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A Toy Biological Modelling Through a Delayed Rational Difference Equation

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Abstract

In modelling of any biological systems, one of the important and fundamental issues is the depth of choice of detail. The relevance goes far beyond mathematical convenience to the heart of understanding the mechanism, specifically, which details at one level are important to the determination of any phenomena at other levels and which can be ignored. The rational difference equations with delay surprisedly gained attention in modelling some of the complicated biological systems over last couple of decades. In this article, a $(k+1)^{th}$ order delayed rational difference equation

$$h_{n+1} = \frac{\alpha + \gamma h_{n-k}}{\beta h_n + \delta h_n h_{n-k} + h_{n-k}}, n = 0, 1, 2, \ldots$$

is considered as a toy model and its behavior is investigated.

Keywords: Bio-System Modelling, Rational difference equation, Local asymptotic stability, Chaotic trajectory and Periodicity.
Mathematics Subject Classification: 39A10, 39A11

1 Introduction and Background

Nature is supernally complex! It is an authors perception that nature encourages people to explore her as closely as they wish but can never be touched. An apparently constant and stable system is often nothing but a balance of tendencies pushing the system towards different directions. The cumulative effect of various interactions and competing tendencies generally make it difficult to analyse the full picture at once. Mathematical language has been designed for precise description of dynamic process in biology. The rationale for constructing mathematical models of reality lies in simple formulas, for instance, that can relate the population of a species in a certain year to that of the following year. This strategy has enjoyed considerable success in natural sciences and there are growing literature exploring their usefulness. In most modern biology fields, it is important to know how populations grow and what factors influence their growth [1]. In this context, mathematical analysis indicates the consequences an equation might have, so that it can be checked against biological observation. Despite ecology knowledge of this kind is important in studies of bacterial growth, wildlife management and harvesting [2].

Popularly, a difference equation is a formula expressing values of some quantity Q in terms of previous values of Q [3] & [4]. The qualitative understanding of difference equations is a fertile research area and increasingly attracts many mathematicians and non-mathematicians as well [5]. The basic questions that arise in this regard are: how does one find an appropriate difference equation to model a situation? how does one understand the behaviour of the difference equation model once it has been formed? There is no scopes of confusion that the theory of difference equations will continue to play an important role in mathematics as a whole. It is found that a dynamic biological system may settle down into a particular pattern regardless of its initial values. The first few steps of iterations though may not really be indicative of what happens over long term. One of the significant biological conclusions from nonlinear models is that a population may exhibit cycles even when the environment is completely unchanging. Complicated behavior can be produced even by simple models. So the natural view that complicated population dynamics is the result of complex iterations and environment fluctuations would have to be abandoned. It is seen over decades, nonlinear difference equations of higher order with delay are of paramount importance in applications [6] & [7]. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics as said earlier. For greater detail of dynamics in biological systems one may refer to the books [8] & [9] and articles [10] & [11] .

In this article, a computational attempt without any rigorous theoretical exercise is made to explore the behavior of such a model which is commonly known as *rational difference equation*. We shall show how the dynamical behavior of a simple difference equation goes from a fixed point, through a sequence of bifurcations, into stable periodic cycles and finally into a regime of apparent chaos. This is illustrated with a particular example of a rational difference equation as stated below, but the emphasis is on the generic character of the process.

Consider the equation

$$h_{n+1} = \frac{\alpha + \gamma h_{n-k}}{\beta h_n + \delta h_n h_{n-k} + h_{n-k}}, n = 0, 1, 2, \ldots \tag{1}$$

where the parameter α, γ, B and D and the initial conditions $h_{-k}, h_{-k+1} \ldots h_0$ are arbitrary real numbers.

The second order rational difference equation Eq.(1) with $(k = 2)$ is studied when the parameters and the initial conditions are non-negative real numbers by *Y. Kostrov and Z. Kudlak* in []. Here we proceed to comprehend the dynamics of the generalized Eq.(1) with delay k.

Before we proceed further here we review a very basic necessary results related to local stability of fixed points.

Definition 1: A difference equation of order $(k + 1)$ is of the form

$$z_{n+1} = f(z_n, z_{n-1}, \ldots, z_{n-k}), n = 0, 1, 2, \ldots \tag{2}$$

where f is a continuous function which maps a subset $\mathbb{D}^{k+1}$ into $\mathbb{D}$ and $\mathbb{D} \subset \mathbb{R}$. A fixed point $\bar{z}$ of the difference equation Eq. (2) is a point that satisfy the condition $\bar{z} = f(\bar{z}, \bar{z}, \ldots, \bar{z})$.

Definition 2: Let $\bar{z}$ be a fixed point of the Eq.(2), then $\bar{z}$ is *locally asymptotically stable* if for every $\epsilon > 0$, there exist a $\delta(\epsilon) > 0$ such that, if $z_{-k}, \ldots, z_{-1}, z_0 \in D$ with $|z_{-k} - \bar{z}| + \cdots + |z_{-1} - \bar{z}| + |z_0 - \bar{z}| < \delta(\epsilon)$, then $|z_n - \bar{z}| < \epsilon$ for all $n \geq -k$.

Definition 3: A sequence ${z_n}_{n=k}^{\infty}$ is said to be *periodic with period* p if $z_{n+p} = z_n$ for all $n \geq -k$. A sequence ${z_n}_{n=k}^{\infty}$ is said to be *periodic* with prime period p if p is the smallest positive integer having this property.

Definition 4: An open ball $B(a,r) \in \mathbb{C}$ is called an *invariant open ball* of Eq.(2) if $z_{-k}, \ldots, z_{-1}, z_0 \in B(a,r)$ then $z_n \in B(a,r)$ for all $n > 0$. That is every solution of Eq.(2) with initial conditions in $B(a,r)$ remains in $B(a,r)$.

Definition 5: The difference equation Eq.(2) is said to be *permanent and bounded* if there exist positive real numbers M and N with $0 < M \leq N < \infty$ such that for any initial conditions $z_{-k}, \ldots, z_{-1}, z_0$ there exists a positive integer P which depends on the initial conditions such that $M \leq |z_n| \leq N$ for all $n \geq P$.

The linearized equation associated with Eq.(2) about the equilibrium point $\bar{z}$ is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{z}, \bar{z}, \ldots, \bar{z})}{\partial u_i} y_{n-i}$$

Its characteristic equation is

$$\lambda^{k+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{z}, \bar{z}, \ldots, \bar{z})}{\partial u_i} \lambda^{n-i}$$

where $n = 0, 1, 2, \ldots$.

Theorem 1.1. *Assume that f is a C^1-function and let $\bar{z}$ a fixed point of Eq.(2). Then the following statements are true:*

- *If all the roots of the characteristic equation lie in the open unit disk $|\lambda| < 1$, then the fixed point $\bar{z}$ of Eq.(2) is locally asymptotically stable.*
- *If at least one root of the characteristic equation has the absolute value greater than one, then the fixed point $\bar{z}$ of Eq.(2) is unstable.*
- *If all the roots of the characteristic equation have the absolute value greater than one, then the fixed point $\bar{z}$ of Eq.(2) is a source.*

Theorem 1.2. *Assume that p, $q \in \mathbb{R}$ and $k \in \mathbb{N}$. Then $|p| + |q| < 1$ is a sufficient condition for asymptotically stability of the difference equation*

$$z_{n+1} - pz_n + qz_{n-k} = 0, n = 0, 1, 2, 3, \ldots$$

Now we shall use these basic theorems to explore the local stability of the fixed points of the Eq.(1).

2 Character of the Rational Function

Here we shall see how does the rational function

$$f(u,v) = \frac{\alpha + \gamma u}{\delta uv + u + \beta v}$$

α, β, γ and δ are real numbers. Here we focus on contour of the function and the forbidden set $\mathbb{F}$ where the function has poles in $\mathbb{R}^2$.
We are looking for the points (u,v) and parameters β and δ such that the rational function has poles and not well defined accordingly. We found such points (u,v) and corresponding parameters β and δ which are plotted in the *Fig. 1*.

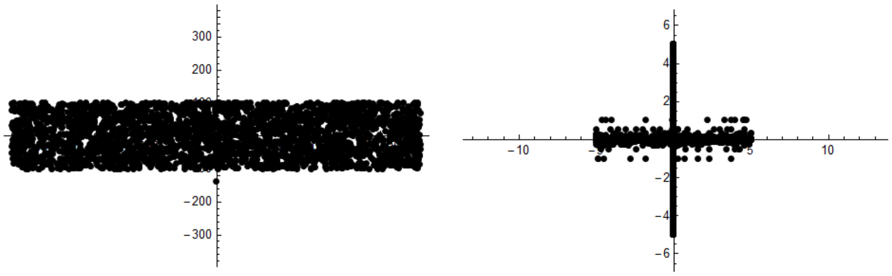


Figure 1: Forbidden set of points (u, v) (Left) and associated parameters plot (β, δ) (Right)

A quick reference of members of the forbidden set $(\mathbb{F}_{\mathbb{R}})$ is given as

$$\left\{u \to 0, v \to -761, \beta \to 0, \delta \to \frac{18}{5}\right\}, \left\{u \to 0, v \to 0, \beta \to -\frac{103}{5}, \delta \to \frac{176}{5}\right\},$$

$$\left\{u \to -83, v \to 71, \beta \to 2, \delta \to \frac{59}{5893}\right\}, \left\{u \to 0, v \to 811, \beta \to 0, \delta \to -\frac{12}{5}\right\},$$

$$\left\{u \to -31, v \to 82, \beta \to -5, \delta \to -\frac{441}{2542}\right\}, \left\{u \to -229, v \to 28, \beta \to \frac{7}{2}, \delta \to -\frac{131}{6412}\right\},$$

$$\left\{u \to -173, v \to 85, \beta \to \frac{31}{10}, \delta \to \frac{181}{29410}\right\}, \left\{u \to 0, v \to -223, \beta \to 0, \delta \to \frac{13}{10}\right\}$$

We are also interested to explore the forbidden set for the complex rational function

$$f(u, v) = \frac{\alpha + \gamma u}{\delta uv + u + \beta v}$$

α, β, γ and δ are complex numbers.
We are looking for the points (u, v) and parameters β and δ such that the rational function has poles and not well defined accordingly. We found such points (u, v) and corresponding parameters which are plotted in the *Fig. 1*.

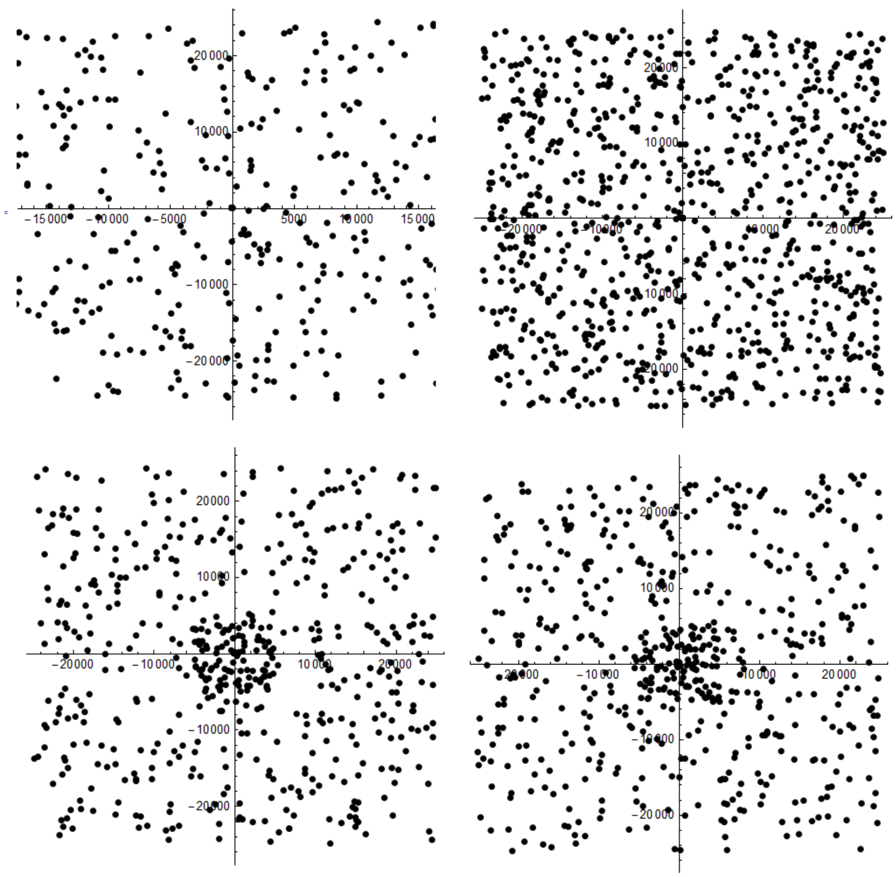


Figure 2: Forbidden set of point u (Top left), v (Top Right) and associated parameters plot β (Bottom Left) and δ (Bottom Right)

A quick reference of members of the forbidden set ($\mathbb{F}_{\mathbb{C}}$) is given as

$$\left\{u \to \frac{1443}{10} + 34i, v \to \frac{1923}{10} - \frac{2323i}{10}, \beta \to -\frac{2449}{10} - \frac{681i}{10}, \delta \to \frac{17100840933528}{9993902925521} + \frac{656645917832i}{9993902925521}\right\},$$

$$\left\{u \to -\frac{787}{5} - \frac{389i}{10}, v \to -\frac{1887}{10} - \frac{867i}{10}, \beta \to -213 + 249i, \delta \to -\frac{100307231345}{111142908363} + \frac{117947092225i}{6537818139}\right\},$$

$$\left\{u \to 0, v \to -\frac{1947}{10} + \frac{23i}{2}, \beta \to 0, \delta \to -\frac{613}{10} + \frac{549i}{10}\right\},$$

and

$$\left\{u \to 0, v \to 0, \beta \to \frac{93}{10} - \frac{39i}{2}, \delta \to -\frac{179}{10} + \frac{417i}{10}\right\}$$

Here we go with specific parameters α, β, γ and δ and see the contour of the function as shown in the following *Table-1.*

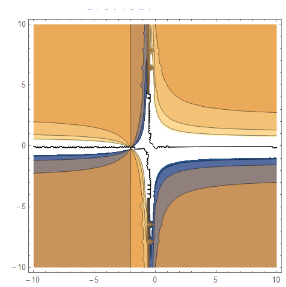
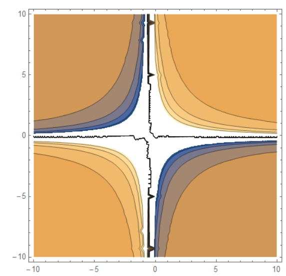
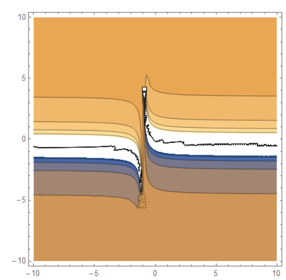
Sl. No.	Parameters	Contour Plot
1	$\alpha = 2, \gamma = 1, \beta = 4, \delta = 8$	
2	$\alpha = 2, \gamma = 0, \beta = 4, \delta = 8$	
3	$\alpha = 2, \gamma = 2, \beta = 2, \delta = 2$	

Table 1: Parameters α, γ β and δ for which the fixed points $\hbar$ is attracting, repelling for different initial values with different delay k.

We are interested to explore the the dynamics of the rational difference equation where (a subset of $\mathbb{R}^2 - \mathbb{F}_{\mathbb{R}}$) the rational function is well defined. We intend to see the dynamics computationally from the delay k perspective.

3 Local Asymptotic Stability of the Equilibrium

The rational difference equation has only one real fixed point ($\hbar$ (say)) and we shall see the local asymptotic stability of the fixed point ($\hbar$). Here we are looking for parameters α, β, γ and δ with delay k, the fixed point ($\hbar$) is locally asymptotically stable. A set of examples are given as follows in the *Table 2*.

Sl. No.	**Parameters** α, γ, β, δ & k	**Remark**
1	$\alpha = -76, \gamma = -59, \beta = -71, \delta = -62$	If the delay k is odd, then the trajectories (for ten different initial values) are attracting to the fixed point 1.01112 which is shown *Fig. 2*. If the delay k is even then the trajectories are diverging as shown in *Fig. 2*.
2	$\alpha = 17, \gamma = -90, \beta = 21, \delta = -92$	If the delay k is odd, then the trajectories (for ten different initial values) are attracting to a period two cycle which is shown *Fig. 2*. If the delay k is even then the trajectories are attracting to the fixed point -0.9683 in *Fig. 2*.

Table 2: Parameters α, γ, β and δ for which the fixed points $\hbar$ is attracting, repelling for different initial values with different delay k.

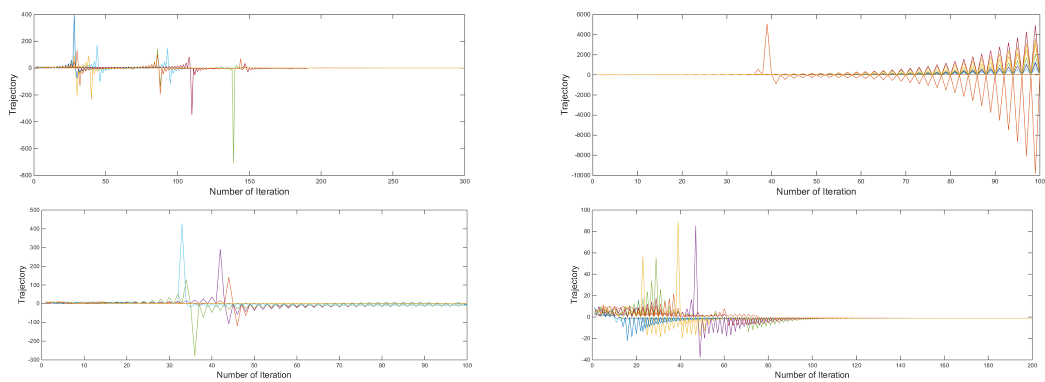


Figure 3: Trajectory plots attracting (Sl No 1) (Above: Left: k is odd, Right: k is even) and repelling (Sl. No. 2) (below: Left: k is odd, Right: k is even) for two different set of parameters with different kinds of delay k.

So far we have seen that fixed point ($\hbar$) either attracting or repelling and forming periodic cycle according to type of delay k. Now we wish to have examples where the trajectory repelling (either unbounded or chaotic) any different kinds of the delay k. Such examples are given in the following. *Table 3*.

Sl. No.	**Parameters** α, γ, β, δ & k	**Remark**
1	$\alpha = -70, \gamma = -68, \beta = 63, \delta = 65$	If the delay k is odd, then the trajectories (for ten different initial values) are repelling and seems chaotic which is shown *Fig. 3*. If the delay k is even then the trajectories are diverging as shown in *Fig. 3*.
2	$\alpha = -66, \gamma = -92, \beta = -63, \delta = -21$	If the delay k is odd, then the trajectories (for ten different initial values) are attracting to a period two cycle which is shown *Fig. 3*. If the delay k is even then the trajectories are chaotic as shown in *Fig. 3*.

Table 3: Parameters α, γ, β and δ for which the fixed points $\hbar$ is attracting, repelling for different initial values with different delay k.

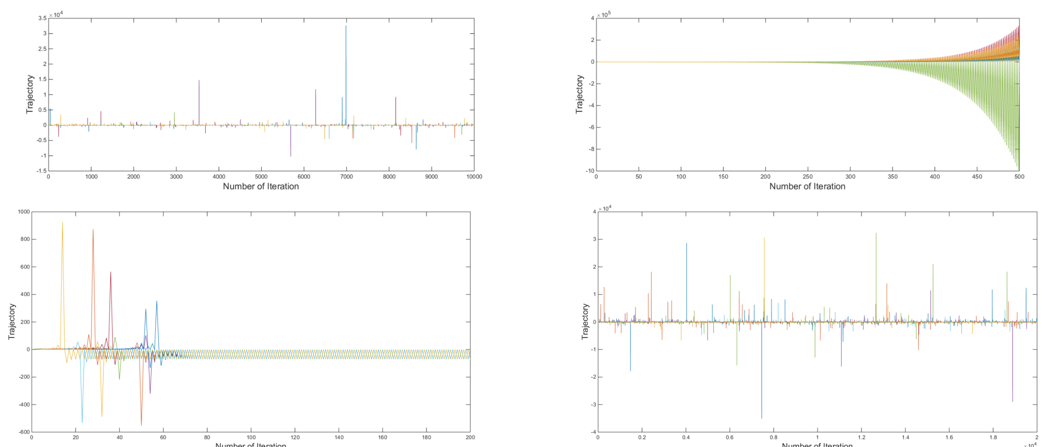


Figure 4: Trajectory plots attracting (Sl No 1) (Above: Left: k is odd, Right: k is even) and repelling (Sl. No. 2) (below: Left: k is odd, Right: k is even) for two different sets of parameters with different kinds of delay k.

In the next two sections, we shall explore the diversity (periodic and chaotic) of dynamics of the rational difference equation Eq.(1) considering the delay k as even and odd separately.

4 Periodic Solutions

By now we are convinced that there are fixed points of the rational difference equation Eq.(1) which are repelling and hence the trajectories are either chaotic or periodic. We are interested to explore different higher (if exists) order periodic solution of the difference equation Eq.(1).

It is an obvious remark that if all the parameters except α are zero, then every solution of the rational difference equation Eq.(1) is periodic and the period ($\geq k$) depends on the delay k.

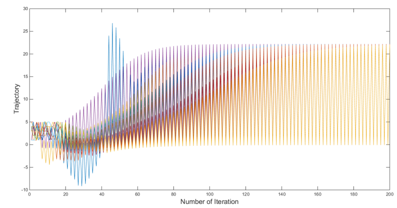
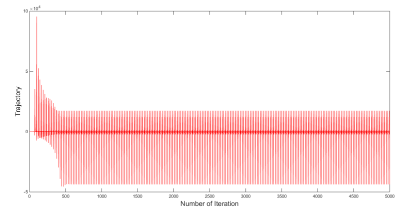
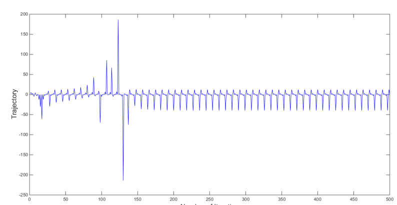
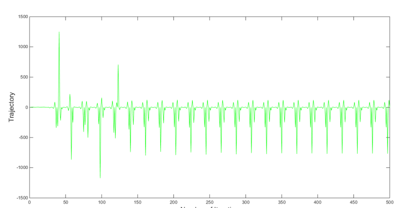
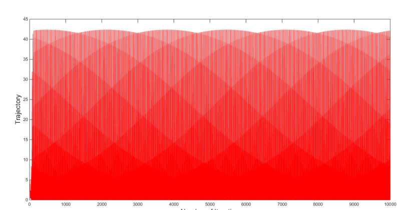
Parameters	**Period**	**Trajectories**
$\alpha = -19, \gamma = 77, \beta = 10, \delta = -26$ for any odd k	2	
$\alpha = 2, \gamma = 0, \beta = 4, \delta = 8,\ k = 17$	24	
$\alpha = -26, \gamma = 83, \beta = 100, \delta = -66,\ k = 4$	9	
$\alpha = 57, \gamma = -52, \beta = 22, \delta = 79,\ k = 13$	22	
$\alpha = 23, \gamma = 71, \beta = 63, \delta = 59,\ k = 14$	24	

Table 4: Parameters, Period and Trajectories with different delay k.

5 Chaotic Solutions

Here we shall explore solutions of the difference equation of type *chaotic* which are neither converging nor forming cycle. A few examples are taken here for demonstration. Over iterations, none of the trajectories for a specific parameter with different initial values is converging/ not forming cycles.

Consider the parameters $\alpha = 9$, $\gamma = 16$, $\beta = -96$, $\delta = -45$ and $k = 28$, the trajectories for many iterations for ten different sets of initial values are seen to be chaotic. The trajectories for 500, 5000, 10000, 80000 and 100000 iterations are shown in *Fig. 4*.

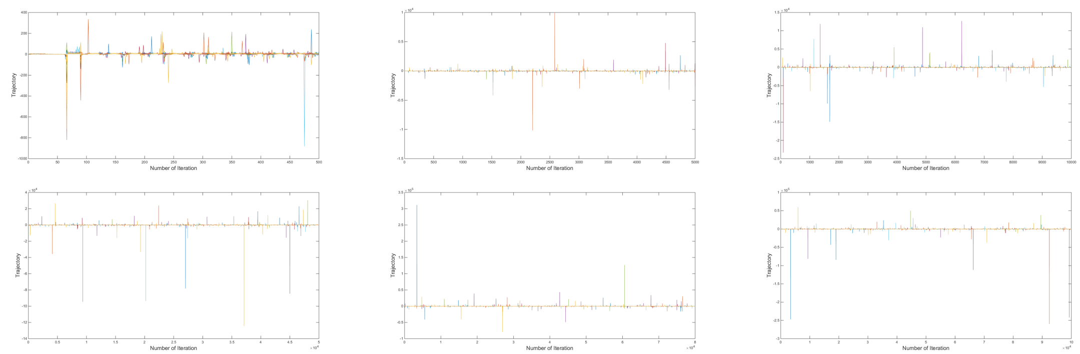


Figure 5: Trajectory plots for 500, 5000, 10000, 80000 and 100000 iterations where delay $k = 28$.

Another similar example is considered here. The parameters $\alpha = -10$, $\gamma = 79$, $\beta = -182$, $\delta = -108$ and $k = 48$, the trajectories for many iterations for ten different set of initial values are seen to be chaotic (ensured by Lyapunav exponent). The trajectories for 500, 5000, 10000, 80000 and 100000 iterations are shown in *Fig. 5*.

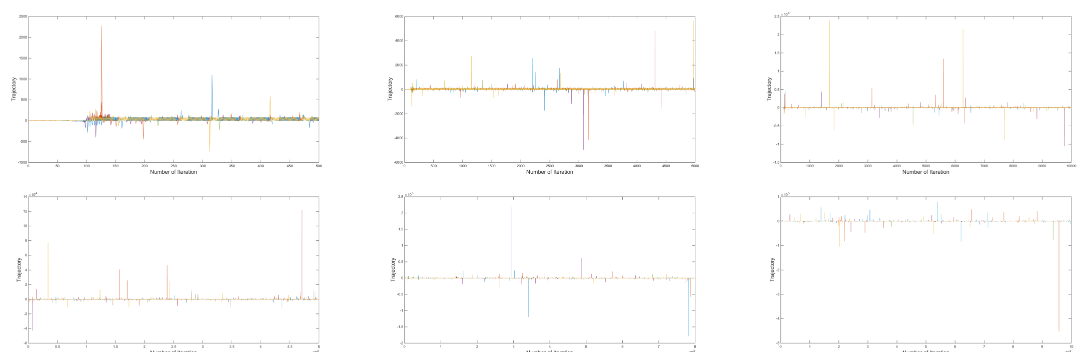


Figure 6: Trajectory plots with for 500, 5000, 10000, 80000 and 100000 iterations where delay $k = 48$.

It is worth noting that the Lyapunav exponent for the two cited examples are found to be positive.
So far we have seen the typical dynamics of the rational difference equation Eq.(1). We shall consider the same equation Eq.(1) but with complex variables (parameters and initial values). In biological context, the rational function can be thought as a response function of a biological system. The variables h_n is a coupled response of two attributes (viz. (temperature, humidity), etc.). The parameters are also acting as coupled two attributes which are associated to the system variables attribute-wise.

In the following section, we shall explore the dynamics of the equation Eq.(1) in complex plane and see the rationale of the system in coupled scenario.

6 A Glimpse of Coupled Dynamics

Here a set of examples is depicted with variety of dynamical behavior of the rational difference equation Eq.(1) where the variables and parameters are complex (having two attributes). In the following *Table-5, 6 & 7*, the attracting, periodic and chaotic dynamics are shown.

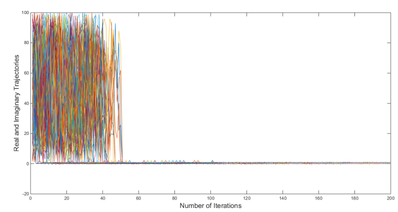
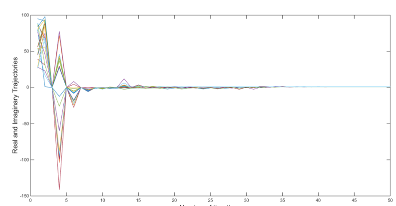
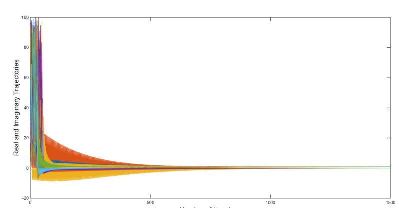
Parameters	**Attracting Nature**	**Trajectories**
$\alpha = 0.706 + 0.6451i, \gamma = 0.5523 + 0.2181i, \beta = 0.7724 + 0.228i, \delta = 0.3709 + 0.89909i$ for any k	Attracting to $0.7962 - 0.0807i$	
$\alpha = 5 - 94i, \gamma = -15 - 67i, \beta = -31 + 24i, \delta = -84 + 25i,\ k = 1$	Attracting to one of its fixed points $0.800033 + 0.859814i$	
$\alpha = 82 - 43i, \gamma = 82 - 43i, \beta = 82 - 43i, \delta = 82 - 43i$, for any k	Attracting to a fixed point	

Table 5: Parameters and Attracting trajectories with different delay k.

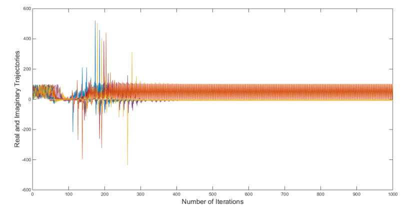
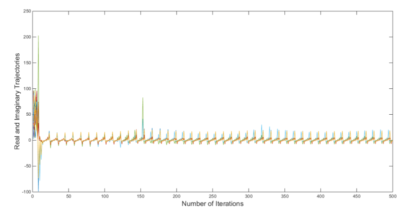
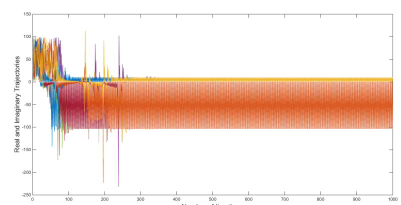
Parameters	Period	Trajectories
$\alpha = 86 - 23i, \gamma = 53 - 90i, \beta = 75 + 44i, \delta = 22 - 88i$ for any odd k	Attracting to period 2 cycle.	
$\alpha = 5 - 94i, \gamma = -15 - 67i, \beta = -31 + 24i, \delta = -84 + 25i,\ k = 5$	Attracting to one of its period 11 cycle	
$\alpha = 34 + 38i, \gamma = -70 + 74i, \beta = 45 + 48i, \delta = 44 + 81i$, for any even k	Attracting to percoid 2 cycle.	

Table 6: Parameters, Periods and Trajectories with different delay k.

The trajectories shown in the *Table-7* are chaotic and that have been confirmed by the positive Lyapunav exponent of the complex(real and imaginary) trajectories [12].

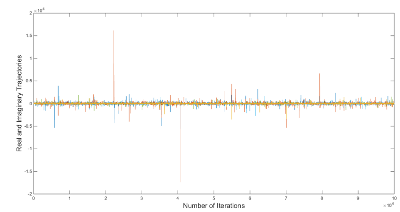
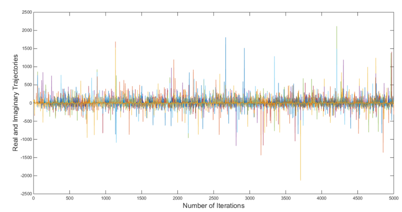
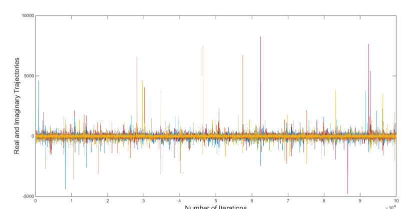
Parameters	**Nature**	**Trajectories**
$\alpha = 86 - 23i, \gamma = 53 - 90i, \beta = 75 + 44i, \delta = 22 - 88i$ for any even k	Chaotic, (Lyapunav Exponent of the real and imaginary trajectories: $(1.345, 0.563)$)	
$\alpha = 5 - 94i, \gamma = -15 - 67i, \beta = -31 + 24i, \delta = -84 + 25i$, $k = 11, 13, 15, 17$ and 19	Chaotic, (Lyapunav Exponent of the real and imaginary trajectories: $(0.5643, 0.4321)$)	
$\alpha = 34 + 38i, \gamma = -70 + 74i, \beta = 45 + 48i, \delta = 44 + 81i$, for any odd k	Chaotic, (Lyapunav Exponent of the real and imaginary trajectories: $(0.7123, 0.98564)$)	

Table 7: Parameters and Chaotic Trajectories with different delay k.

Here we observe the examples as adumbrated in the *Table-5, 6 & 7* and make following remarks from delay k perspective. When $\alpha = 5 - 94i, \gamma = -15 - 67i, \beta = -31 + 24i, \delta = -84 + 25i$ and if the delay k is 1, then the trajectories for different initial values are attracting to a fixed point and while the delay k is 5 then the trajectories are forming periodic cycle of length 11 as shown in the second example of each *Table-5 & 6*. And if the delay $k = 11, 13, 15, 17$ and 19 the trajectories are chaotic which is seen in the *Table 7*.

In another example when the parameters are set to be $\alpha = 34 + 38i, \gamma = -70 + 74i, \beta = 45 + 48i, \delta = 44 + 81i$ then for any even k the trajectories are periodic of period 2 and if the delay k is even then the trajectories are chaotic as shown in *Table-6 & 7*.

7 Special Kind of Complex Dynamics

In this section, we have specified parameters and seen the special kind of dynamics which ar shown in the *Table 8*.

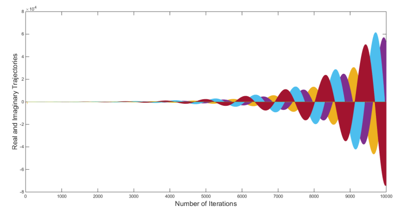
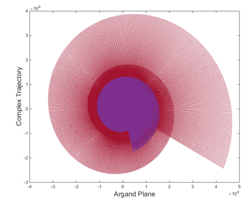
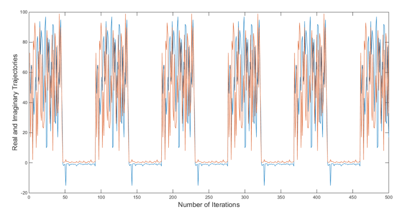
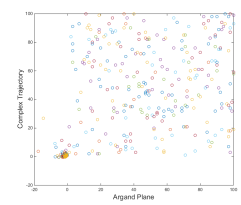
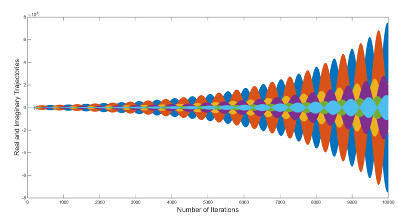
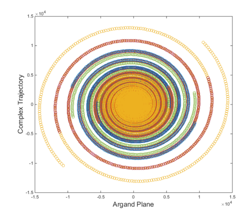
Parameters	Nature	Trajectories
$\alpha = -13-87i, \gamma = -13-87i, \beta = -13-87i, \delta = -13-87i$ for any k and then the trajectories are diverging monotonically		
$\alpha = -95-17i, \gamma = 0, \beta = 0, \delta = 0,$ for any k, trajectory is converging to a periodic cycle of high length.		
$\alpha = 34+38i, \gamma = -\alpha, \beta = -\alpha, \delta = -\alpha$, for any even k Gradually diverging		

Table 8: Special Cases of Complex Parameters and Trajectories with different delay k.

8 Special Kind of Real Dynamics

In this section, we have specified parameters and seen the special kind of dynamics which ar shown in the *Table 8*.

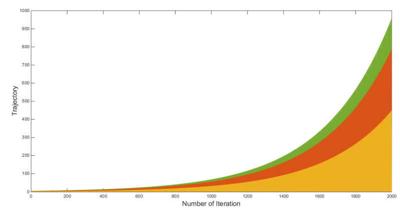
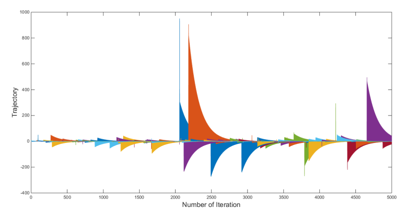
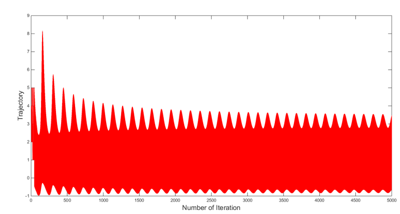
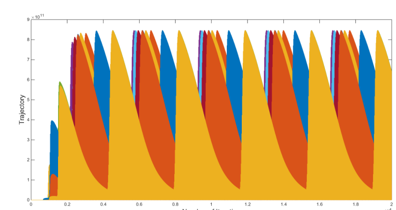
Parameters	**Nature**
$\alpha = -170, \gamma = \alpha, \beta = -191, \delta = \beta$ for any $k = 2$ and then the trajectories are gradually diverging monotonically	
$\alpha = -56, \gamma = \alpha, \beta = 70, \delta = \beta$ for any $k = 2$ and then the trajectories are chaotic with self similarity (fractal-like).	
$\alpha = 95, \gamma = \alpha, \beta = -40, \delta = \beta$ for any $k = 43$ and then the trajectory is periodic with high periodicity).	
$\alpha = 53, \gamma = \frac{1}{\alpha}, \beta = 1, \delta = \frac{1}{\alpha}$ for any $k = 15$ and then the trajectories are quasi-periodic with very high quasi-periodicity).	

Table 9: Special Cases of Real Parameters and Trajectories with different delay k.

9 Summary and Future Endeavours

From the very computational evidences as seen in this article, we are almost surely convinced that the dynamics of the rational difference equation Eq.(1) is rich with typical variety of dynamics starting from attracting trajectories, higher order periodic solutions to chaotic trajectories. The delay term k governs the dynamical system and *'even' & 'odd'* character of the delay term control the behavior as seen in the entire set of examples. We understand that the typical mathematical stability analysis (which is an exercise!) would give more concrete impressions about the dynamics but the computational evidence is more close to the *happenings!* and hence we had urged to experience it. In comprehending the dynamics

of biological systems, the study of rational difference equations is still in its infancy. Although in last two decades, the theoretical understanding of second and third order rational difference equations are sufficiently enriched.
In our future endeavours, we wish to see the dynamics of the most generalized rational difference equation

$$h_{n+1} = \frac{\alpha + \gamma h_{n-k}}{\beta h_n + \delta h_n h_{n-l} + h_{n-l}}, n = 0, 1, 2, \ldots$$

where k and l are two different delay terms and it indeed demands similar computational analysis.

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