

Idempotent generators of quadratic residue cyclic codes of length $4p^nq^m$

Dr. Ranjeet Singh*, Monika[#]

*Department of Mathematics, Govt. College Siwani, Haryana, India

#Department of Mathematics, Govt. College Matlaura, Haryana, India

Abstract

We consider the ring $R_{4p^nq^m} = GF(1)[x] / (x^{4p^nq^m} - 1)$, where p, q and ℓ be distinct odd primes (ℓ is of the type 4k+1), ℓ is quadratic residue modulo $2p^n$ as well as modulo $2q^m$ ($n, m \geq 1$) with $\gcd(\phi(2p^n)/2, \phi(2q^m)/2) = 1$. Explicit expressions for all the $8mn+8n+8m+4$ primitive idempotents are obtained.

MSC:

Primary 11T30

Secondary 94B15, 11T71, 20C05, 16S34.

Keywords: primitive idempotents; quadratic residue; cyclotomic cosets; cyclic codes.

1. Introduction

Let $GF(\ell)$ be a field of odd prime order ℓ of the type 4k+1. Let $\eta \geq 1$ be an integer with $\gcd(\ell, \eta) = 1$. Let $R_\eta = GF(\ell)[x] / (x^\eta - 1)$. The minimal cyclic codes of length η over $GF(\ell)$ are ideals of the ring R_η generated by the primitive idempotents. Arora and Pruthi [1,2] obtained the primitive idempotents in R_η For $\eta = 2, 4, p^n, 2p^n$ where p is an odd prime and ℓ is primitive root mod η . When $\eta = p^nq$ where p, q are distinct odd primes and ℓ is a primitive root mod p^n and q both with $\gcd(\phi(p^n)/2, \phi(q)/2) = 1$, the primitive idempotent in R_η have been obtained by, G.K.Bakshi and Madhu Raka [4]. $\eta = p^nq^m$, where p, q and ℓ be distinct odd primes (ℓ is of the type 4k+1), ℓ is quadratic residue modulo p^n as well as modulo q^m ($n, m \geq 1$) with $\gcd(\phi(p^n)/2, \phi(q^m)/2) = 1$, the primitive idempotent in R_η have been obtained by, Ranjeet Singh and Manju Pruthi [5]. $\eta = 2p^nq^m$, where p, q and ℓ be distinct odd primes, ℓ is quadratic residue modulo $2p^n$ as well as modulo q^m ($n, m \geq 1$) with $\gcd(\phi(p^n)/2, \phi(q^m)/2) = 1$, the primitive idempotent in R_η have been obtained by, Ranjeet Singh [6]. In this paper, we consider the case when $\eta = 4p^nq^m$ where p, q and ℓ be distinct odd primes (ℓ is of the type 4k+1), ℓ is quadratic residue modulo $2p^n$ as well as modulo $2q^m$ ($n, m \geq 1$) with $\gcd(\phi(2p^n)/2, \phi(2q^m)/2) = 1$. We obtain explicit expressions for all the $8mn+8n+8m+4$ primitive idempotents in $R_{4p^nq^m}$ (see theorem 2.3).

2. Primitive Idempotents in $R_{4p^nq^m} = GF(1)[x] / (x^{4p^nq^m} - 1)$

2.1. For $0 \leq s \leq \eta - 1$, let $C_s = \{s, sl, sl^2, \dots, sl^{t_s-1}\}$, where t_s is the least positive integer such that

$sl^{t_s} \equiv s \pmod{\eta}$ be the cyclotomic coset containing s , if α denotes a primitive η th root of unity in some extension field of $GF(\ell)$ then the polynomial $M^s(x) = \prod_{i \in C_s} (x - \alpha^i)$ is the minimal polynomial of α^s over

$GF(\ell)$. Let M_s be the minimal ideal in R_η generated by $\frac{x^\eta - 1}{M^s(x)}$ and θ_s be the primitive idempotent of M_s

then we know by (Theorem1, [4]) the primitive idempotent θ_s corresponding to the cyclotomic coset C_s containing s in $R_{4p^nq^m}$ is given by $\theta_s = \sum_{i=0}^{4p^nq^m-1} \varepsilon_i x^i$, where $\varepsilon_i = \frac{1}{4p^nq^m} \sum_{j \in C_s} \alpha^{-ij} \quad \forall i \geq 0$. Thus to describe θ_s it

becomes necessary to compute ε_i . To compute ε_i numerically, we consider the case when $-C_1 = C_{3ab}$ and we get that

$$\varepsilon_i = \frac{1}{4p^nq^m} \sum_{j \in C_s} \alpha^{-ij} = \frac{1}{4p^nq^m} \sum_{j \in C_{3s}} \alpha^{ij} \quad \forall i \geq 0.$$

Lemma2.2. Let p, q, l be distinct odd primes (l is of the type $4k+1$) and ℓ is quadratic residue modulo $2p^n$ as well as modulo $2q^m$ ($n, m \geq 1$) with $\gcd(\phi(2p^n)/2, \phi(2q^m)/2) = 1$. Then

$$0(l)_{4p^{n-j}q^{m-k}} = \frac{\phi(4p^{n-j}q^{m-k})}{8}, \text{ for all } j, k, 0 \leq j \leq n-1, 0 \leq k \leq m-1.$$

Theorem2.3. The $8mn + 8n + 8m + 4$ primitive idempotents corresponding to cyclotomic cosets

$$\begin{aligned} & C_o, C_{p^nq^m}, C_{2p^nq^m}, C_{3p^nq^m}, C_{p^nq^m}, C_{p^nq^m}, C_{2p^nq^j}, C_{3p^nq^j}, C_{4p^nq^j}, C_{bp^nq^j}, C_{2bp^nq^j}, C_{3bp^nq^j}, \\ & C_{4bp^nq^j}, C_{p^iq^m}, C_{2p^iq^m}, C_{3p^iq^m}, C_{4p^iq^m}, C_{ap^iq^m}, C_{2ap^iq^m}, C_{3ap^iq^m}, C_{4ap^iq^m}, C_{p^iq^j}, C_{2p^iq^j}, \\ & , C_{3p^iq^j}, C_{4p^iq^j}, C_{ap^iq^j}, C_{2ap^iq^j}, C_{3ap^iq^j}, C_{4ap^iq^j}, C_{bp^iq^j}, C_{2bp^iq^j}, C_{3bp^iq^j}, C_{4bp^iq^j}, C_{abp^iq^j}, \\ & C_{2abp^iq^j}, C_{3abp^iq^j}, C_{4abp^iq^j}, \end{aligned}$$

$0 \leq j \leq n-1, o \leq k \leq m-1$ in $R_{4p^nq^m}$ are

$$(i) \quad \theta_0(x) = \frac{1}{4p^nq^m} (1 + x + x^2 + \dots + x^{4p^nq^m-1}).$$

(ii)

$$\begin{aligned} \theta_{p^nq^m}(x) = & \frac{1}{4p^nq^m} \left\{ \sum_{(i,r)=(0,0)}^{(n,m)} (\sigma_{4(i,r)}(x) + \sigma_{4a(i,r)}(x) - \sigma_{2(i,r)}(x) - \sigma_{2a(i,r)}(x)) + \right. \\ & \sum_{(i,r)=(0,0)}^{(n,m)} (\sigma_{4ab(i,r)}(x) + \sigma_{4b(i,r)}(x) - \sigma_{2b(i,r)}(x) - \sigma_{2ab(i,r)}(x)) + \\ & \sum_{(i,r)=(0,0)}^{(n,m)} i(\sigma_{3(i,r)}(x) + \sigma_{3a(i,r)}(x) - \sigma_{(i,r)}(x) - \sigma_{a(i,r)}(x)) \\ & \left. + \sum_{(i,r)=(0,0)}^{(n,m)} i(\sigma_{3ab(i,r)}(x) + \sigma_{3b(i,r)}(x) - \sigma_{ab(i,r)}(x) - \sigma_{b(i,r)}(x)) \right\} \end{aligned}$$

(iii) Replacing $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $-\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $-\sigma_{(i,r)}(x)$ and

we get the required expression for $\theta_{3p^nq^m}(x)$.

(iv) Replacing $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $-\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $\sigma_{(i,r)}(x)$ and

we get the required expression for $\theta_{3p^nq^m}(x)$.

(v) For $0 \leq k \leq m-1$,

$$\begin{aligned} \theta_{p^nq^k}(x) = & \frac{1}{4p^nq^{k+1}} \left\{ \sum_{(i,r)=(0,0)}^{(n-1,m-k-1)} \eta_0^*(\sigma_{(i,r)}(x) + \sigma_{ab(i,r)}(x) + \sigma_{3(i,r)}(x) + \sigma_{3ab(i,r)}(x)) \right. \\ & + \sum_{(i,r)=(0,0)}^{(n-1,m-k-1)} \xi_1^*(\sigma_{2(i,r)}(x) + \sigma_{2ab(i,r)}(x) + \sigma_{4(i,r)}(x) + \sigma_{4ab(i,r)}(x)) \\ & + \sum_{(i,r)=(0,0)}^{(n-1,m-k-1)} \eta_1^*(\sigma_{a(i,r)}(x) + \sigma_{b(i,r)}(x) + \sigma_{3a(i,r)}(x) + \sigma_{3b(i,r)}(x)) \\ & + \sum_{(i,r)=(0,0)}^{(n-1,m-k-1)} \xi_0^*(\sigma_{2a(i,r)}(x) + \sigma_{2b(i,r)}(x) + \sigma_{4a(i,r)}(x) + \sigma_{4b(i,r)}(x)) \\ & + \frac{\phi(q^{m-k})}{8p^nq^m} \left\{ \sum_{(i,r)=(0,m-k)}^{(n-1,m-1)} (\sigma_{4(i,r)}(x) + \sigma_{4ab(i,r)}(x) - \sigma_{2(i,r)}(x) - \sigma_{2ab(i,r)}(x)) \right. \\ & + \sum_{(i,r)=(0,m-k)}^{(n-1,m-1)} (\sigma_{4a(i,r)}(x) + \sigma_{4b(i,r)}(x) - \sigma_{2a(i,r)}(x) - \sigma_{2b(i,r)}(x)) \\ & + \sum_{(i,r)=(0,m-k)}^{(n-1,m-1)} i(\sigma_{3(i,r)}(x) + \sigma_{3ab(i,r)}(x) - \sigma_{(i,r)}(x) - \sigma_{ab(i,r)}(x)) \\ & + \sum_{(i,r)=(0,m-k)}^{(n-1,m-1)} i(\sigma_{3a(i,r)}(x) + \sigma_{3b(i,r)}(x) - \sigma_{a(i,r)}(x) - \sigma_{b(i,r)}(x)) \\ & + \sum_{(i,r)=(n,m-k)}^{(n,m-1)} (\sigma_{4(i,r)}(x) + \sigma_{4b(i,r)}(x) - \sigma_{2(i,r)}(x) - \sigma_{2b(i,r)}(x)) \\ & + \sum_{(i,r)=(n,m-k)}^{(n,m-1)} i(\sigma_{3(i,r)}(x) + \sigma_{3b(i,r)}(x) - \sigma_{(i,r)}(x) - \sigma_{b(i,r)}(x)) \\ & \left. + \sum_{(i,r)=(0,m)}^{(n-1,m)} [\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))] \right\} \end{aligned}$$

$$+ \sum_{(i,r)=(0,m)}^{(n-1,m)} [\sigma_{4a(i,r)}(x) - \sigma_{2a(i,r)}(x) + i(\sigma_{3a(i,r)}(x) - \sigma_{a(i,r)}(x))] \\ + (\sigma_{4(n,m)}(x) - \sigma_{2(n,m)}(x) + i(\sigma_{3(n,m)}(x) - \sigma_{(n,m)}(x)))$$

(vi) Similarly (v), Replacing η_0^* by ξ_0^* and ξ_1^* by ξ_0^* and η_1^* by ξ_0^* and ξ_0^* by ξ_1^* and η_1^* by ξ_1^* , $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $-\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $-\sigma_{(i,r)}(x)$ and we get the required expression for $\theta_{2p^nq^k}(x)$.

(vii) Similarly (v) Replacing ξ_1^* by ξ_0^* and ξ_0^* by ξ_1^* , $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $-\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $\sigma_{(i,r)}(x)$ and we get the required expression for $\theta_{3p^nq^k}(x)$.

(viii) Similarly (v) Replacing η_0^* by ξ_0^* and ξ_1^* by ξ_0^* and η_1^* by ξ_1^* and ξ_0^* by ξ_1^* , $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $\sigma_{(i,r)}(x)$ and we get the required expression for $\theta_{4p^nq^k}(x)$.

(ix) Similarly (v), Replacing η_0^* by η_1^* and ξ_1^* by ξ_0^* in $\theta_{p^nq^k}(x)$ and we get the required expression for $\theta_{bp^nq^k}(x)$.

(x) Similarly (vi), Replacing ξ_1^* by ξ_0^* and ξ_0^* by ξ_1^* in $\theta_{2p^nq^k}(x)$ and we get the required expression for $\theta_{2bp^nq^k}(x)$.

(xi) Similarly (vii), Replacing η_0^* by η_1^* and ξ_0^* by ξ_1^* and η_1^* by η_0^* and ξ_1^* by ξ_0^* in $\theta_{3p^nq^k}(x)$ and we get the required expression for $\theta_{3bp^nq^k}(x)$.

(xii) Similarly (viii), Replacing ξ_1^* by ξ_0^* and ξ_0^* by ξ_1^* in $\theta_{4p^nq^k}(x)$ and we get the required expression for $\theta_{4bp^nq^k}(x)$.

(xiii) For $0 \leq j \leq n-1$

$$\theta_{p^j q^m}(x) = \frac{1}{4p^{j+1}q^m} \left\{ \sum_{r=0}^{m-1} \eta_0(\sigma_{(n-j-1,r)}(x) + \sigma_{3(n-j-1,r)}(x) + \sigma_{ab(n-j-1,r)}(x) + \sigma_{3ab(n-j-1,r)}(x)) \right. \\ \left. + \sum_{r=0}^{m-1} \xi_1(\sigma_{2(n-j-i,r)}(x) + \sigma_{4(n-j-i,r)}(x) + \sigma_{2ab(n-j-i,r)}(x) + \sigma_{4ab(n-j-i,r)}(x)) \right. \\ \left. + \sum_{r=0}^{m-1} \eta_1(\sigma_{a(n-j-i,r)}(x) + \sigma_{b(n-j-i,r)}(x) + \sigma_{3a(n-j-i,r)}(x) + \sigma_{3b(n-j-i,r)}(x)) \right. \\ \left. + \sum_{r=0}^{m-1} \xi_0(\sigma_{2a(n-j-i,r)}(x) + \sigma_{4a(n-j-i,r)}(x) + \sigma_{2b(n-j-i,r)}(x) + \sigma_{4b(n-j-i,r)}(x)) \right\} \\ + \frac{\phi(p^{n-j})}{8p^nq^m} \left\{ \sum_{(i,r)=(n-j,0)}^{(n-1,m-1)} [\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))] \right\}$$

$$\begin{aligned}
 & + \sum_{(i,r)=(n-j,0)}^{(n-1,m-1)} [\sigma_{4ab(i,r)}(x) - \sigma_{2ab(i,r)}(x) + i(\sigma_{3ab(i,r)}(x) - \sigma_{ab(i,r)}(x))] \\
 & + \sum_{(i,r)=(n-j,0)}^{(n-1,m-1)} [\sigma_{4a(i,r)}(x) - \sigma_{2a(i,r)}(x) + i(\sigma_{3a(i,r)}(x) - \sigma_{a(i,r)}(x))] \\
 & + \sum_{(i,r)=(n-j,0)}^{(n-1,m-1)} [\sigma_{4b(i,r)}(x) - \sigma_{2b(i,r)}(x) + i(\sigma_{3b(i,r)}(x) - \sigma_{b(i,r)}(x))] \\
 & + \sum_{r=0}^{(m-1)} [\sigma_{4(n,r)}(x) - \sigma_{2(n,r)}(x) + i(\sigma_{3(n,r)}(x) - \sigma_{(n,r)}(x))] \\
 & + \sum_{r=0}^{(m-1)} [\sigma_{4b(n,r)}(x) - \sigma_{2b(n,r)}(x) + i(\sigma_{3b(n,r)}(x) - \sigma_{b(n,r)}(x))] \\
 & + \sum_{(i,r)=(n-j,m)}^{(n-1,m)} [\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))] \\
 & + \sum_{(i,r)=(n-j,m)}^{(n-1,m)} [\sigma_{4a(i,r)}(x) - \sigma_{2a(i,r)}(x) + i(\sigma_{3a(i,r)}(x) - \sigma_{a(i,r)}(x))] \\
 & + (\sigma_{4(n,m)}(x) - \sigma_{2(n,m)}(x) + i(\sigma_{3(n,m)}(x) - \sigma_{(n,m)}(x)))
 \end{aligned}$$

(xiv) Similarly (xiii), Replacing $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $\sigma_{(i,r)}(x)$

and ξ_0 by ξ_1 , η_1 by ξ_1 , η_0 by ξ_0 , ξ_1 by ξ_0 , η_1 by ξ_0 , we get the required expression for $\theta_{2p^j q^m}(x)$.

(xv) Similarly (xiii) Replacing $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $\sigma_{(i,r)}(x)$

and ξ_0 by ξ_1 , ξ_1 by ξ_0 , we get the required expression for $\theta_{3p^j q^m}(x)$.

(xvi) Similarly (xiii) Replacing $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $-\sigma_{3(i,r)}(x)$ by $\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $\sigma_{(i,r)}(x)$ and

ξ_0 by ξ_1 , η_1 by ξ_1 , η_0 by ξ_0 , ξ_1 by ξ_0 , η_1 by ξ_1 we get the required expression for $\theta_{4p^j q^m}(x)$.

(xvii) Similarly (xiii) Replacing η_0 by η_1 , ξ_1 by ξ_0 , we get the required expression for $\theta_{ap^j q^m}(x)$.

(xviii) Similarly (xiv) Replacing ξ_0 by ξ_1 , ξ_1 by ξ_0 , we get the required expression for $\theta_{2ap^j q^m}(x)$.

(xix) Similarly (xv) Replacing η_0 by η_1 , ξ_0 by ξ_1 , η_1 by η_0 , ξ_1 by ξ_0 , we get the required expression

for $\theta_{3ap^j q^m}(x)$.

(xx) Similarly (xvi) Replacing ξ_0 by ξ_1 , ξ_1 by ξ_0 , we get the required expression for $\theta_{4ap^j q^m}(x)$.

(xi) For $0 \leq j \leq n-1$, $0 \leq k \leq m-1$

$$\begin{aligned}
 \theta_{p^j q^k}(x) = & \frac{1}{4p^n q^m} \left\{ \frac{1}{p^j q^k} \sum_{(i,r)=(0,0)}^{(n-j-1,m-k-1)} [C_{(i+j,r+k)}(\sigma_{(i,r)}(x) + \sigma_{ab(i,r)}(x)) + B_{(i+j,r+k)}^*(\sigma_{2(i,r)}(x) + \sigma_{2ab(i,r)}(x))] \right. \\
 & A_{(i+j,r+k)}(\sigma_{3(i,r)}(x) + \sigma_{3ab(i,r)}(x)) + D_{(i+j,r+k)}^*(\sigma_{4(i,r)}(x) + \sigma_{4ab(i,r)}(x))] \\
 & + \frac{1}{p^j q^k} \sum_{(i,r)=(0,0)}^{(n-j-1,m-k-1)} [D_{(i+j,r+k)}(\sigma_{a(i,r)}(x) + \sigma_{b(i,r)}(x)) + A_{(i+j,r+k)}^*(\sigma_{2a(i,r)}(x) + \sigma_{2b(i,r)}(x))] \\
 & + B_{(i+j,r+k)}(\sigma_{3a(i,r)}(x) + \sigma_{3b(i,r)}(x)) + C_{(i+j,r+k)}^*(\sigma_{4a(i,r)}(x) + \sigma_{4b(i,r)}(x))] \\
 & + \frac{\phi(p^{n-j})q^{m-k-1}}{2} \left[\sum_{(i,r)=(n-j,m-k-1)}^{(n-1,m-k-1)} (\sigma_{(i,m-k-1)}(x) + \sigma_{2(i,m-k-1)}(x) + \sigma_{3(i,m-k-1)}(x) + \sigma_{4(i,m-k-1)}(x)) \right. \\
 & + \sum_{(i,r)=(n-j,m-k-1)}^{(n-1,m-k-1)} (\sigma_{a(i,m-k-1)}(x) + \sigma_{2a(i,m-k-1)}(x) + \sigma_{3a(i,m-k-1)}(x) + \sigma_{4a(i,m-k-1)}(x)) \\
 & + \sum_{(i,r)=(n-j,m-k-1)}^{(n-1,m-k-1)} (\sigma_{b(i,m-k-1)}(x) + \sigma_{2b(i,m-k-1)}(x) + \sigma_{3b(i,m-k-1)}(x) + \sigma_{4b(i,m-k-1)}(x)) \\
 & + \sum_{(i,r)=(n-j,m-k-1)}^{(n-1,m-k-1)} (\sigma_{ab(i,m-k-1)}(x) + \sigma_{2ab(i,m-k-1)}(x) + \sigma_{3ab(i,m-k-1)}(x) + \sigma_{4ab(i,m-k-1)}(x))] \\
 & + p^{n-j-1} \frac{\phi(q^{m-k})}{2} \left[\sum_{(i,r)=(n-j-1,m-k)}^{(n-j-1,m-1)} (\sigma_{(n-j-1,r)}(x) + \sigma_{2(n-j-1,r)}(x) + \sigma_{3(n-j-1,r)}(x) + \sigma_{4(n-j-1,r)}(x)) \right. \\
 & + \sum_{(i,r)=(n-j-1,m-k)}^{(n-j-1,m-1)} (\sigma_{a(n-j-1,r)}(x) + \sigma_{2a(n-j-1,r)}(x) + \sigma_{3a(n-j-1,r)}(x) + \sigma_{4a(n-j-1,r)}(x)) \\
 & + \sum_{(i,r)=(n-j-1,m-k)}^{(n-j-1,m-1)} (\sigma_{b(n-j-1,r)}(x) + \sigma_{2b(n-j-1,r)}(x) + \sigma_{3b(n-j-1,r)}(x) + \sigma_{4b(n-j-1,r)}(x)) \\
 & + \sum_{(i,r)=(n-j-1,m-k)}^{(n-j-1,m-1)} (\sigma_{ab(n-j-1,r)}(x) + \sigma_{2ab(n-j-1,r)}(x) + \sigma_{3ab(n-j-1,r)}(x) + \sigma_{4ab(n-j-1,r)}(x))] + \\
 & \frac{\phi(4p^{n-j}q^{m-k})}{4} \left[\sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)} (\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))) \right. \\
 & + \sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)} (\sigma_{4a(i,r)}(x) - \sigma_{2a(i,r)}(x) + i(\sigma_{3a(i,r)}(x) - \sigma_{a(i,r)}(x))) \\
 & + \sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)} (\sigma_{4b(i,r)}(x) - \sigma_{2b(i,r)}(x) + i(\sigma_{3b(i,r)}(x) - \sigma_{b(i,r)}(x))) \\
 & + \sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)} (\sigma_{4ab(i,r)}(x) - \sigma_{2ab(i,r)}(x) + i(\sigma_{3ab(i,r)}(x) - \sigma_{ab(i,r)}(x))) +
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{(i,r)=(n,m-k)}^{(n,m-1)} (\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))) \\
 & + \sum_{(i,r)=(n,m-k)}^{(n,m-1)} (\sigma_{4b(i,r)}(x) - \sigma_{2b(i,r)}(x) + i(\sigma_{3b(i,r)}(x) - \sigma_{b(i,r)}(x))) + \\
 & \sum_{(i,r)=(n-j,m)}^{(n-1,m)} (\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))) + \\
 & \sum_{(i,r)=(n-j,m)}^{(n-1,m)} (\sigma_{4a(i,r)}(x) - \sigma_{2a(i,r)}(x) + i(\sigma_{3a(i,r)}(x) - \sigma_{a(i,r)}(x))) + \sigma_{4(n,m)}(x) - \sigma_{2(n,m)}(x) \\
 & + i(\sigma_{3(n,m)}(x) - \sigma_{(n,m)}(x)) \}
 \end{aligned}$$

(xxii) On the similar lines as in (xxi), we can find $\theta_{2p^j q^k}(x)$ by replacing $C_{(i+j,r+k)} = A^*_{(i+j,r+k)}$,

$$\begin{aligned}
 B^*_{(i+j,r+k)} &= C^*_{(i+j,r+k)}, A_{(i+j,r+k)} = B^*_{(i+j,r+k)}, D_{(i+j,r+k)} = B^*_{(i+j,r+k)}, \text{ and } A^*_{(i+j,r+k)} = D^*_{(i+j,r+k)}, B_{(i+j,r+k)} = \\
 A^*_{(i+j,r+k)} &\& \frac{\phi(4p^{n-j}q^{m-k})}{4} = -\frac{\phi(p^{n-j}q^{m-k})}{4} \text{ and vice versa in } \theta_{p^j q^k}(x).
 \end{aligned}$$

(xxiii) On the similar lines as in (xxi), we can find $\theta_{3p^j q^k}(x)$ by replacing $C_{(i+j,r+k)} = A_{(i+j,r+k)}$,

$$\begin{aligned}
 B_{(i+j,r+k)} &= D_{(i+j,r+k)}, B^*_{(i+j,r+k)} = A^*_{(i+j,r+k)}, C^*_{(i+j,r+k)} = D^*_{(i+j,r+k)} \text{ and } -\frac{\phi(4p^{n-j}q^{m-k})}{4} = \frac{\phi(4p^{n-j}q^{m-k})}{4} \text{ vice versa in} \\
 \theta_{p^j q^k}(x).
 \end{aligned}$$

(xxiv) On the similar lines as in (xxi), we can find $\theta_{4p^j q^k}(x)$ by replacing $C_{(i+j,r+k)} = C^*_{(i+j,r+k)}$, $A_{(i+j,r+k)} = B^*_{(i+j,r+k)}$, $B_{(i+j,r+k)} = C^*_{(i+j,r+k)}$, $D_{(i+j,r+k)} = D^*_{(i+j,r+k)}$ and $\frac{\phi(4p^{n-j}q^{m-k})}{4} = \frac{\phi(4p^{n-j}q^{m-k})}{4}$ and viceversa in $\theta_{p^j q^k}(x)$.

(xxv) On the similar lines as in (xxi), we can find $\theta_{ap^j q^k}(x)$ by replacing $C_{(i+j,r+k)} = D_{(i+j,r+k)}$,

$$B^*_{(i+j,r+k)} = A^*_{(i+j,r+k)}, A_{(i+j,r+k)} = B_{(i+j,r+k)}, D^*_{(i+j,r+k)} = C^*_{(i+j,r+k)}, \text{ and viceversa in } \theta_{p^j q^k}(x).$$

(xxvi) On the similar lines as in (xxii), we can find $\theta_{2ap^j q^k}(x)$ by replacing $A^*_{(i+j,r+k)} = B^*_{(i+j,r+k)}$,

$$C^*_{(i+j,r+k)} = D^*_{(i+j,r+k)}, \text{ and viceversa in } \theta_{2p^j q^k}(x).$$

(xxvii) On the similar lines as in (xxiii), we can find $\theta_{3ap^j q^k}(x)$ by replacing $A_{(i+j,r+k)} = B_{(i+j,r+k)}$, $C_{(i+j,r+k)} = D_{(i+j,r+k)}$, $A^*_{(i+j,r+k)} = B^*_{(i+j,r+k)}$, $C^*_{(i+j,r+k)} = D^*_{(i+j,r+k)}$, and viceversa in $\theta_{3p^j q^k}(x)$.

(xxviii) On the similar lines as in (xxiv), we can find $\theta_{4ap^j q^k}(x)$ by replacing $C^*_{(i+j,r+k)} = D^*_{(i+j,r+k)}$ and Viceversa

in $\theta_{4p^j q^k}(x)$.

(xxix) On the similar lines as in (xxv), we can find $\theta_{bp^j q^k}(x)$ by replacing $D_{(i+j,r+k)} = X_{(i+j,r+k)}$, $B_{(i+j,r+k)} = E$

$A^*_{(i+j,r+k)} = Y^*_{(i+j,r+k)}$, $D^*_{(i+j,r+k)} = X^*_{(i+j,r+k)}$, and viceversa in $\theta_{ap^j q^k}(x)$.

(xxx) On the similar lines as in (xxvi), we can find $\theta_{2bp^j q^k}(x)$ by replacing $B^*_{(i+j,r+k)} = E^*_{(i+j,r+k)}$, $D^*_{(i+j,r+k)} = X^*$

$(i+j,r+k)$, $A^*_{(i+j,r+k)} = Y^*_{(i+j,r+k)}$, $C^*_{(i+j,r+k)} = Z^*_{(i+j,r+k)}$ and viceversa in $\theta_{2ap^j q^k}(x)$.

(xxxi) On the similar lines as in (xxvii), we can find $\theta_{3bp^j q^k}(x)$ by replacing $B_{(i+j,r+k)} = E_{(i+j,r+k)}$, $B^*_{(i+j,r+k)} = E^*$

$(i+j,r+k)$, $D_{(i+j,r+k)} = X_{(i+j,r+k)}$, $D^*_{(i+j,r+k)} = X^*_{(i+j,r+k)}$ and viceversa in $\theta_{3ap^j q^k}(x)$.

(xxxii) On the similar lines as in (xxviii), we can find $\theta_{4bp^j q^k}(x)$ by replacing $D^*_{(i+j,r+k)} = X^*_{(i+j,r+k)}$, $C^*_{(i+j,r+k)}$

$= Z^*_{(i+j,r+k)}$ and viceversa in $\theta_{4ap^j q^k}(x)$.

(xxxiii) On the similar lines as in (xxi), we can find $\theta_{abp^j q^k}(x)$ by replacing $C_{(i+j,r+k)} = Z_{(i+j,r+k)}$, $B^*_{(i+j,r+k)} = E^*$

$(i+j,r+k)$, $A_{(i+j,r+k)} = Y_{(i+j,r+k)}$, $D^*_{(i+j,r+k)} = X^*_{(i+j,r+k)}$ and viceversa in $\theta_{pj q^k}(x)$.

(xxxiv) On the similar lines as in (xxii), we can find $\theta_{2abp^j q^k}(x)$ by replacing $A^*_{(i+j,r+k)} = Y^*_{(i+j,r+k)}$, $C^*_{(i+j,r+k)} = Z^*$

$(i+j,r+k)$, $B^*_{(i+j,r+k)} = E^*_{(i+j,r+k)}$, $D^*_{(i+j,r+k)} = X^*_{(i+j,r+k)}$ and viceversa in $\theta_{2pj q^k}(x)$.

(xxxv) On the similar lines as in (xxiii), we can find $\theta_{3abp^j q^k}(x)$ by replacing $A_{(i+j,r+k)} = Y_{(i+j,r+k)}$, $A^*_{(i+j,r+k)} = Y^*$

$(i+j,r+k)$, $D_{(i+j,r+k)} = X_{(i+j,r+k)}$, $C^*_{(i+j,r+k)} = Z^*_{(i+j,r+k)}$ and viceversa in $\theta_{3pj q^k}(x)$.

(xxxvi) On the similar lines as in (xxiv), we can find $\theta_{4abp^j q^k}(x)$ by replacing $C^*_{(i+j,r+k)} = Z^*_{(i+j,r+k)}$, $D^*_{(i+j,r+k)}$

$= X^*_{(i+j,r+k)}$ and viceversa in $\theta_{4pj q^k}(x)$.

where $A_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{-1+r+\gamma+\delta}{4} \right)$, $B_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{-1+r-\delta-\gamma}{4} \right)$

$C_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{-1-r-\gamma+\delta}{4} \right)$, $D_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{-1-r+\gamma-\delta}{4} \right)$

$A^*_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{1+r+\gamma+\delta}{4} \right)$, $B^*_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{1+r-\delta-\gamma}{4} \right)$

$C^*_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{1-r-\gamma+\delta}{4} \right)$, $D^*_{(n-1,m-1)} = p^{n-1} q^{m-1} \left(\frac{1-r+\gamma-\delta}{4} \right)$

$$\eta_0 = \sum_{s=0}^{\frac{\phi(p)}{2}-1} (\alpha^{4p^{n-1}q^m})^s, \quad \eta_1 = \sum_{s=0}^{\frac{\phi(p)}{2}-1} (\alpha^{4p^{n-1}q^m})^{al^s}, \quad \xi_0 = \sum_{s=0}^{\frac{\phi(p)}{2}-1} (\alpha^{p^{n-1}q^m})^{l^s},$$

$$\xi_1 = \sum_{s=0}^{\frac{\phi(p)}{2}-1} (\alpha^{p^{n-1}q^m})^{al^s} \text{ where } r^2 = -q, \quad \gamma^2 = -p, \quad \delta^2 = pq.$$

5. References

- [1] S.K.Arora, M.Pruthi, “Minimal Cyclic Codes of prime power length”, Finite Fields Appl.3 (1997)99-113.
- [2] S.K. Arora, M. Pruthi, “Minimal Cyclic Codes of length $2p^n$ ”, Finite Fields Appl. 5 (1999) 177-187.
- [3] Anuradha Sharma, G.K.Bakshi, V.C. Dumir, M. Raka, “Cyclotomic Numbers and Primitive idempotents in the ring $GF(l)[x]/<x^{p^n} - 1>$ ”, Finite Fields Appl. 10 (2004) 653-673.
- [4] G.K.Bakshi, Madhu Raka, “Minimal cyclic codes of length p^nq ”, Finite Fields Appl.9 (2003) 432-448.
- [5] Ranjeet Singh, M.Pruthi, “Primitive Idempotents of Irreducible Quadratic Residue Cyclic Codes of Length p^nq^m ” International Journal of Algebra vol.5 (2011) 285-294.
- [6] Ranjeet Singh, “Some results in the Ring $R_{2p^nq^m} = GF(l)[x]/(x^{2p^nq^m} - 1)$

International Journal of Mathematics Trends and Technology Vol (35) No.1(2016) 32-37.