Some bistar related square sum graphs

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Abstract

A graph G = (V, E) with p vertices and q edges is said to be square sum graph, if there exists a bijection mapping $f : V(G) \to \{0, 1, 2, \dots, p-1\}$ such that the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. A graph with square sum labeling is called square sum graph. In this paper we prove that restricted square graph, splitting graph and shadow graph of $B_{n,n}$ are square sum. We also prove that restricted total, restricted middle and degree splitting graph of $B_{n,n}$, duplication of vertex and arbitrary super subdivision of $B_{n,n}$ are square sum graph.

Key words: Square sum graphs, Bistar graph. AMS Subject classification number: 05C78.

1 Introduction

Labeled graphs have broad range of applications in different fields such as coding theory, particularly in the development of missile guidance codes, design of radar type codes and convolution codes with optimal autocorrelation properties. According to Beineke and Hegde [8], graph labeling is described as a frontier between number theory and structure of graphs. Labeling in graph was initiated by Rosa[1] in 1967 and an enormous body of literature has grown around the subject, especially in the last three decades, which is regularly updated by J. A. Gallian[5] and it is published by *The Electronic Journal of Combinatorics*.

In this paper we consider simple, finite, connected and undirected graph G = (V, E) with order p and size q is also denoted as G = (p, q) graph. We refer to Bondy and Murty[4] for the standard terminology and notations related to graph theory and David M. Burton [2] for Number theory. Square

sum labeling is defined by Ajitha, Arumugam and Germina[11]. In this section we mention some definitions and preliminaries which are helpful for the present investigations.

Definition 1.1 ([11]). A graph G = (V, E) with p vertices and q edges is said to be a square sum graph if there exists a bijection mapping $f : V(G) \rightarrow \{0, 1, 2, ..., p - 1\}$ such that the induced function $f^* : E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective.

Definition 1.2 ([5]). Bistar $B_{n,n}$ is the graph obtained by joining the center (apex) vertices of two copies of $K_{1,n}$ by an edge. The vertex set of $B_{n,n}$ is $V(B_{n,n}) = \{u, v, u_i, v_i/1 \le i \le n\}$, where u, v are apex vertices and u_i, v_i are pendent vertices. The edge set of $B_{n,n}$ is $E(B_{n,n}) = \{uv, uu_i, vv_i/1 \le i \le n\}$. So, $|V(B_{n,n})| = 2n+2$ and $|E(B_{n,n})| = 2n+1$.

Theorem 1.3 ([11]). Trees are square sum.

Theorem 1.4 ([11]). K_n is square sum if and only if $n \leq 5$.

Theorem 1.5 ([11]). The cycle-cactus $C_k^{(n)}$, consisting of n copies of C_k , concatenated at exactly one vertex, is a square sum.

Theorem 1.6 ([11]). The complete lattice grids $L_{m,n} = P_m \times P_n$ are square sum.

Theorem 1.7 ([7]). Shadow graph of P_n and $K_{1,n}$, split graph of P_n and $K_{1,n}$ are square sum.

Theorem 1.8 ([6]). Total graph of path P_n and cycle C_n , middle graph of path P_n and cycle C_n are square sum.

Theorem 1.9 ([3]). The graph obtained by duplication of any vertex in K_n is a square sum for $n \leq 7$.

Theorem 1.10 ([3]). The Petersen graph P(5,2) is square sum.

Theorem 1.11 ([3]). Cycle C_n with $\left\lfloor \frac{n}{2} \right\rfloor$ concurrent chords is a square sum for all n.

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2 Square Sum Labeling in Context of Some Operations on $B_{n,n}$

Definition 2.1 ([10]). The restricted square of $B_{n,n}$ is a graph G with vertex set $V(G) = V(B_{n,n})$ and edge set $E(G) = E(B_{n,n}) \cup \{uv_i, vu_i/1 \le i \le n\}$.

Theorem 2.2. The restricted square of bistar $B_{n,n}$ is a square sum graph.

Proof. Let G be the restricted square of bistar $B_{n,n}$ with vertex set $V(G) = V(B_{n,n})$ and edge set $E(G) = E(B_{n,n}) \cup \{uv_i, vu_i/1 \leq i \leq n\}$. So, |V(G)| = 2n + 2 and |E(G)| = 4n + 1. We define a bijection $f: V(G) \to \{0, 1, 2, \dots, 2n + 1\}$ as follows.

$$\begin{split} f(u) &= 1, \\ f(u_i) &= 2i; \ 1 \leq i \leq n. \\ f(v) &= 0, \\ f(v_i) &= 2i + 1; \ 1 \leq i \leq n. \end{split}$$

Then there are five types of edge labels in G.

- (1) $\{f^*(uv)\}$ is 1.
- (2) $\{f^*(uu_i), 1 \le i \le n\}$ is in ascending order of the form $4k + 1, k \in N$.
- (3) $\{f^*(vv_i), 1 \le i \le n\}$ is in ascending order of the form $4k + 1, k \in N$.
- (4) $\{f^*(vu_i), 1 \le i \le n\}$ is in ascending order of the form $4k, k \in N$.
- (5) $\{f^*(uv_i), 1 \le i \le n\}$ is in ascending order of the form $4k + 2, k \in N$.

It is clear that labels of types (1), (4) and (5) are distinct.

It is also easy to see that the labels of types (1), (4) and (5) are distinct from the labels of types (2) and (3). The square of odd numbers never equal to sum of square of even number and 1, because if they are equal then there exists some positive integer k and m such that $(2k + 1)^2 = (2m)^2 + 1$. So, $k(k + 1) = m^2$. which is not true [2].

So, labels of the types (2) and (3) are distinct.

Hence labels of above all types are internally as well as externally distinct.

Therefore, from above defined function f, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence restricted square of bistar $B_{n,n}$ is a square sum graph.

Example 2.3. Square sum labeling in restricted square of bistar $B_{5,5}$ is shown in following Figure 1.



Figure 1:

Definition 2.4 ([10]). The splitting graph S'(G) of a graph G is constructed by adding to each vertex v, a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G, i.e. N(v) = N(v').

Theorem 2.5. The splitting graph of bistar $B_{n,n}$ is a square sum graph.

Proof. Let $G = S'(B_{n,n})$ be a splitting graph of bistar $B_{n,n}$ with vertex set $V(G) = V(B_{n,n}) \cup$ $\{u', v', u'_i, v'_i/1 \leq i \leq n\}$ and edge set E(G) = $E(B_{n,n}) \cup \{uv', vu', uu'_i, vv'_i, u'u_i, v'v_i/1 \leq i \leq n\}$. So, |V(G)| = 4n + 4 and |E(G)| = 6n + 3. We define a bijection $f : V(G) \rightarrow \{0, 1, 2 \dots, 4n + 3\}$ as follows.

$$f(u) = 4n + 2,$$

$$f(u') = 0,$$

$$f(u_i) = 2i; 1 \le i \le n,$$

$$f(u'_i) = 2n + 2i; 1 \le i \le n.$$

$$f(v) = 1,$$

$$f(v') = 4n + 3,$$

$$f(v_i) = 2n + 2i + 1; 1 \le i \le n,$$

$$f(v'_i) = 2i + 1; 1 \le i \le n.$$

So, from above defined function f, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence splitting graph of $B_{n,n}$ is square sum. \Box

Example 2.6. Square sum labeling in splitting graph of $B_{5,5}$ is shown in following Figure 2.

Definition 2.7 ([10]). The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G'' and join each vertex u' in G' to the neighbors of the corresponding vertex v' in G''.

Theorem 2.8. The shadow graph of bistar $B_{n,n}$ is a square sum graph.

Proof. Let $G = D_2(B_{n,n})$ be a shadow graph of bistar $B_{n,n}$ with vertex set



Figure 2:

 $\begin{array}{rcl} V(G) &=& V(B_{n,n}) \,\cup\, \{u',v',u'_i,v'_i/1 &\leq i \leq n\} & \text{and edge set } E(G) &=& E(B_{n,n}) \,\cup\, \\ \{u'u_i,uu'_i,u'v,u'v',v'u,v'v_i,vv'_i,u'u'_i,v'v'_i/1 \leq i \leq n\}. & \text{So}, \, |V(G)| = 4n + 4 \text{ and } |E(G)| = 8n + 4. \text{ We define a bijection } f : V(G) \to \{0,1,2\ldots,4n+3\} \text{ as follows.} \end{array}$

$$f(u) = 4n + 2,$$

$$f(u') = 0,$$

$$f(u_i) = 2n + 2i; 1 \le i \le n,$$

$$f(u'_i) = 2i; 1 \le i \le n.$$

$$f(v) = 4n + 3,$$

$$f(v') = 1,$$

$$f(v_i) = 2n + 2i + 1; 1 \le i \le n,$$

$$f(v'_i) = 2i + 1; 1 \le i \le n.$$

So, from above defined function f, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence shadow graph of bistar $B_{n,n}$ is a square sum graph. \Box

Example 2.9. Square sum labeling in shadow graph of $B_{5,5}$ is shown in the following Figure 3.



Figure 3:

Definition 2.10 ([9]). Let G(V, E) be a graph with $V = S_1 \bigcup S_2 \bigcup S_3 \ldots S_t \bigcup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V - \bigcup_{i=1}^t S_i$. The degree splitting graph of G denoted by DS(G) is obtained from G by adding vertices $w_1, w_2, w_3 \ldots w_t$ and joining to each vertex of S_i , for $1 \le i \le t$.

Theorem 2.11. The degree splitting graph of $B_{n,n}$ is a square sum graph.

Proof. Let $G = DS(B_{n,n})$ be the degree splitting graph of $B_{n,n}$ with vertex set $V(G) = V(B_{n,n}) \cup$ $\{w_1, w_2, u_i, v_i/1 \le i \le n\}$ and edge set E(G) = $\{w_1u_i, w_1v_i, uu_i, vv_i, uv, uw_2, vw_2/1 \le i \le n\}$. So, |V(G)| = 2n + 4 and |E(G)| = 4n + 3. We define a bijection $f: V(G) \to \{0, 1, 2, \ldots, 2n + 3\}$ as follows. f(u) = 2n + 2.

$$f(w_i) = 2i; 1 \le i \le n.$$

$$f(v_i) = 2i + 1,$$

$$f(v_i) = 2i - 1; 1 \le i \le n.$$

$$f(w_1) = 0,$$

$$f(w_2) = 2n + 3.$$

So, from above defined function f, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence degree splitting graph of $B_{n,n}$ is a square sum graph. \Box

Example 2.12. Square sum labeling in degree splitting graph of $B_{5,5}$ is shown in the following Figure 4.



Figure 4:

Definition 2.13 ([5]). A graph obtained from given graph G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i , $1 \le i \le q$ is called arbitrary super subdivision of G.

Theorem 2.14. The arbitrary super subdivision of $B_{n,n}$ is a square sum graph.

Proof. Let G be arbitrary super subdivision of $B_{n,n}$ constructed as follows:

- (1) Edge $e_i = (u, u_i)$ is replaced by complete bipartite graph $K_{2,mi}$ and u_{ij} be the vertices of m_i vertex section, $1 \le i \le n$ and $1 \le j \le m_i$.
- (2) Edge $e_i = (v, v_i)$ is replaced by complete bipartite graph $K_{2,ri}$ and v_{ik} be the vertices of r_i vertex section, $1 \le i \le n$ and $1 \le k \le r_i$.
- (3) Edge e = (u, v) is replaced by complete bipartite graph $K_{2,s}$ and w_l be the vertices of svertex section, $1 \le l \le s$.

Let $cm_0 = 0$, where cm_i = cumulative values of m_i and $cr_0 = 0$, where cr_i = cumulative values of r_i for $1 \le i \le n$.

Let $M = \sum_{i=1}^{n} m_i$ and $R = \sum_{i=1}^{n} r_i$. Then V(G) = 2n + 2 + M + s + R, E(G) = 2(M + s + R). We define a bijection $f: V(G) \to \{0, 1, 2, \dots, 2n + M + s + R + 1\}$ as follows.

 $f(u_i) = i - 1; \ 1 \le i \le n,$ $f(u_{ij}) = n - 1 + cm_{i-1} + j; \ 1 \le i \le n \text{ and}$ $1 \le j \le m_i,$ $f(u) = n + cm_n.$

 $f(w_l) = n + cm_n + l; \ 1 \le l \le s.$

$$f(v) = n + cm_n + s + 1,$$

 $f(v_{ik}) = n + cm_n + s + 1 + cr_{i-1} + k; \ 1 \le i \le n$ and $1 \le k \le r_i$,

 $f(v_i) = n + cm_n + s + 1 + cr_n + i; 1 \le i \le n.$ So, from above defined function f, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) =$ $(f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence arbitrary super subdivision of $B_{n,n}$ is a square sum graph. \Box

Example 2.15. Square sum labeling in arbitrary super subdivision of $B_{5,5}$ is shown in the following Figure 5.



Figure 5:

vertices which are adjacent to v are now adjacent to v'.

Theorem 2.17. Duplication of any vertex of bistar $B_{n,n}$ is a square sum graph.

Proof. The following two cases are to be considered for duplication of any vertex of $B_{n,n}$.

- (1) Duplication of any pendent vertex of $B_{n,n}$. In this case, the graph becomes a tree. So, it is a square sum graph[11].
- (2) Duplication of apex vertex of $B_{n,n}$.

Let G be a graph obtained by duplication of apex vertex v (or u) in $B_{n,n}$ with vertex set $V(G) = V(B_{n,n}) \cup \{v'\}$ and edge set E(G) = $E(B_{n,n}) \cup \{v_i v', uv'\}$. So, |V(G)| = 2n + 3and |E(G)| = 3n + 2. We define a bijection $f: V(G) \to \{0, 1, 2..., 2n+2\}$ as follows. f(u) = 1, $f(u_i) = 2i; 1 \le i \le n.$ f(v) = 2n + 2,f(v') = 0, $f(v_i) = 2i + 1; 1 \le i \le n.$ So, from above defined function f, the induced function $f^*: E(G) \to \mathbb{N}$ defined by $f^*(uv) =$ $(f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence graph obtained by duplication of apex vertex in $B_{n,n}$ is a square sum graph.

Example 2.18. Square sum labeling in duplication of apex vertex of $B_{5,5}$ is shown in the following Figure 6.



We define the restricted total graph of $B_{n,n}$ as follows.

Definition 2.16 ([5]). Duplication of a vertex vof a graph G produces a new graph G' by adding a vertex v' with N(v') = N(v). In other words a vertex v' is said to be a duplication of v if all the the edges of $B_{n,n}$ and edge set $= E(B_{n,n}) \cup \{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \le i \le n\}.$

Theorem 2.20. The restricted total graph of $B_{n,n}$ is a square sum graph.

Proof. Let G be a restricted total graph of $B_{n,n}$. $V(G) = V(B_{n,n}) \cup E(B_{n,n}) =$ $\{u, v, w, u_i, v_i, u'_i, v'_i/1 \leq i \leq n\}$, where u and v are apex vertices, u_i and v_i are pendent vertices, w, u'_i and v'_i are vertices related to edges. $E(G) = E(B_{n,n}) \cup$ $\{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \leq i \leq n\}$. So, |V(G)| = 4n + 3 and |E(G)| = 8n + 3. We define a bijection $f: V(G) \rightarrow \{0, 1, 2 \dots, 4n+2\}$ as follows. f(u) = 1,

f(u) = 1, $f(u_i) = 2(i-1); \ 1 \le i \le n,$ $f(u'_i) = 2n + 2(i-1); \ 1 \le i \le n.$ f(v) = 4n + 2, $f(v_i) = 2i + 1; \ 1 \le i \le n,$ $f(v'_i) = 2n + 2i + 1; \ 1 \le i \le n.$ f(w) = 4n.

So, from above defined function f, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence restricted total graph of $B_{n,n}$ is a square sum graph.

Example 2.21. Square sum labeling in restricted total graph of $B_{5,5}$ is shown in the following Figure 7.



Figure 7:

We define the restricted middle graph of $B_{n,n}$ as follows.

Definition 2.22. The restricted middle graph of $B_{n,n}$ is a graph with vertex set $= V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v_i, u'_i, v'_i/1 \leq i \leq n\},$

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where u and v are apex vertices, u_i and v_i are pendent vertices, w, u'_i and v'_i are vertices corresponding to the edges of $B_{n,n}$ and edge set = $\{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \le i \le n\}.$

Corollary 2.23. The restricted middle graph of bistar $B_{n,n}$ is a square sum graph.

Example 2.24. Square sum labeling in restricted middle graph of $B_{6,6}$ is shown in the following Figure 8.



Figure 8:

3 Conclusion

It is very interesting to study graphs which admit square sum labeling. Here we discover some bistar related square sum graphs also we define restricted total and restricted middle graph of bistar graph. To investigate equivalent results for different graph families is an open area of research.

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