Convex Structures in Normed Spaces

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Abstract: *A convexity on a nonempty set X is a collection C of subsets of X which is closed under intersection and union of sets totally ordered under inclusion. The norm on a Normed linear space X defines a convexity on X. In this paper we discuss the geometrical properties of convexities defined by different norms on* $\mathbf{R}^2$

Keywords: *Convexity, interval Convexity, Normed spaces etc.*

1. INTRODUCTION

The Concept of Convexity is a very old topic which can be traced back to the period of Plato. Motivated by the properties of convex bodies such as the five platonic Solids and other Polyhedra, study of Convex sets in R^n was systematically studied by Newton Minkowski, Helly, Radon and Others [3,7]. It is well known that, a subset C of a real vector space is convex if and only if it contains with any pair of points in C the entire line segment joining them. It can be easily observed that intersection of any family of convex sets is convex, even though the intersection may be empty. This lead to the development of Abstract convexity spaces in which a convexity $\mathcal{C}$on a set X is a family of subsets of X which contains the null set and X, and which is closed under intersection and union of sets totally ordered under inclusion. Given a set $A \subset X$, By convex hull of A denoted by $Co(A)$, we mean the smallest convex set containing A. Convexity in Graphs was studied by the author [7], M changat [6] John [1] and others. An out standing survey of developments in the area of convexity is given by Van de Vel [8]. We use the following basic definitions and results

***Definition* 1.**

Let X be a Linear space over a field K. A set $E \subset X$ is convex if, $tx + (1-t)y \in E$ whenever $x, y \in E$ and $t \in (0,1)$.

***Definition* 2** [1, 4].

Let **X** be a Linear space. A norm on X is a function $\|.\|$: **X** $\to$ R, satisfying the following properties.

For all x, y, z in **X** and k in the field K.

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if x = 0. (Positive definiteness)
2. $\|x + y\| \leq \|x\| + \|y\|$.(Triangle inequality)
3. $\|kx\| = |k| \, \|x\|$

***Definition* 3.**

Let X be any set. An interval function on X is a function $I : X \times X \to P(X)$ Such that

1. $I(a, b) = I(b, a)$
2. $a, b \in I(a, b)$

A Convexity is called **interval convexity** if its convexity is induced by an interval function. That is, $A \subset X$ is convex if $I(a, b) \subset A$ whenever $a, b \in A$.

Minkowsky inequality [4] : If $a_1, a_2. \ldots\ldots a_n$ and If $b_1, b_2. \ldots\ldots b_n$ are complex numbers and $1 \leq p < \infty$, Then,

$$\left(\sum_1^n |a_j + b_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_1^n |a_j|^p\right)^{\frac{1}{p}} + \left(\sum_1^n |b_j|^p\right)^{\frac{1}{p}}$$

2. INTERVAL CONVEXITY ON NORMED SPACES

Let X be a normed linear space. Define $I: X \times X \to P(X)$ as,

$I(x, y) = Seg\,(x, y) = \{z \in E : d(x, y) = d(x, z) + d(z, y).\}$

Then, I defines an interval function on X and hence a convexity. This is different from the usual definition of convex sets in linear spaces.

Remark:

In R^n, we usually define a set to be convex, if $tx + (1-t)y \in E$ whenever x, y $\in$ E and t $\in$ (0, 1).When R^n is given the Euclidean norm, $E \subset R^n$, x, y $\in$ E and t $\in$ (0, 1). Then,

$\|x - (tx + (1-t)y)\| = \|(1-t)x - (1-t)y\| = (1\text{-}t)\,\|x - y)\|$

Similarly, $\|(tx + (1-t)y) - y\| = t\,\|x - y\|$

Hence, $\|x - y\| = \|x - (tx + (1-t)y)\| + \|(tx + (1-t)y) - y\|$

Conversely if $\|x - y\| = \|x - z\| + \|z - y\|$

Then, $x - z = k(z - y)$ for some $k \geq 0$

That is, $z = \frac{1}{1+k}x + \frac{k}{1+k}y$. Put $t = \frac{1}{1+k}$. Then $t \in (0,1)$ and z = tx + (1-t)y

Example:

Let R^n be given the norms $\|.\|_1$, $\|.\|_\infty$ and the Euclidean norm, $\|.\|_2$ are defined by,

For $x = (x(1)$, x(2),……x(n)),

$\|x\|_1 = \sum_1^n |x(i)|$,

$\|x\|_\infty = Max\{ |x(1)|, |x(12)|, \dots |x(n)|\}$ and

$\|x\|_2 = \sqrt{\sum_1^n |x(i)|^2}$

Let n = 2, x = (1,1), $y = ((-1,-1)$

Then the segment $seg(x,y)$ with respect to $\|x\|_2$ is the line segment joining x and y. But when we consider $\|x\|_1$

$seg(x,y) = \{ z = (z(1), z(2)) : \|x-y\|_1 = \|x-z\|_1 + \|z-y\|_1 \}$

$= \{ z = (z(1), z(2)) : |x(1)-y(1)| + |x(2)-y(2)| = |x(1)-z(1)| + |x(2)-z(2)| + |z(1)-y(1)| + |z(2)-y(2)|\}$.

$= \{ z = (z(1), z(2)) : 4 = |1-z(1)| + |1-z(2)| + |z(1)+1| + |z(2)+1|\}$.

$= \{ z = (z(1), z(2)) : -1 \le z(1) \le 1,$

$-1 \le z(1) \le 1\}$

Remark 1:

If x = (x(1), x(2)) and y = (y(1), y(2)),

Seg (x, y) = { z = (z(1), z(2)) : z(1) is between x(1) and y(1), z(2) is between x(2) and y(2) }

Remark 2:

When we consider $\|x\|_\infty$,

x = (0,0) , y = (1,1). Then ,

$seg(x,y) = \{ z = (z(1), z(2)) : \|x-y\|_\infty = \|x-z\|_\infty + \|z-y\|_\infty \}$

$\|x-y\|_\infty = 1$

Now, For any $z = (z(1), z(2))$,

$1 = |x(1)-y(1)|$

$\le |x(1)-z(1)| + |z(1)-y(1)|$

$\le |z(1)| + |1-z(1)|$

$\le$ Max { $|z(1)|, |z(2)|$ } +

Max { $|1-z(1)|, |1-z(2)|$ }.

Now let, z ∈ $seg(x,y)$

Then, z(i) , 1 – z(i) ≥ 0 , for i = 1,2 and .

Max{z(1), z(2)} + Max { 1- z(1), 1- z(2) } = 1

If z(1) ≤ z(2) , Then, 1 – z(2) ≤ 1- z(1).

Hence

1-z(1) + z(2) = 1. That is, z(1) = z(2)

Thus,

$Seg(x,y) = \{ z = (z(1), z(2)) : \|x-y\|_\infty = \|x-z\|_\infty + \|z-y\|_{\infty 1}$

$= \{ z = (z(1), z(2)) : z(1) = z(2),\ 0 \le z(i) \le 1, i = 1, 2.\}$, is the line segment joining x and y.

Now we prove the following Theorem.

Theorem 2.1:

Let 1 < p < ∞ and a function f be defined as,

$$f(x,y) = (x^p + y^p)^{\frac{1}{p}} + ((1-x)^p + (1-y)^p)^{\frac{1}{p}},$$

0 ≤ x, y ≤ 1.

Then, f has a minimum at (x, y) if and only if

x = y,

Proof :

First suppose that **x = y.**

Then, f(x, y) = $2^{\frac{1}{p}}(x+1-x) = 2^{\frac{1}{p}}$,

But by Minkowski inequality, f (x, y) ≥ $2^{\frac{1}{p}}$ (a_1 = x, b_1 = 1-x and a_2 = y, b_2 = 1-y). Hence f has a minimum at (x, y) if x = y.

Conversely suppose that f has a minimum at (x, y). Then $\frac{\partial f}{\partial x} = 0$ and $\frac{\delta f}{\delta y}$ =0

$$\frac{\partial f}{\partial x} = 0 \Rightarrow$$

$$\frac{1}{p}(x^p + y^p)^{\frac{1-p}{p}} p.x^{p-1} - \frac{1}{p}((1-x)^p + (1-y)^p)^{\frac{1-p}{p}} p.(1-x)^{p-1} = 0.$$

That is,

$$(x^p + y^p)^{\frac{1-p}{p}} x^{p-1} = ((1-x)^p + (1-y)^p)^{\frac{1-p}{p}} (1-x)^{p-1}.$$

Then,

$$(x^p + y^p)^{\frac{1}{p}} (1-x) = ((1-x)^p + (1-y)^p)^{\frac{1}{p}} x .$$

That is,

$(x^p + y^p)^{\frac{1}{p}}$ - x $(x^p + y^p)^{\frac{1}{p}}$

$$= x\,((1-x)^p + (1-y)^p))^{\frac{1}{p}}$$

That is,

$$((1-x)^p + (1-y)^p))^{\frac{1}{p}} = (1\text{-}x)\left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}$$

$$= \left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}} - (x^p + y^p)^{\frac{1}{p}}, \text{ if } x \in (0, 1)$$

Hence,

$$f(x,y) = (x^p + y^p)^{\frac{1}{p}} + ((1-x)^p + (1-y)^p)^{\frac{1}{p}}$$

$= \left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}$ if $x \in (0, 1)$. If $x = 0$, $f(x,y) = 2^{\frac{1}{p}}$ if and only if $y = 0$

Now,

$$f(1,1) = 2^{\frac{1}{p}} = f(0,0) \Rightarrow$$

$$2^{\frac{1}{p}} = \left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}, \ x \in (0, 1), \Rightarrow$$

$$1 + (\tfrac{y}{x})^p = 2 \Rightarrow$$

$$\tfrac{y}{x} = 1. \Rightarrow$$

$$y = x.$$

■

The following generalizes Theorem 2.1

Theorem 2.2:

Let $1 < p < \infty$, $a,b \in R$, $b \neq 0$ and $f: R^2 \to R$ be defined as,

$$f(x,y) = (|x^p| + |y^p|)^{\frac{1}{p}} + (|(a-x)^p| + |(b-y)^p|)^{\frac{1}{p}},$$

Then, f has a minimum at (x, y) if and only if $x = \frac{a}{b}y$, x between 0 and a, y between 0 and b.

Proof :

But by Minkowski inequality,

$$f(x,y) \geq (|a^p| + |b^p|)^{\frac{1}{p}}$$

First suppose that $\mathbf{x} = \frac{a}{b}y$, x between 0 and a, y between 0 and b.

Then,

$$f(x,y) = \left(\left|\tfrac{a}{b}\right|^p + 1\right)^{\frac{1}{p}} |y| + \left(\left|\tfrac{a}{b}\right|^p + 1\right)^{\frac{1}{p}} |b-y|$$

$= \left(\left|\tfrac{a}{b}\right|^p + 1\right)^{\frac{1}{p}} |b|$. (since, $0 < y < b$ or $b < y < 0$, we have, $|y| + |b-y| = |b|$)

$$= (|a^p| + |b^p|)^{\frac{1}{p}}.$$

Thus f has a minimum at (x, y) if $x = \frac{a}{b}y$

Note that if , $0 < y < b$, then $0 < x < a$ if $a > 0$ and $a < x < 0$ if $a < 0$. Similarly the other case. If $a = 0$ the case is trivial.

Conversely suppose that f has a minimum at (x, y)

Case 1: $0 < x < a$, $0 < y < b$

Then $f(x,y) = (x^p + y^p)^{\frac{1}{p}} + ((a-x)^p + (b-y)^p)^{\frac{1}{p}}$

Then $\frac{\partial f}{\partial x} = 0$ and $\frac{\delta f}{\delta y} = 0$

$\frac{\partial f}{\partial x} = 0$ imply that,

$$\frac{1}{p}(x^p + y^p)^{\frac{1-p}{p}} p.x^{p-1} - \frac{1}{p}(a-x)^p +)^{\frac{1-p}{p}} p.(a-x)^{p-1} = 0.$$

That is,

$$(x^p + y^p)^{\frac{1-p}{p}} x^{p-1} = ((a-x)^p + (b-y)^p))^{\frac{1-p}{p}} (a-x)^{p-1}$$

That is,

$$(x^p + y^p)^{\frac{1}{p}} (a\text{-}x) = ((1-x)^p + (1-y)^p))^{\frac{1}{p}} x$$

That is,

$a\,(x^p + y^p)^{\frac{1}{p}}$ - x $(x^p + y^p)^{\frac{1}{p}}$

$$= x\,((a-x)^p + (b-y)^p))^{\frac{1}{p}}$$

That is,

$$((a-x)^p + (b-y)^p))^{\frac{1}{p}} = (a-x)\left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}$$

$$= a\left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}} - (x^p + y^p)^{\frac{1}{p}},$$

Hence ,

$$f(x,y) = (x^p + y^p)^{\frac{1}{p}} + ((1-x)^p + (1-y)^p)^{\frac{1}{p}}$$

$$= a\left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}$$

Now,

$$f(0,0) \quad = (a^p + b^p)^{\frac{1}{p}} = (|a^p| + |b^p|)^{\frac{1}{p}}$$

$$= f(a,b)$$

Hence,

$$(a^p + b^p)^{\frac{1}{p}} = a\left(1 + (\tfrac{y}{x})^p\right)^{\frac{1}{p}}$$

That is $\quad \mathbf{x} = \frac{a}{b}\mathrm{y}$

The other cases can be proved in similar way.

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Corollary:

Let $1 < p < \infty$, a, b, c, d $\in$ R , b$\neq$ d and

f: $R^2 \rightarrow R$ be defined as,

$$f(x,y) = (|(a-x)^p| + |(b-y)^p|)^{\frac{1}{p}} + (|(x-c)^p| + |(y-d)^p|)^{\frac{1}{p}},$$

Then, f has a minimum at (x, y) if and only if

x - a $= \frac{c-a}{d-b}$y, x between a and c, y between b and d

The minimum value of $f(x,y)$ is

$$f(a,b) = f(c,d) = (|(a-c)^p| + |(b-d)^p|)^{\frac{1}{p}}$$

■

`The above results prove that on R^2 induced by the norm $\|.\|_p$,1< p < ∞, for x, y in R^2,

Seg (x,y) = { x + (1-t)(y-x) : t$\in$ (0,1)}. Hence we have proved the following Theorem.

Theorem 2.3:

Let C be the interval convexity on R^2 induced by the norm , $\|.\|_p$1< p < ∞.
Then C coincides with the Euclidean convexity.

Theorem 2.4.

Let $\mathcal{C}$ be the interval convexity on R^2 induced by the norm , $\|.\|_\infty$. Then $\mathcal{C}$ does not coincide with the Euclidean convexity.

Proof :

Let x = (0, 0), y = (1, 0) Then,

$$\|x - y\|_\infty = 1$$

Let z = $(\frac{1}{2}, \frac{1}{2})$. Then z is not on the line joining x and y. But

$$\|x - z\|_\infty + \|z - y\|_\infty = \frac{1}{2} + \frac{1}{2} = 1$$ Hence z belong to the convex hull of {x,y}. Thus the line segment joining x and y is not a member of $\mathcal{C}$

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Theorem 2.5. :

Let C be the interval convexity on R^2 induced by the norm , $\|.\|_\infty$. Let x = (0, 0) and y = (a, b). and let α = max{|a|,|b|} Then, The convex hull of { x, y} is the area bounded by the straight lines y = ± x, y = x + b – a and y+x = b-a

Proof:

Case1 **:** (a, b) is in the first quadrant. Then a, b ≥ 0.

Let a ≥ b.

Then $\|x - y\|_\infty$ = a

A point z = (s, t) is in the convex hull of x and y if and only if

Max { |s|, |t|} + Max { |a-s|, |b-t|} = a ,that is if and only if

|t| ≤ |s|, and |b-t| ≤ |a-s| that is if and only if –s ≤ t ≤ s and s-a ≤ b-t ≤ a-s, that is if and only if –s ≤ t ≤ s and b-a +s ≤ t ≤ a+ b-s.

Now let b ≥ a. Then, $\|x - y\|_\infty$ = b and a point z = (s, t) is in the convex hull of x and y if and only if

Max { |s|, |t|} + Max { |a-s|, |b-t|} = b ,that is if and only if

|s| ≤ |t|, and |a-s| ≤ |b-t| that is if and only if –t ≤ s ≤ t and t-b ≤ a-s ≤ b-t'

That is–t ≤ s ≤ t and a-b +t ≤ s ≤ a+ b-t

Case2 **:** (a, b) is in the II quadrant. Then a ≤ 0, b ≥ 0. Then,

Let |a| ≥ b.

Then $\|x - y\|_\infty$ = -a

A point z = (s, t) is in the convex hull of x and y if and only if

Max { |s|, |t|} + Max { |a-s|,|b-t|} = Max { -s,|t|} + Max {s-a ,|b-t|}= - a, that is if and only if

|t| ≤ -s , and |b-t| ≤ |a-s| that is if and only if s ≤ t ≤ -s and a-s ≤ b-t ≤ s-a, that is if and only if –s ≤ t ≤ s and a+b –s ≤ t ≤ b-a +s.

Similarly, the other cases can be proved.

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Corollary:

The convex hull of any two points (a, b) and (c,d) is the rectangle bounded by the straight lines $|a-x| = |b-y|$ and $|c-x| = |d-y|$

Theorem2.6:

The area bounded by a circle C in R^2 is convex with respect the convexities induced by$\|.\|_p$, $1 < p < \infty$. But not convex with respect to the convexity induced by $\|.\|_\infty$ or $\|.\|_1$

Proof :

With respect to convexities induced by $\|.\|_p$, $1 < p < \infty$., it is clear. Now let R^2 be given the norm $\|.\|_1$.

Consider the set, $S = \{ (x, y): \ x^2 + y^2 \leq 1 \}$.

Then $z_1 = (1, 0)$, and $z_2 = (0, 1)$ are in S. Also we can see that $(1, 1) \in Co(\{ z_1, z_2\})$, but (1, 1)is not appoint in S. Hence S is not convex with respect to $\|.\|_1$.

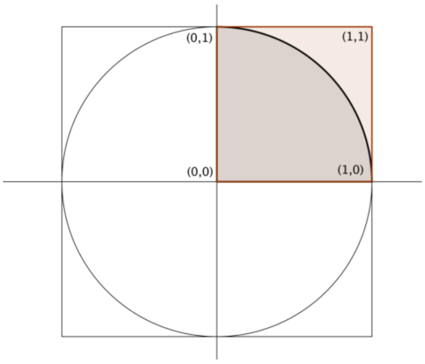

Fig. 1

Similarly if R^2 is given the norm $\|.\|_\infty$, and Let $S' = \{ (x, y): \ x^2 + y^2 \leq 2 \}$.

Then take $z_1 = (1, -1)$ and $z_2 = (1, 1)$. Then z_1 and z_2 are in S' ,

(2, 0) is in the convex hull of z_1 and z_2 , but (2, 0) is not in S.

■

Note : If the points A and B are on the line y = x or y = -x , Then the convex hull is the line segment joining them.

REFERENCES

[1]. J John , V Mary Gleeta, '*The forcing Monophonic Hull Number of a Graph"* IJMTT Vol.3. 2012.

[2] Kesavan S. "*Topics in Functional Analysis And Applications"* John Wiley and Sons Ltd

[3]. Lay S. R. "*Convex Sets and Applications*" Dover Publications Ltd

[4]. Leonard Berkowitz "*Convexity and Optimizations in R^n"*

[5]. Limaye B. V " *Functional Analysis*" New Age International Publishers

[6] M Changat '*On Triangle Path Convexity inGraphs*" Discrete Mathematics.206(1-3) 91-95,1999.

[7]. Parvathy K S *"Studies on Convex Structures with Emphasis on Convexity in Graphs"* PhD thesis.

[8]. Van de Vel *"Theory of Convex Structures"* North Holland.