Symmetric Skew Reverse n-Derivations on Prime Rings and Semiprime rings

[1]B. Satyanarayana, [2]Mohammad Mastan
Department of Mathematics
Acharya Nagarjuna University
Nagarjuna Nagar – 522 510
A.P., India.

Abstract: *Let $n \geq 2$ be any fixed positive integer and δ denote the trace of symmetric skew reverse n-derivation $\Delta: R^n \rightarrow R$, associated with an antiautomorphism α^* .Let I be any Ideal of R.(1)If R is non commutative prime ring such that $[\delta(x),\alpha^*(x)]=0$, for all $x \in I$ then $\Delta = 0$ in R.(2)Let R be non commutative semiprime ring such that δ is commuting on I and $[\delta(x),\alpha^*(x)] \in Z(R)$, for all $x \in I$ then $[\delta(x), \alpha^*(x)] = 0$ for all $x \in I$.*

Key words : *Prime ring , semiprime ring , commuting mapping , centralizing mapping , derivation, skew derivation, reverse derivation , skew reverse derivation , automorphism , antiautomorphism.*

1. INTRODUCTION

The concept of reverse derivations of prime rings was introduced by Bresar and Vukman [2]. Relations between derivations and reverse derivations with examples were given by Samman and Alyamani [17]. Recently there has been a great deal of work done by many authors on commutativity and centralizing mappings on prime rings and semi prime rings in connection with derivations, skew derivations, reverse derivations, skew reverse derivations [1,4-6,7-12,15-17,19-22]. Vukman [19-22], Mohammad Ashraf [1], Jung and Park [7] have studied the concepts of symmetric biderivations, 3-derivations, 4-derivations and n-derivations. Ajda Fosner [5], Faiza Shujat and Abuzaid Ansari [17], Jayasubba Reddy et.al [8] and Basudeb Dhara and Faiza Shujat [4], Yadav and Sharma [22] have extended and studied the concepts of symmetric skew 3-derivations, 4-derivations and n-derivations. Recently Jayasubba Reddy et.al, [9-12] have studied the concepts of symmetric skew reverse derivations. Inspired by these works we have made an attempt to introduce the notion of symmetric skew reverse n- derivations and prove some properties, which may be a contribution to the theory of commuting and centralizing mappings of prime rings and semi prime rings.

2. Preliminaries

Throughout this paper R will represent a ring with center Z (R). Let n be any integer. A ring R is said to be n- torsion free, if for all $x, y \in R$, $n x = 0 \Longrightarrow x = 0$. Recall that a ring R is said to be prime, if $a R b = 0$, $\forall$ $a, b \in R \Rightarrow$ either a=0 (or) b=0 and a ring R is said to be semiprime, if $a R a = 0$, $\forall$ $a \in R \Rightarrow$ a=0. Let us denote the commutator xy - yx by [x , y].In this paper we extensively make use of the commutator identities $[xy ,z]=[x , z] y + x [y , z]$ and $[x , yz] = [x , y] z + y [x ,z]$. Let I be any non-empty two sided ideal of R. Then the mapping $\delta : R \rightarrow R$ is said to be commuting on I, if $[\delta(x) , x] = 0$, $\forall x \in I$ and centralizing on I if, $[\delta(x) , x] \in Z(R)$, $\forall x \in I$. An additive mapping $D:R \rightarrow R$ is called a derivation if, $D(xy)=D(x) y + x D(y)$ holds $\forall$ x, y $\in$ R and is called a reverse derivation, if $D(xy)=D(y) x + y D(x)$ holds $\forall$ x , y $\in$ R. An additive mapping $D:R \rightarrow R$ is called a skew derivation (α-derivation) associated with an automorphism α if, $D(xy)=D(x) y + \alpha(x) D(y)$ holds $\forall$ x , y $\in$ R and is called symmetric skew reverse derivation (α^*- derivation) associated with an antiautomorphism α^* if, $D(xy)=D(y) x + \alpha^*(y) D(x)$, holds $\forall$ x, y $\in$ R

Definition. 2.1. Let $n \geq 2$ be any fixed positive integer and a mapping $\Delta : R^n \rightarrow R$ is said to be symmetric (permuting), if the equation

$\Delta(x_1, x_2, x_3, ..., x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ holds

$\forall x_i \in R$ and for every permutation $\pi \in S_n$, the Symmetric permutation group of order n.

Definition.2.2. Let $n \geq 2$ be any fixed positive integer and a mapping $\delta:R \rightarrow R$ is defined as $\delta(x)= \Delta(x,x,x,\ldots,x)$,where $\Delta:R^n \rightarrow R$, is called the trace of the symmetric mapping Δ.

Definition2.3. An n – additive mapping $\Delta: R^n \rightarrow R$ is said to be symmetric skew n – derivation, associated with an automorphism α, if for every i = 1, 2, 3..., n and for all $x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_n \in R$, the mapping $x \mapsto \Delta(x_1, x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ is a skew derivation of R associated with the automorphism α . In particular, for all $x, y, x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n \in R$,

$\Delta(x_1, x_2, ..., x_{i-1}, xy, x_{i+1}, ..., x_n) =$

$\Delta(x_1, x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_n)\, y$

$+\alpha(x)\, \Delta(x_1, x_2, ..., x_{i-1}, y, x_{i+1}, ..., x_n)$.

Definition.2.4. An n – additive mapping $\Delta: R^n \to R$ is said to be symmetric skew reverse n – derivation, associated with an antiautomorphism α^*, if for every i=1,2,3...,n and for all $x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n \in R$, the mapping $x \mapsto \Delta(x_1, x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ is a skew reverse derivation of R associated with the antiautomorphism α^*. In particular, for all $x, y, x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n \in R$,

$\Delta(x_1, x_2, ..., x_{i-1}, xy, x_{i+1}, ..., x_n) =$

$\Delta(x_1, x_2, ..., x_{i-1}, y, x_{i+1}, ..., x_n)\, x\, +$

$\alpha^*(y)\, \Delta(x_1, x_2, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$

Further, $\Delta(0, x_2, x_2, ..., x_n) = \Delta(0+0, x_2, ..., x_n)$

$= \Delta(0, x_2, x_3, ..., x_n) + \Delta(0, x_2, x_3, ..., x_n)$. Hence

$\Delta(0, x_2, x_3, ..., x_n) = 0$

$\Rightarrow \Delta(x_1 - x_1, x_2, x_3, ..., x_n) = 0$

$\Rightarrow \Delta(x_1, x_2, ..., x_n) + \Delta(-x_1, x_2, ..., x_n) = 0$

$\Rightarrow \Delta(-x_1, x_2, ..., x_n) = -\Delta(x_1, x_2, ..., x_n)$.

Clearly, Δ is an odd function if n is odd and is an even function, if n is even. If $\Delta: R^n \to R$ is a symmetric map which is n – additive then the trace δ of Δ satisfy the relation:

$\delta(x+y) = C_0\Delta(x, x, ,..., x) + C_1\Delta(x, x, ,..., x, y)$

$+C_2\, \Delta(x, x, ,..., x, y) + ... +$

$C_r\Delta$(x,x,...,n-r times ,y,y,...,r-times) $+ \ldots +$

$C_n\, \Delta(y, y, y, ..., y)$

i.e $\delta(x+y) = C_0\delta(x) + C_1\Delta(x, x, x, ..., x, y)$

$+ C_2\, \Delta(x, x, ,..., x, y) + ... +$

$C_r\Delta$(x,x,...,n-r times ,y,y,...,r-times) $+ \ldots +$

$C_n\delta(y),\ \forall\, x, y \in R$

For symmetric biderivations δ satisfies the relation $\delta(x+y) = \delta(x) + 2\Delta(x, y) + \delta(y), \forall\, x, y \in R$

For symmetric 3-derivations δ satisfies the relation $\delta(x+y) = \delta(x) + 3\Delta(x, x, y) + 3\Delta(x, y, y) + \delta(y)$, $\forall\, x, y \in R$

For symmetric 4-derivations δ satisfies the relation: $\delta(x+y) = \delta(x) + 4\Delta(x, x, x, y) + 6\Delta(x, x, y, y)$ $+4\Delta(x, y, y, y) + \delta(y), \forall\, x, y \in R$.

For convenience let us define $\delta(x+y)$ as follows:

$\delta(x+y) = \sum_{r=0}^{n} c_r\, f_r(x,y)$.

$= \delta(x) + \sum_{r=1}^{n-1} c_r\, f_r(x,y) + \delta(y)$, $\forall\, x, y \in R$, where

$f_r(x,y) = \Delta$(x,x,x,...,n-r times , y,y,y,...,r times),

$C_r = \dfrac{n!}{r!(n-r)!}$.

Example2.5. Let F be a field and α^* be an antiautomorphism of F

Suppose that $R = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in F \right\}$. Clearly, the set R of all 2 x 2 matrices w.r.t matrix addition and matrix multiplication is a commutative ring. Let us define a map

$\alpha^*: R \to R$ such that $\alpha^*\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \alpha^*(x) & 0 \\ 0 & 0 \end{bmatrix}, x, y \in F$.

Obviously α^* is an antiautomorphism.

Next let us denote $A_k = \begin{bmatrix} x_k & y_k \\ 0 & 0 \end{bmatrix} \in R$, k=1,2,3,....,n .

Now let us define a mapping $\Delta: R^n \to R$ such that

$\Delta(A_1, A_2, ..., A_n) = \begin{bmatrix} 0 & \alpha^*(x_1)\alpha^*(x_2)...\alpha^*(x_n) \\ 0 & 0 \end{bmatrix}$

$\Delta(A_1 A_1^1, A_2, ..., A_n) = \begin{bmatrix} 0 & \alpha^*(x_1 x_1^1)\alpha^*(x_2)...\alpha^*(x_n) \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & \alpha^*(x_1^1)\alpha^*(x_1)\alpha^*(x_2)...\alpha^*(x_n) \\ 0 & 0 \end{bmatrix}$

Now, $\Delta(A_1^1, A_2, ..., A_n)A_1 =$

$\begin{bmatrix} 0 & \alpha^*(x_1^1)\alpha^*(x_2)......\alpha^*(x_n) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and

$\alpha^*(A_1^1)\Delta(A_1, A_2, ..., A_n)$

$= \begin{bmatrix} 0 & \alpha^*(x_1^1)\alpha^*(x_1)\alpha^*(x_2)...\alpha^*(x_n) \\ 0 & 0 \end{bmatrix}$.

$\therefore \Delta(A_1 A_1^1, A_2, ..., A_n) = \Delta(A_1^1, A_2, ..., A_n)A_1$

$+\alpha^*(A_1^1)\Delta(A_1, A_2, ..., A_n)$. Hence from the above it is clear that the n – additive map Δ is a symmetric skew reverse n – derivation associated with an antiautomorphism α^*.

Now let us quote some important lemmas, which are useful in proving the main results.

Lemma. 2.6. [3] Let n be a fixed positive integer and R be an n! – torsion free ring .Suppose that $y_1, y_2, y_3, ..., y_n \in R$, which satisfy the relation:

$\lambda^1 y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \ldots + \lambda^n y_n = 0$ for $\lambda = 1,2,3,\ldots,n$ .Then $y_i = 0, \forall i$.

Lemma. 2.7. [14] Let R be a prime ring. Let D: R→R be a derivation and a ∈ R. If a D(x) = 0 or D(x) a = 0, ∀ x ∈ R, then we have either a = 0 or D = 0.

Lemma 2.8. Let R be a prime ring. If a [x , b] = 0 or [x , b] a = 0, ∀ x ∈ R then either a = 0 or b ∈ Z(R), the center of the ring R.

Proof. Replacing x by xy in the relation a [x , b] = 0, ⟹ a [xy , b] = 0 ⟹ ax [y , b] + a [x , b] y = 0 ⟹ ax [y , b] = 0 , ∀ x,y ∈ R. Thus, a R [y, b] = 0, y ∈ R. Since R is prime then either a = 0 or b ∈ Z(R).

3. The main results:

Theorem.3.1. Let n ≥2 be any fixed positive integer. Let R be a non commutative n! – torsion free prime ring and I be any non – zero two sided ideal of R .Suppose that there exist a symmetric skew reverse n-derivation $\Delta: R^n \to R$ associated with an antiautomorphism α*. Let δ denote the trace of Δ such that [δ(x), $\alpha^*(x)$] = 0, for all x ∈ I then $\Delta(x_1, x_2, x_3, \ldots, x_n) = 0$, $x_i \in R$

Proof. Our supposition is that there exist a symmetric skew reverse n – derivation $\Delta: R^n \to R$ associated with an antiautomorphism α^* such that

$[\delta(x), \alpha^*(x)] = 0, \ \forall x \in I.$ (1)

Substituting x by x+μy (1 ≤ μ ≤ n), in (1), we have

$$[\delta(x+\mu y), \alpha^*(x+\mu y)] = 0 \ \forall x, y \in I$$

$$\Rightarrow [\sum_{r=0}^{n} c_r f_r(x,\mu y), \ \alpha^*(x) + \alpha^*(\mu y)] = 0$$

$$\Rightarrow [\delta(x) + \delta(\mu y) + \sum_{r=1}^{n-1} c_r f_r(x,\mu y) \ , \ \alpha^*(x) + \mu\alpha^*(y)] = 0$$

$$\Rightarrow [\delta(x), \alpha^*(x)] + [\delta(x), \mu\alpha^*(y)] + \mu^n[\delta(y), \alpha^*(x)] + \mu^{n+1}[\delta(y), \alpha^*(y)] + {} + [\sum_{r=1}^{n-1} c_r f_r(x,\mu y), \ \alpha^*(x)] + [\sum_{r=1}^{n-1} c_r f_r(x,\mu y), \ \mu\alpha^*(y)] = 0$$

$$\Rightarrow \mu\left\{[\delta(x), \alpha^*(y)] + [c_1 f_1(x,y), \ \alpha^*(x)]\right\} + \mu^2\left\{[c_2 f_2(x,y), \alpha^*(x)] + [c_1 f_1(x,y), \alpha^*(y)]\right\} + \mu^3\left\{[c_3 f_3(x,y), \alpha^*(x)] + [c_2 f_2(x,y), \alpha^*(y)]\right\} + \ldots + \mu^n[c_{n-1} f_{n-1}(x,y), \ \alpha^*(y)] + \mu^n[\delta(y), \ \alpha^*(x)] = 0. \quad (2)$$

Applying Lemma 2.6 to equation (2),

$$[c_1 f_1(x,y), \ \alpha^*(x)] + [\delta(x), \alpha^*(y)] = 0. \quad (3)$$

Now replacing y by yx in equation (3) ,

$$[c_1\{f_1(x,x)y + \alpha^*(x) f_1(x,y)\}, \alpha^*(x)] + [\delta(x), \alpha^*(x)\alpha^*(y)] = 0$$

$$\Rightarrow c_1\delta(x)[y, \alpha^*(x)] + \alpha^*(x)[c_1 f_1(x,y), \alpha^*(x)] + \alpha^*(x)[\delta(x), \alpha^*(y)] = 0$$

$$\Rightarrow \alpha^*(x)\left\{[c_1 f_1(x,y), \alpha^*(x)] + [\delta(x), \alpha^*(y)]\right\} + c_1\delta(x)[y, \alpha^*(x)] = 0$$

$$\Rightarrow c_1\delta(x)[y, \alpha^*(x)] = 0$$

Using n!-torsion freeness,

$$\delta(x)[y, \alpha^*(x)] = 0, \forall x, y \in I \quad (4)$$

Now replacing y by yr in equation (4) ,

$$\delta(x)[yr, \alpha^*(x)] = 0 \ , \ \forall x, y \in I \ \ r \in R$$

$$\Rightarrow \delta(x)[y, \alpha^*(x)]r + \delta(x)y[r, \alpha^*(x)] = 0$$

$$\Rightarrow \delta(x)y[r, \alpha^*(x)] = 0, \forall x, y \in I \text{ and } r \in R \quad (5)$$

Since R is prime then by using Lemma 2.8 to equation either $\delta(x) = 0, \forall x \in I \setminus Z$ (6)

(or) $\alpha^*(x) \in Z(R), \forall x \in I \setminus Z$.

Let $x \in I \cap Z$, $y \in I$ and $y \notin Z$ then $y + \mu x \in I \setminus Z$

From relation (6), we have,

$$\delta(y + \mu x) = 0$$

$$\Rightarrow \sum_{s=0}^{n} c_s f_s(y, \mu x) = 0$$

$$\Rightarrow c_0\delta(y) + c_n\delta(\mu x) + \sum_{s=1}^{n-1} \mu^s c_s f_s(y, x) = 0$$

$$\Rightarrow c_n\mu^n\delta(x) + \sum_{s=1}^{n-1} \mu^s c_s f_s(y, x) = 0$$

$$\Rightarrow \mu^1 c_1 f_1(y,x) + \mu^2 c_2 f_2(y,x) + \mu^3 c_3 f_3(y,x) + \ldots\ldots + \mu^{n-1} c_{n-1} f_{n-1}(y,x) + c_n\mu^n\delta(x) = 0 \quad (7)$$

Again applying Lemma 2.6 to equation (7),

$$c_1 f_1(y,x) = 0 \Rightarrow f_1(y,x) = 0$$

$$\Rightarrow \Delta(y, y, y, \ldots, y, x) = 0$$

$$c_2 f_2(y,x) = 0 \Rightarrow f_2(y,x) = 0$$

$$\Rightarrow \Delta(y, y, y, \ldots, y, x, x) = 0 \ldots,$$

$$c_{n-1} f_{n-1}(y,x) = 0 \Rightarrow f_{n-1}(y,x) = 0$$

$$c_n\delta(x) = 0 \Rightarrow \Delta(x, x, x, \ldots, x, x) = 0.$$

Hence from the above we have,

$$\delta(x) = 0 \quad , \ \forall \ x \in \mathrm{I} \cap \mathrm{Z} \tag{8}$$

Now for each value of l=1, 2, 3,..., n let us denote

$\mathrm{T}_l(x) = \Delta(x, x, x, ..., x, x_{l+1}, x_{l+2}, x_{l+3}, ..., x_n)$,

where $x, x_i \in \mathrm{I}$, $i = l+1, l+2, ..., n$

$$\Rightarrow \mathrm{T}_n(x) = \delta(x) = 0 \ , \ \forall \ x \in \ \mathrm{I} \tag{9}$$

Let η (1≤η≤n) be any positive integer .Then from (9),

$\mathrm{T}_n(\eta x + x_n) = 0$

$\Rightarrow \mathrm{T}_n(x_n) + \mathrm{T}_n(\eta x) + \sum_{l=1}^{n-1} \eta^l c_l \mathrm{T}_l(x) = 0$

$\Rightarrow \delta(x_n) + \eta^n \delta(x) + \sum_{l=1}^{n-1} \eta^l c_l \mathrm{T}_l(x) = 0$

$\Rightarrow \sum_{l=1}^{n-1} \eta^l c_l \mathrm{T}_l(x) = 0 \ , \ \forall \ \ x, x_n \in \mathrm{I}$

$$\Rightarrow \eta^1 c_1 \mathrm{T}_1(x) + \eta^2 c_2 \mathrm{T}_2(x) + \eta^3 c_3 \mathrm{T}_3(x) + ... + \eta^{n-1} c_{n-1} \mathrm{T}_{n-1}(x) = 0. \tag{10}$$

Again applying Lemma 2.8 to equation (10),

$c_1 \mathrm{T}_1(x) = 0 \Rightarrow \mathrm{T}_1(x) = 0 \Rightarrow \Delta(x, x_2, x_3, ..., x_n) = 0$

$c_2 \mathrm{T}_2(x) = 0 \Rightarrow \mathrm{T}_2(x) = 0 \Rightarrow \Delta(x, x, x_3, ..., x_n) = 0...,$

$c_{n-1} \mathrm{T}_{n-1}(x) = 0 \Rightarrow \mathrm{T}_{n-1}(x) = 0 \Rightarrow \Delta(x, x, x, ..., x, x_n) = 0$

Hence from the above, we have $\mathrm{T}_{n-1}(x) = 0$, ∀ x ∈ I (11)

Again let τ (1≤τ≤n-1) be any positive integer. Then from (11)

$\mathrm{T}_{n-1}(\tau x + x_{n-1}) = 0, \forall x, x_{n-1} \in \mathrm{I}$

$\Rightarrow \mathrm{T}_{n-1}(\tau x) + \mathrm{T}_{n-1}(x_{n-1}) + \sum_{t=1}^{n-2} \eta^t c_t \mathrm{T}_t(x) = 0$

$$\Rightarrow \tau^1 c_1 \mathrm{T}_1(x) + \tau^2 c_2 \mathrm{T}_2(x) + \tau^3 c_3 \mathrm{T}_3(x) + ... + \tau^{n-2} c_{n-2} \mathrm{T}_{n-2}(x) = 0 \ , \forall \ x \in \mathrm{I}. \tag{12}$$

Again applying Lemma (1) to result (12) we have,

$\Rightarrow \Delta(x, x, x, ...x, x_{n-1}, x_n) = 0$

$$\Rightarrow \quad \mathrm{T}_{n-2}(x) = 0, \forall \ x \in \mathrm{I} \tag{13}$$

Continuing the above process, finally we obtain $\mathrm{T}_1(x) = 0$, ∀ x ∈ I.

$$\Rightarrow \Delta(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0, \forall \ x_i \in \mathrm{I}\,. \tag{14}$$

Now replacing x_1 by $x_1 p_1$, where $p_1 \in$ R in (14),

$\Delta(x_1 p_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$

$\Rightarrow \Delta(p_1, x_2, x_3, ..., x_{n-1}, x_n) x_1 + \alpha(p_1) \Delta(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$

$$\Rightarrow \Delta(p_1, x_2, x_3, ..., x_{n-1}, x_n) x_1 = 0. \tag{15}$$

Now applying Lemma 2.8 to equation (15),

$\Delta(p_1, x_2, ..., x_{n-1}, x_n) = 0, \ p_1 \in \mathrm{R}, \forall x_i \in \mathrm{I}$

Now replacing x_2 by $x_2 p_2$, where $p_2 \in$ R in (15),

$\Delta(p_1, p_2, x_3, ..., x_{n-1}, x_n) = 0$, where $p_1, p_2 \in \mathrm{R}, \ \forall x_i \in \mathrm{I}$

Repeating the above process we finally obtain,

$\Delta(p_1, p_2, p_3, ..., p_{n-1}, p_n) = 0 \ , \ p_i \in \mathrm{R}$

Hence the proof of the theorem is completed.□

Theorem.3.2. Let n ≥2 be any fixed positive integer. Let R be non commutative n! – torsion free semiprime ring and I be any non – zero two sided ideal of R. Suppose that there exist a symmetric skew reverse n-derivation Δ: $\mathrm{R}^n \to \mathrm{R}$ associated with an antiautomorphism α^*. Let δ denote the trace of Δ such that δ is commuting on I and $[\delta(x), \alpha^*(x)] \in Z(R)$, for all x ∈ I then $[\delta(x), \alpha^*(x)] = 0$, ∀ x ∈ I.

Proof. Our supposition is that there exist a symmetric skew reverse n – derivation Δ: $\mathrm{R}^n \to \mathrm{R}$ associated with an antiautomorphism α^* such that [δ(x), x] = 0, for all x ∈ I and $[\delta(x), \alpha^*(x)] \in Z(R)$, ∀ x∈I.

Let μ (1≤μ≤n) be any positive integer. Substituting x by x+μy in (1) we have,

$[\delta(x+\mu y), \alpha^*(x+\mu y)] \in \mathrm{Z}, \ \forall \ x, y \in \mathrm{I}$

$\Rightarrow [\sum_{r=0}^{n} c_r f_r(x, \mu y), \ \alpha^*(x) + \alpha^*(\mu y)] \in \mathrm{Z}$

$\Rightarrow \mu \left\{[\delta(x), \alpha^*(y)] + [c_1 f_1(x,y), \ \alpha^*(x)]\right\}$

$+\mu^2 \left\{[c_2 f_2(x,y), \alpha^*(x)] + [c_1 f_1(x,y), \alpha^*(y)]\right\} + ...$

$+\mu^n \left\{[c_{n-1} f_{n-1}(x,y), \alpha^*(y)] + [\delta(y), \alpha^*(x)]\right\}$

$$+\mu^{n+1}[\delta(y), \alpha^*(y)] + [\delta(x), \alpha^*(x)] \in \mathrm{Z}, \forall \ x, y \in \mathrm{I} \tag{17}$$

Now commuting (17) with δ(x),

$\mu\left\{\left[[\delta(x), \alpha^*(y)] + [c_1 f_1(x,y), \alpha^*(x)], \delta(x)\right]\right\}$

$+\mu^2 \left[[c_2 f_2(x,y), \alpha^*(x)] + [c_1 f_1(x,y), \alpha^*(y)], \delta(x)\right] + ... +$

$+\mu^n \left[[c_{n-1} f_{n-1}(x,y), \alpha^*(y)] + [\delta(y), \alpha^*(x)], \delta(x)\right]$

$+\mu^{n+1}\left[[\delta(y), \alpha^*(y)], \delta(x)\right]$

$$+\left[[\delta(x), \alpha^*(x)], \delta(x)\right] = 0 \ , \forall \ x, y \in \mathrm{I} \tag{18}$$

Applying Lemma. 2.6 to equation (18),we have

$$\left[\left\{[c_1 f_1(x,y), \alpha^*(x)] + [\delta(x), \alpha^*(y)]\right\}, \delta(x)\right] = 0 \tag{19}$$

Now replacing y by x^2 in equation (19), we get

$\left[[c_1 f_1(x, x^2), \alpha^*(x)], \delta(x)\right] + \left[[\delta(x), \alpha^*(x^2)], \delta(x)\right] = 0$

$$\Rightarrow\left[\left\{(c_1+1)\alpha^*(x)[\delta(x),\alpha^*(x)]\right\},\delta(x)\right]$$

$$+\left[[\delta(x),\alpha^*(x)]\alpha^*(x),\delta(x)\right]$$

$$+\left[c_1[\delta(x),\alpha^*(x)]x,\delta(x)\right]$$

$$+\left[c_1\delta(x)[x,\alpha^*(x)],\delta(x)\right]=0$$

$$\Rightarrow (c_1+1)[\alpha^*(x),\delta(x)][\delta(x),\alpha^*(x)]$$

$$+(c_1+1)\alpha^*(x)\left[[\delta(x),\alpha^*(x)],\delta(x)\right]$$

$$+[\delta(x),\alpha^*(x)][\alpha^*(x),\delta(x)]+\left[[\delta(x),\alpha^*(x)],\delta(x)\right]\alpha^*(x)$$

$$+c_1[\delta(x),\alpha^*(x)][x,\delta(x)]+\left[c_1[\delta(x),\alpha^*(x)],\delta(x)\right]x$$

$$+c_1\delta(x)\left[[x,\alpha^*(x)],\delta(x)\right]+c_1[\delta(x),\delta(x)][x,\alpha^*(x)]=0$$

$$\Rightarrow (c_1+1)[\delta(x),\alpha^*(x)]^2+[\delta(x),\alpha^*(x)]^2$$

$$+c_1\delta(x)\left[\delta(x),\{x\alpha^*(x)-\alpha^*(x)x\}\right]=0$$

$$\Rightarrow (c_1+2)[\delta(x),\alpha^*(x)]^2+c_1\delta(x)\left[x,[\delta(x),\alpha^*(x)]\right]=0$$

$$\Rightarrow (c_1+2)[\delta(x),\alpha^*(x)]^2=0,\ \forall\, x\in I \tag{20}$$

Again commuting equation (17) with $\alpha^*(x)$ and using Lemma.2.6,

$$\left[[c_1 f_1(x,y),\alpha^*(x)]+[\delta(x),\alpha^*(y)],\alpha^*(x)\right]=0 \tag{21}$$

Now replacing y by yx in the above equation (21) ,

$$\left[[c_1 f_1(x,yx),\alpha^*(x)],\alpha^*(x)\right]+\left[[\delta(x),\alpha^*(yx)],\alpha^*(x)\right]=0$$

$$\Rightarrow\left[\left\{[c_1 f_1(x,x)y+\alpha^*(x)f_1(x,y),\alpha^*(x)]\right\},\alpha^*(x)\right]$$

$$+\left[[\delta(x),\alpha^*(x)\alpha^*(y)],\alpha^*(x)\right]=0$$

$$\Rightarrow\left[[c_1\delta(x)y,\alpha^*(x)],\alpha^*(x)\right]$$

$$+\left[[c_1\alpha^*(x)f_1(x,y),\alpha^*(x)],\alpha^*(x)\right]$$

$$+\left[[\delta(x),\alpha^*(x)\alpha^*(y)],\alpha^*(x)\right]=0$$

$$\Rightarrow\left[c_1[\delta(x),\alpha^*(x)]y,\alpha^*(x)\right]+\left[c_1\delta(x)[y,\alpha^*(x)],\alpha^*(x)\right]$$

$$+\left[c_1[\alpha^*(x),\alpha^*(x)]f_1(x,y),\alpha^*(x)\right]$$

$$+\left[c_1\alpha^*(x)[f_1(x,y),\alpha^*(x)],\alpha^*(x)\right]$$

$$+\left[[\delta(x),\alpha^*(x)]\alpha^*(y),\alpha^*(x)\right]$$

$$+\left[\alpha^*(x)[\delta(x),\alpha^*(y)],\alpha^*(x)\right]=0$$

$$\Rightarrow\left[c_1[\delta(x),\alpha^*(x)]y,\alpha^*(x)\right]$$

$$+\left[c_1\delta(x)[y,\alpha^*(x)],\alpha^*(x)\right]$$

$$+\left[c_1\alpha^*(x)[f_1(x,y),\alpha^*(x)],\alpha^*(x)\right]$$

$$+\left[[\delta(x),\alpha^*(x)]\alpha^*(y),\alpha^*(x)\right]$$

$$+\left[\alpha^*(x)[\delta(x),\alpha^*(y)],\alpha^*(x)\right]=0$$

$$\Rightarrow c_1[\delta(x),\alpha^*(x)][y,\alpha^*(x)]$$

$$+c_1\left[[\delta(x),\alpha^*(x)],\alpha^*(x)\right]y$$

$$+c_1\delta(x)\left[[y,\alpha^*(x)],\alpha^*(x)\right]+c_1[\delta(x),\alpha^*(x)][y,\alpha^*(x)]$$

$$+c_1\alpha^*(x)\left[[f_1(x,y),\alpha^*(x)],\alpha^*(x)\right]$$

$$+c_1[\alpha^*(x),\alpha^*(x)][f_1(x,y),\alpha^*(x)]$$

$$+[\delta(x),\alpha^*(x)][\alpha^*(y),\alpha^*(x)]$$

$$+\left[[\delta(x),\alpha^*(x)],\alpha^*(x)\right]\alpha^*(y)$$

$$+\alpha^*(x)\left[[\delta(x),\alpha^*(y)],\alpha^*(x)\right]$$

$$+[\alpha^*(x),\alpha^*(x)][\delta(x),\alpha^*(y)]=0$$

$$\Rightarrow 2c_1[\delta(x),\alpha^*(x)][y,\alpha^*(x)]$$

$$+c_1\delta(x)\left[[y,\alpha^*(x)],\alpha^*(x)\right]$$

$$+[\delta(x),\alpha^*(x)][\alpha^*(y),\alpha^*(x)]=0. \tag{22}$$

Now replacing y by $\delta(x)[\delta(x),\alpha^*(x)]$ in relation (22),

$$2c_1[\delta(x),\alpha^*(x)]\left[\delta(x)[\delta(x),\alpha^*(x)],\alpha^*(x)\right]$$

$$+c_1\delta(x)\left[\left\{[\delta(x)[\delta(x),\alpha^*(x)],\alpha^*(x)]\right\},\alpha^*(x)\right]$$

$$+[\delta(x),\alpha^*(x)]\left[[\delta(x),\alpha^*(x)]\delta(x),\alpha^*(x)\right]=0$$

$$\Rightarrow c_1\delta(x)\left[[\delta(x),\alpha^*(x)]^2,\alpha^*(x)\right]$$

$$+2c_1[\delta(x),\alpha^*(x)]^3+[\delta(x),\alpha^*(x)]^3=0$$

$$\Rightarrow (2c_1+1)[\delta(x),\alpha^*(x)]^3=0,\ \forall\, x\in I \tag{23}$$

$$\Rightarrow (2c_1+1)[\delta(x),\alpha^*(x)]^2 R(2c_1+1)[\delta(x),\alpha^*(x)]^2=0$$

$\forall\, x\in I$. Since R is semiprime then,

$$(2c_1+1)\,[\delta(x),\alpha^*(x)]^2=0,\ \forall\, x\in I \tag{24}$$

Now combining (20) and (24), $[\delta(x), \alpha^*(x)]^2 = 0$, $\forall\, x \in \mathrm{I}$. As the center of the semiprime ring contains no non-zero nilpotent elements we have $[\delta(x), \alpha^*(x)] = 0$, for all $x \in \mathrm{I}$. Hence the proof of the theorem is completed.□

Theorem.3.3. Let $n \geq 2$ be any fixed positive integer. Let R be an n! – torsion free prime ring and I be any non zero two sided ideal of R. Suppose that there exist a non zero symmetric skew reverse n-derivation $\Delta: R^n \to R$, associated with an antiautomorphism α^*. Let δ denote the trace of Δ such that δ is commuting on I and $[\delta(x), \alpha^*(x)] \in Z(R)$, for all $x \in \mathrm{I}$ then R must be commutative.

Proof. On contrary suppose that R is a non commutative prime ring. Then from Theorem.3.2, we have $[\delta(x), \alpha^*(x)] = 0$, for all $x \in \mathrm{I}$. Now from Theorem.3.1, we have $\Delta = 0$, $\forall\, x \in R$, which is a contradiction. Hence R must be commutative prime ring.□

Conflict of Interests:

The authors declare that there is no conflict of interests.

References:

[1] Mohammad Ashraf, "On symmetric biderivations in rings", Rend.Istit.Mat.Univ, Trieste, 31 (1999), 25-36.

[2] M. Bresar and J. Vukman, "On some additive mappings in rings with involution", Aequationes. Math. 38 (1989), 178-185.

[3] L.O. Chung and Jiang Luh, "Semiprime rings with nilpotent derivatives", Cannad.Math.Bull., 24 (1981), 415-421.

[4] Basudeb Dhara and Faiza Shujat, "Symmetric skew n-derivations in prime and semi prime rings", Southeast Asian Bull. of Math., 1-9 , (2017).

[5] Ajda Fosner, "Prime and semiprime rings with symmetric skew 3-derivations", Aequationes .Math. 87 (2014), 191-200.

[6] I.N. Herstein.., "Jordan derivations of prime rings", Proc.Amer.Math.Soc. 8 (1957), 1104-1110.

[7] Y.S. Jung. and K.H.Park, "On prime and semiprime rings with permuting 3-derivations", Bull.Korean.Math.Soc., 44 (2007), 789- 794.

[8] C. Jayasubba Reddy, S.Vasanth Kumar and S.Mallikarjuna Rao, "Symmetric skew 4- derivations on prime rings", Global Jr. of pure and appl. Math., 12 (2016), 1013-1018.

[9] C. Jayasubba Reddy, V.Vijaya Kumar and K.Hemavati, "Prime ring with symmetric skew 3-reverse derivation", IJMCAR, 4 (2014), 69-73.

[10] C. Jayasubba Reddy, V.Vijaya Kumar and S.Mallikarjuna Rao, "Semiprime ring with Symmetric skew 3-reverse derivations", Trans. on Math., 1 (2015), 11-15.

[11] C. Jayasubba Reddy, V.Vijaya Kumar and S.Mallikarjuna Rao, "Prime ring with symmetric skew 4-reverse derivation", Int. J. of Pure Algebra, 6 (2016), 251-254.

[12] C. Jayasubba Reddy, S.Vasanth Kumar and K. Subbarayudu, "Symmetric skew 4-reverse derivations on prime rings", IOSR Jr. of Math., 12 (2016), 60-63.

[13] Joseph H Mayne, "Centralizing mappings of prime rings", Cannad.Math.Bull, 27 (1983), 122-126.

[14] E.C. Posner, "Derivations in prime rings", Amer.Math.Soc, 8 (1957), 1093-1100.

[15] K.H. Park,, "On 4-permuting 4-derivations in prime and semiprimerings",. Korea.Soc. .Educ.Ser. B.Pure.Appl.Math, 14 (2007), 271-278.

[16] K.H. Park, "On prime and semiprime rings with symmetric n-derivations", J.Chungcheong Math.Soc., 22 (2009), 451-458.

[17] Faiza Shujat, Abuzaid Ansari, "Symmetric skew 4-derivations on prime rings", J. Math. Compt.Sci., 4 (2014), 649-656.

[18] M Samman and N Alyamani, "Derivations and Reverse derivations", Int.Math.Forum, 2 (2007), 1895-1902.

[19] J. Vukman, "Commuting and centralizing mappings in prime rings", Proc.Amer.Math.Soc, 109 (1990), 47-52.

[20] J. Vukman, "Symmetric biderivations on prime and semiprime rings", Aequationes Math.Soc. 38 (1989), 245-254.

[21] J. Vukman, "Two results concerning symmetric biderivations on prime rings", Aequationes. Math, 49 (1990), 181-189.

[22] V.K. Yadav. and R.K. Sharma., "Skew n-derivations on prime and semi prime rings", Ann.Univ.Ferrara, (2016), 1-12.