,"Exponential change point model", *statistics in Medicine,* Vol.14,pp.2173-2190. 1995

[5] R. Suresh Kumar,"Shock Model when the threshold has a change of Distribution after a change point."*Journal of Indian Acada Math,* Vol.28,(No.1):pp.73-84.2006

[6] S. Sachithanantham, V. Ganesan and R. Sathiyamoorthi,"A model for Optimal Reserve inventory Between Two Machines in Series."*Journal of Indian Acad. Math.* Vol.29.No. 1 pp 59-70.2007

[7] S. Ramathilagam, L. Henry and S. Sachithanantham, "A model for optimal Reserve inventory between two machines in series with repair time undergoes a parametric change."*Ultra scientist* vol.26 (3) P:227-237. 2014