Common Fixed Point Theorem in G-Metric Spaces

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Abstract *- In this paper, we prove some common fixed point theorems for pair of compatible mappings in G-metric spaces.*

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1. Introduction.

Mustafa and Sims [5] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa and Sims[4, 5, 6] and Mustafa et al. [7, 8] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [1] initiated the study of a common fixed point theory in generalized metric spaces. Saadati et al. [9] proved some fixed point results for contractive mappings in partially ordered G-metric spaces.

2. Basic definitions and preliminaries.

Definition 2.1[5]. Let X be a non empty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following conditions:

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$(symmetry in all three variables),

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2[5]. Let (X, G) be a G-metric space, $\{x_n\}$ a sequence of points in G- convergent to x if $\lim_{n\to\infty} G(x, x_n, x_m) = 0$; that is for each $\varepsilon > 0$ there exists an N such that

$G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

We say that x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.1[5]. Let (X, G) be a G-metric space. Then the following are equivalent:

(i) $\{x_n\}$ is G- convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$,

Definition 1.3[5]. Let (X, G) be a G-metric space, then a sequence $\{x_n\}$ is called G-Cauchy if for each $\varepsilon > 0$, there exists an N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$.

Proposition 1.2[5]. In a G-metric space(X, G), the following are equivalent:

(i) The sequence $\{x_n\}$ is G-Cauchy,

(ii) for each $\varepsilon > 0$ there exists an N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$,

(iii) $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_G).

Proposition 1.3[5]. Let (X, G) be a G-metric space.Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 1.4[5]. A G-metric space (X,G) is called a symmetric G-metric space if

$G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Proposition 1.4[5]. Every G-metric space (X,G) defines a metric space (X,d_G)

(i) $d_G(x,y) = G(x,y,y) + G(y,x,x)$ for all $x,y \in X$.

If (X,G) is a symmetric G-metric space, then

(ii) $d_G(x,y) = 2G(x,y,y)$ for all $x,y \in X$.

However, if (X,G) is not symmetric, then it follows from the G-metric properties that

(iii) $\frac{3}{2}\, G(x,y,y) \le d_G(x,y) \le 3G(x,y,y)$ for all $x,y \in X$.

Definition 1.5[5]. A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in(X,G).

Proposition 1.5[5]. A G-metric space (X,G) is G-complete if and only if (X,d_G) is a complete metric space.

Proposition 1.6[5]. Let (X,G) be a G-metric space, then for any $x,y,z,a \in X$ it follows that:

(i) If $G(x,y,z) = 0$, then $x = y = z$,
(ii) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$,
(iii) $G(x,y,y) \le 2G(y,x,x)$,
(iv) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$
(v) $G(x,y,z) \le \frac{2}{3}[G(x,y,a) + G(x,a,z)$
(vi) $+ G(a,y,z)]$,
(vii) $G(x,y,z) \le G(x,a,a) + G(y,a,a)$
(viii) $+ G(z,a,a)$,
(ix) $|G(x,y,z) - G(x,y,a)|$
$\le \max\{G(a,z,z), G(z,a,a)\}$,
(x) $|G(x,y,z) - G(x,y,a)| \le G(x,a,z)$,
(xi) $|G(x,y,z) - G(y,z,z)|$
(xii) $\le \max\{G(x,z,z), G(z,x,x)\}$,
(xiii) $|G(x,y,y) - G(y,x,x)|$
(xiv) $\le \max\{G(y,x,x), G(x,y,y)\}$

Jungck[2] introduced the concept of compatible mappings. Afterwards many researchers used this concept in fixed point theory. Manro et al.[3] introduced the concept of compatible maps in G-metric space.

Definition 1.5[3]. Let f and g are mappings from a G-metric space (X,G) into itself. The maps f and g are said to be compatible map if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ or $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$

whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

3. Main Theorem.

Theorem 3.1. Let (X,G) be a complete G-metric spaces and let A and S be two self maps of a G-metric space (X,G) satisfying

(3.1.1) $A \subset S$

(3.1.2) $G(Ax,Ay,Az) \le a_1\, G(Sx,Sy,Sz)$

$$+a_2\, G(Sx,Ax,Sx) + a_3\, G(Sx,Ay,Sz) + a_4 G(Sy,Ax,Sz),$$

where $a_i \ge 0$(for i=1, 2, 3, 4) and

$a_1 + a_2 + a_3 + a_4 < 1$,

(3.1.3) one of A or S is continuous and pair (A,S) is compatible maps.

Then A and S have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X and we can choose a sequence $\{y_n\}$ in X such that

$$y_n = Ax_n = Sx_{n+1}, \quad n = 0,1,2\ldots$$

From (3.1.2), we have

$$G(Ax_n, Ax_{n+1}, Ax_{n+1}) \le a_1\, G(Sx_n, Sx_{n+1}, Sx_{n+1})$$
$$+a_2\, G(Sx_n, Ax_n, Sx_n)$$
$$+a_3\, G(Sx_n, Ax_{n+1}, Sx_{n+1})$$
$$+a_4\, G(Sx_{n+1}, Ax_n, Sx_{n+1})$$

$$\leq a_1\, G(Ax_{n-1},Ax_n,Ax_n)$$
$$+a_2\, G(Ax_{n-1},Ax_n,Ax_{n-1})$$
$$+a_3\, G(Ax_{n-1},Ax_{n+1},Ax_n)$$
$$+a_4\, G(Ax_n,Ax_n,Ax_n)$$

$$G(Ax_n,Ax_{n+1},Ax_{n+1})$$
$$\leq (a_1+2a_2)G(Ax_{n-1},Ax_n,Ax_n)$$
$$+a_3\, G(Ax_{n-1},Ax_n,Ax_{n+1}) \qquad (3.1.4)$$

By rectangular inequality of G-metric space

$$G(Ax_{n-1},Ax_n,Ax_{n+1}) \leq G(Ax_{n-1},Ax_n,Ax_n)$$
$$+\, G(Ax_n,Ax_n,Ax_{n+1})$$
$$\leq G(Ax_{n-1},Ax_n,Ax_n)$$
$$+2\, G(Ax_n,Ax_{n+1},Ax_{n+1}),$$

[by proposition(1.6)].

From inequality (3.1.4), we have

$$G(Ax_n,Ax_{n+1},Ax_{n+1})$$
$$\leq (a_1+2a_2)G(Ax_{n-1},Ax_n,Ax_n)$$
$$+\,a_3\begin{bmatrix} G(Ax_{n-1},Ax_n,Ax_n) \\ +2\, G(Ax_n,Ax_{n+1},Ax_{n+1}) \end{bmatrix}$$

$$G(Ax_n,Ax_{n+1},Ax_{n+1})$$
$$\leq \left(\frac{a_1+2a_2+a_3}{1-2a_3}\right) G(Ax_{n-1},Ax_n,Ax_n)$$

$$G(Ax_n,Ax_{n+1},Ax_{n+1}) \leq \delta\, G(Ax_{n-1},Ax_n,Ax_n),$$

where $\delta = \frac{a_1+2a_2+a_3}{1-2a_3} < 1$.

Similarly,

$$G(Ax_n,Ax_{n+1},Ax_{n+1}) \leq \delta^n\, G(Ax_{n-1},Ax_n,Ax_n).$$

So that for any $m > n$, $m,n \in N$,

$$G(y_n,y_m,y_m) \leq G(y_n,y_{n+1},y_{n+1}) + G(y_{n+1},y_{n+2},y_{n+2}) + \cdots + G(y_{m-1},y_m,y_m)$$
$$\leq (\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})G(y_0,y_1,y_1)$$
$$\leq \frac{\delta^n}{1-\delta} G(y_0,y_1,y_1) \to 0$$

as $m,n \to \infty$.

Thus sequence $\{y_n\}$ is a G-Cauchy sequence in X and since X is complete G-metric space, therefore sequence $\{y_n\}$ converges to point u in X such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{n+1} = u.$$

Since the mapping A or S is continuous, one can assume that A is continuous, therefore $\lim_{n\to\infty} ASx_{n+1} = \lim_{n\to\infty} AAx_n = Au$. Further, A and S are compatible, therefore

$$\lim_{n\to\infty} G(SAx_n,ASx_n,ASx_n) = 0$$

implies that $\lim_{n\to\infty} SAx_n = Au$.

Consider

$$G(AAx_n,Ax_n,Ax_n) \leq a_1 G(SAx_n,Sx_n,Sx_n)$$
$$+a_2 G(SAx_n,AAx_n,SAx_n)$$
$$+a_3 G(SAx_n,Ax_n,Sx_n)$$
$$+a_4\, G(Sx_n,AAx_n,Sx_n).$$

Letting $n \to \infty$, we get

$$G(Au,u,u) \leq a_1\, G(Au,u,u) + a_2\, G(Au,Au,Au)$$
$$+a_3\, G(Au,u,u) + a_4\, G(u,Au,u)$$
$$\leq (a_1+a_3+a_4)\, G(Au,u,u).$$

Implies that $G(Au,u,u) \leq 0$ so that $Au = u$.

Again consider

$$G(SAx_n,Au,Au) \leq G(ASx_n,Au,Au)$$
$$\leq a_1\, G(SSx_n,Su,Su)$$
$$+a_2\, G(SSx_n,ASx_n,SSx_n)$$

$$+a_3\ G(SSx_n, Au, Su)$$

$$+\ a_4\ G(Su, ASx_n, Su).$$

Letting $n \to \infty$, we get

$$G(Su,u,u) \le a_1\ G(Su,Su,Su) + a_2\ G(Su,u,Su)$$

$$+a_3\ G(Su,u,Su) + a_4\ G(u,u,Su)$$

$$G(Su,u,u) \le\ 2a_2\ G(Su,u,u) + 2a_3\ G(Su,u,u)$$

$$+a_4\ G(Su,u,u)$$

$$G(Su,u,u) \le\ (2a_2 + 2a_3 + a_4)\ G(Su,u,u).$$

Implies that $G(Su,u,u) \le 0$ so that $Su = u$.

Hence $Au = Su = u$. Therefore u is a common fixed point of A and S.

Suppose that $v(\ne u)$ be another common fixed point of A and S. Then $G(u,v,v) > 0$.

Consider

$$G(u,v,v) = G(Au,Av,Av)$$

$$\le a_1\ G(Su,Sv,Sv) + a_2\ G(Su,Au,Su)$$

$$+a_3\ G(Su,Av,Sv) + a_4\ G(Sv,Au,Sv)$$

$$\le a_1\ G(u,v,v) + a_2\ G(u,u,u)$$

$$+a_3\ G(u,v,v) + a_4\ G(v,u,v)$$

$$\le (a_1\ + a_3\ + a_4)G(u,v,v) < G(u,v,v),$$

which is a contradiction,

so that $u = v$.

Hence u is a unique common fixed point of A and S.

Remark 3.1. All the conditions of theorem 3.1 remain true, if we replace the contraction condition (3.1.2) by one of the following conditions.

(3.1.5) $G(Ax,Ay,Az)$

$$\le k\ max \begin{Bmatrix} G(Sx,Sy,Az), G(Sx,Ax,Sx), G(Sy,Ay,Sy), \\ \frac{1}{2}[G(Sx,Ay,Sz) + G(Sy,Ax,Sz)] \end{Bmatrix}$$

(3.1.6) $G(Ax,Ay,Az)$

$$\le k\ max \begin{Bmatrix} G(Sx,Sy,Az), \frac{1}{2}[G(Sx,Ax,Sx) + G(Sy,Ay,Sy)], \\ \frac{1}{2}[G(Sx,Ay,Sz) + G(Sy,Ax,Sz)] \end{Bmatrix}$$

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