# Some bistar related square sum graphs

G. V. Ghodasara [1], Mitesh J. Patel [2]
[1] *Assistant Professor in Mathematics, H. & H. B. Kotak Institute of Science, Rajkot - 360001, Gujarat - INDIA*
*gaurang_enjoy@yahoo.co.in*
[1] *Assistant Professor in Mathematics, Tolani College of Arts and Science , Adipur - Kachchh - 370205, Gujarat - INDIA*
*miteshmaths1984@gmail.com*

## Abstract

A graph $G = (V, E)$ with $p$ vertices and $q$ edges is said to be square sum graph, if there exists a bijection mapping $f : V(G) \rightarrow \{0, 1, 2, \ldots, p-1\}$ such that the induced function $f^* : E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$ , for every $uv \in E(G)$ is injective. A graph with square sum labeling is called square sum graph. In this paper we prove that restricted square graph, splitting graph and shadow graph of $B_{n,n}$ are square sum. We also prove that restricted total, restricted middle and degree splitting graph of $B_{n,n}$, duplication of vertex and arbitrary super subdivision of $B_{n,n}$ are square sum graph.

**Key words:** Square sum graphs, Bistar graph.
**AMS Subject classification number:** 05C78.

## 1 Introduction

Labeled graphs have broad range of applications in different fields such as coding theory, particularly in the development of missile guidance codes, design of radar type codes and convolution codes with optimal autocorrelation properties. According to Beineke and Hegde [8], graph labeling is described as a frontier between number theory and structure of graphs. Labeling in graph was initiated by Rosa[1] in 1967 and an enormous body of literature has grown around the subject, especially in the last three decades, which is regularly updated by J. A. Gallian[5] and it is published by *The Electronic Journal of Combinatorics*.

In this paper we consider simple, finite, connected and undirected graph $G = (V, E)$ with order $p$ and size $q$ is also denoted as $G = (p, q)$ graph. We refer to Bondy and Murty[4] for the standard terminology and notations related to graph theory and David M. Burton [2] for Number theory. Square sum labeling is defined by Ajitha, Arumugam and Germina[11]. In this section we mention some definitions and preliminaries which are helpful for the present investigations.

**Definition 1.1** ([11]). *A graph $G = (V, E)$ with $p$ vertices and $q$ edges is said to be a square sum graph if there exists a bijection mapping $f : V(G) \rightarrow \{0, 1, 2, \ldots, p-1\}$ such that the induced function $f^* : E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective.*

**Definition 1.2** ([5]). *Bistar $B_{n,n}$ is the graph obtained by joining the center (apex) vertices of two copies of $K_{1,n}$ by an edge. The vertex set of $B_{n,n}$ is $V(B_{n,n}) = \{u, v, u_i, v_i / 1 \le i \le n\}$, where $u$, $v$ are apex vertices and $u_i$, $v_i$ are pendent vertices. The edge set of $B_{n,n}$ is $E(B_{n,n}) = \{uv, uu_i, vv_i / 1 \le i \le n\}$. So, $|V(B_{n,n})| = 2n+2$ and $|E(B_{n,n})| = 2n+1$.*

**Theorem 1.3** ([11]). *Trees are square sum.*

**Theorem 1.4** ([11]). *$K_n$ is square sum if and only if $n \le 5$.*

**Theorem 1.5** ([11]). *The cycle-cactus $C_k^{(n)}$, consisting of $n$ copies of $C_k$, concatenated at exactly one vertex, is a square sum.*

**Theorem 1.6** ([11]). *The complete lattice grids $L_{m,n} = P_m \times P_n$ are square sum.*

**Theorem 1.7** ([7]). *Shadow graph of $P_n$ and $K_{1,n}$, split graph of $P_n$ and $K_{1,n}$ are square sum.*

**Theorem 1.8** ([6]). *Total graph of path $P_n$ and cycle $C_n$, middle graph of path $P_n$ and cycle $C_n$ are square sum.*

**Theorem 1.9** ([3]). *The graph obtained by duplication of any vertex in $K_n$ is a square sum for $n \le 7$.*

**Theorem 1.10** ([3]). *The Petersen graph $P(5, 2)$ is square sum.*

**Theorem 1.11** ([3]). *Cycle $C_n$ with $[\frac{n}{2}]$ concurrent chords is a square sum for all $n$.*

# 2 Square Sum Labeling in Context of Some Operations on $B_{n,n}$

**Definition 2.1** ([10]). *The restricted square of $B_{n,n}$ is a graph $G$ with vertex set $V(G) = V(B_{n,n})$ and edge set $E(G) = E(B_{n,n}) \cup \{uv_i, vu_i/1 \leq i \leq n\}$.*

**Theorem 2.2.** *The restricted square of bistar $B_{n,n}$ is a square sum graph.*

*Proof.* Let $G$ be the restricted square of bistar $B_{n,n}$ with vertex set $V(G) = V(B_{n,n})$ and edge set $E(G) = E(B_{n,n}) \cup \{uv_i, vu_i/1 \leq i \leq n\}$. So, $|V(G)| = 2n+2$ and $|E(G)| = 4n+1$. We define a bijection $f : V(G) \to \{0, 1, 2, \dots, 2n+1\}$ as follows.
$f(u) = 1$,
$f(u_i) = 2i; 1 \leq i \leq n$.
$f(v) = 0$,
$f(v_i) = 2i+1; 1 \leq i \leq n$.
Then there are five types of edge labels in $G$.

(1) $\{f^*(uv)\}$ is 1.

(2) $\{f^*(uu_i), 1 \leq i \leq n\}$ is in ascending order of the form $4k+1$, $k \in N$.

(3) $\{f^*(vv_i), 1 \leq i \leq n\}$ is in ascending order of the form $4k+1$, $k \in N$.

(4) $\{f^*(vu_i), 1 \leq i \leq n\}$ is in ascending order of the form $4k$, $k \in N$.

(5) $\{f^*(uv_i), 1 \leq i \leq n\}$ is in ascending order of the form $4k+2$, $k \in N$.

It is clear that labels of types (1), (4) and (5) are distinct.
It is also easy to see that the labels of types (1), (4) and (5) are distinct from the labels of types (2) and (3). The square of odd numbers never equal to sum of square of even number and 1, because if they are equal then there exists some positive integer $k$ and $m$ such that $(2k+1)^2 = (2m)^2 + 1$. So, $k(k+1) = m^2$. which is not true [2].
So, labels of the types (2) and (3) are distinct.
Hence labels of above all types are internally as well as externally distinct.
Therefore, from above defined function $f$, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence restricted square of bistar $B_{n,n}$ is a square sum graph. □

**Example 2.3.** *Square sum labeling in restricted square of bistar $B_{5,5}$ is shown in following Figure 1.*

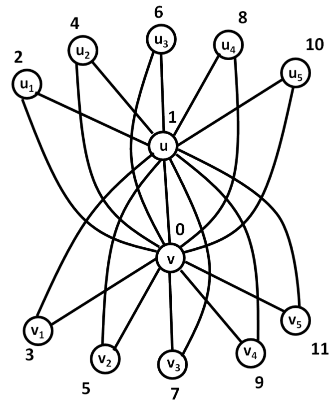


Figure 1:

**Definition 2.4** ([10]). *The splitting graph $S'(G)$ of a graph $G$ is constructed by adding to each vertex $v$, a new vertex $v'$ such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$, i.e. $N(v) = N(v')$.*

**Theorem 2.5.** *The splitting graph of bistar $B_{n,n}$ is a square sum graph.*

*Proof.* Let $G = S'(B_{n,n})$ be a splitting graph of bistar $B_{n,n}$ with vertex set $V(G) = V(B_{n,n}) \cup \{u', v', u'_i, v'_i/1 \leq i \leq n\}$ and edge set $E(G) = E(B_{n,n}) \cup \{uv', vu', uu'_i, vv'_i, u'u_i, v'v_i/1 \leq i \leq n\}$. So, $|V(G)| = 4n+4$ and $|E(G)| = 6n+3$. We define a bijection $f : V(G) \to \{0, 1, 2 \dots, 4n+3\}$ as follows.
$f(u) = 4n+2$,
$f(u') = 0$,
$f(u_i) = 2i; 1 \leq i \leq n$,
$f(u'_i) = 2n+2i; 1 \leq i \leq n$.
$f(v) = 1$,
$f(v') = 4n+3$,
$f(v_i) = 2n+2i+1; 1 \leq i \leq n$,
$f(v'_i) = 2i+1; 1 \leq i \leq n$.
So, from above defined function $f$, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence splitting graph of $B_{n,n}$ is square sum. □

**Example 2.6.** *Square sum labeling in splitting graph of $B_{5,5}$ is shown in following Figure 2.*

**Definition 2.7** ([10]). *The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$ and join each vertex $u'$ in $G'$ to the neighbors of the corresponding vertex $v'$ in $G''$.*

**Theorem 2.8.** *The shadow graph of bistar $B_{n,n}$ is a square sum graph.*

*Proof.* Let $G = D_2(B_{n,n})$ be a shadow graph of bistar $B_{n,n}$ with vertex set

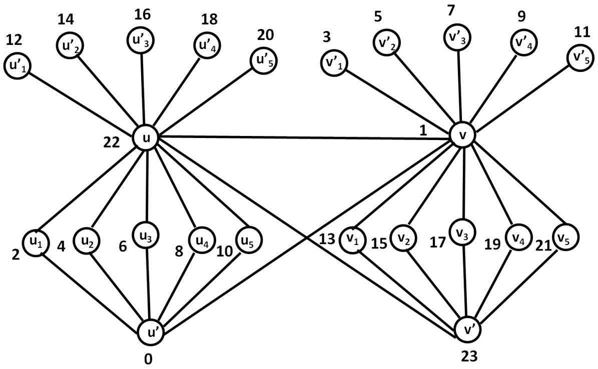


Figure 2:

$V(G) = V(B_{n,n}) \cup \{u', v', u_i', v_i'/1 \leq i \leq n\}$ and edge set $E(G) = E(B_{n,n}) \cup \{u'u_i, uu_i', u'v, u'v', v'u, v'v_i, vv_i', u'u_i', v'v_i'/1 \leq i \leq n\}$. So, $|V(G)| = 4n + 4$ and $|E(G)| = 8n + 4$. We define a bijection $f : V(G) \to \{0, 1, 2 \ldots, 4n + 3\}$ as follows.

$f(u) = 4n + 2$,
$f(u') = 0$,
$f(u_i) = 2n + 2i;\ 1 \leq i \leq\ n$,
$f(u_i') = 2i;\ 1 \leq i \leq\ n$.
$f(v) = 4n + 3$,
$f(v') = 1$,
$f(v_i) = 2n + 2i + 1; 1 \leq i \leq\ n$,
$f(v_i') = 2i + 1; 1 \leq i \leq\ n$.

So, from above defined function $f$, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence shadow graph of bistar $B_{n,n}$ is a square sum graph. □

**Example 2.9.** *Square sum labeling in shadow graph of $B_{5,5}$ is shown in the following Figure 3.*

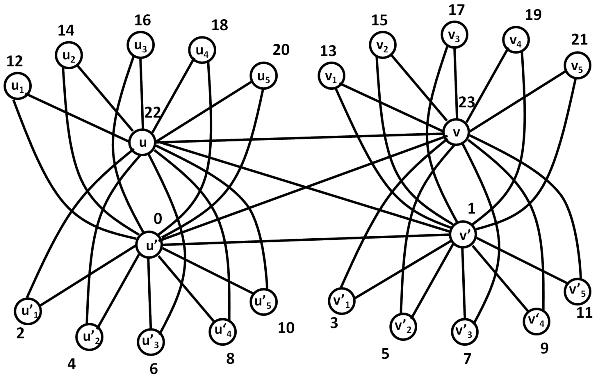


Figure 3:

**Definition 2.10** ([9]). *Let $G(V, E)$ be a graph with $V = S_1 \bigcup S_2 \bigcup S_3 \ldots S_t \bigcup T$ where each $S_i$ is a set of vertices having at least two vertices of the same degree and $T = V - \bigcup_{i=1}^{t} S_i$. The degree splitting graph of $G$ denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, w_3 \ldots w_t$ and joining to each vertex of $S_i$, for $1 \leq i \leq t$.*

**Theorem 2.11.** *The degree splitting graph of $B_{n,n}$ is a square sum graph.*

*Proof.* Let $G = DS(B_{n,n})$ be the degree splitting graph of $B_{n,n}$ with vertex set $V(G) = V(B_{n,n}) \cup \{w_1, w_2, u_i, v_i/1 \leq i \leq n\}$ and edge set $E(G) = \{w_1u_i, w_1v_i, uu_i, vv_i, uv, uw_2, vw_2/1 \leq i \leq n\}$. So, $|V(G)| = 2n + 4$ and $|E(G)| = 4n + 3$. We define a bijection $f : V(G) \to \{0, 1, 2 \ldots, 2n + 3\}$ as follows.

$f(u) = 2n + 2$,
$f(u_i) = 2i;\ 1 \leq i \leq\ n$.
$f(v) = 2n + 1$,
$f(v_i) = 2i - 1; 1 \leq i \leq\ n$.
$f(w_1) = 0$,
$f(w_2) = 2n + 3$.

So, from above defined function $f$, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence degree splitting graph of $B_{n,n}$ is a square sum graph. □

**Example 2.12.** *Square sum labeling in degree splitting graph of $B_{5,5}$ is shown in the following Figure 4.*

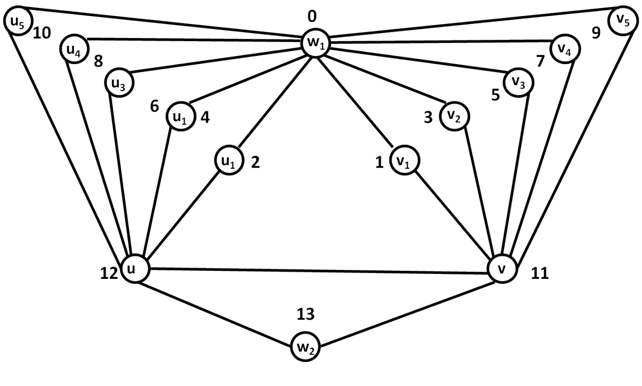


Figure 4:

**Definition 2.13** ([5]). *A graph obtained from given graph $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for some $m_i$, $1 \leq i \leq q$ is called arbitrary super subdivision of $G$.*

**Theorem 2.14.** *The arbitrary super subdivision of $B_{n,n}$ is a square sum graph.*

*Proof.* Let $G$ be arbitrary super subdivision of $B_{n,n}$ constructed as follows:

(1) Edge $e_i = (u, u_i)$ is replaced by complete bipartite graph $K_{2,mi}$ and $u_{ij}$ be the vertices of $m_i$ vertex section, $1 \le i \le n$ and $1 \le j \le m_i$.

(2) Edge $e_i = (v, v_i)$ is replaced by complete bipartite graph $K_{2,ri}$ and $v_{ik}$ be the vertices of $r_i$ vertex section, $1 \le i \le n$ and $1 \le k \le r_i$.

(3) Edge $e = (u, v)$ is replaced by complete bipartite graph $K_{2,s}$ and $w_l$ be the vertices of $s$ vertex section, $1 \le l \le s$.

Let $cm_0 = 0$, where $cm_i$ = cumulative values of $m_i$ and $cr_0 = 0$, where $cr_i$ = cumulative values of $r_i$ for $1 \le i \le n$.
Let $M = \sum_{i=1}^{n} m_i$ and $R = \sum_{i=1}^{n} r_i$. Then $V(G) = 2n+2+M+s+R, E(G) = 2(M+s+R)$.
We define a bijection $f : V(G) \to \{0, 1, 2 \ldots, 2n + M + s + R + 1\}$ as follows.
$f(u_i) = i - 1;\ 1 \le i \le n$,
$f(u_{ij}) = n - 1 + cm_{i-1} + j;\ 1 \le i \le n$ and $1 \le j \le m_i$,
$f(u) = n + cm_n$.
$f(w_l) = n + cm_n + l;\ 1 \le l \le s$.
$f(v) = n + cm_n + s + 1$,
$f(v_{ik}) = n + cm_n + s + 1 + cr_{i-1} + k;\ 1 \le i \le n$ and $1 \le k \le r_i$,
$f(v_i) = n + cm_n + s + 1 + cr_n + i;\ 1 \le i \le n$.
So, from above defined function $f$, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence arbitrary super subdivision of $B_{n,n}$ is a square sum graph. □

**Example 2.15.** *Square sum labeling in arbitrary super subdivision of $B_{5,5}$ is shown in the following Figure 5.*

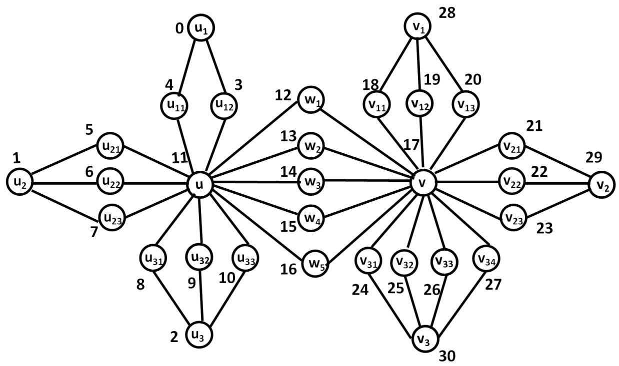


Figure 5:

**Definition 2.16** ([5]). *Duplication of a vertex $v$ of a graph $G$ produces a new graph $G'$ by adding a vertex $v'$ with $N(v') = N(v)$. In other words a vertex $v'$ is said to be a duplication of $v$ if all the vertices which are adjacent to $v$ are now adjacent to $v'$.*

**Theorem 2.17.** *Duplication of any vertex of bistar $B_{n,n}$ is a square sum graph.*

*Proof.* The following two cases are to be considered for duplication of any vertex of $B_{n,n}$.

(1) Duplication of any pendent vertex of $B_{n,n}$.
In this case, the graph becomes a tree. So, it is a square sum graph[11].

(2) Duplication of apex vertex of $B_{n,n}$.
Let $G$ be a graph obtained by duplication of apex vertex $v$ (or $u$) in $B_{n,n}$ with vertex set $V(G) = V(B_{n,n}) \cup \{v'\}$ and edge set $E(G) = E(B_{n,n}) \cup \{v_i v', uv'\}$. So, $|V(G)| = 2n + 3$ and $|E(G)| = 3n + 2$. We define a bijection $f : V(G) \to \{0, 1, 2 \ldots, 2n + 2\}$ as follows.
$f(u) = 1$,
$f(u_i) = 2i;\ 1 \le i \le n$.
$f(v) = 2n + 2$,
$f(v') = 0$,
$f(v_i) = 2i + 1; 1 \le i \le n$.
So, from above defined function $f$, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence graph obtained by duplication of apex vertex in $B_{n,n}$ is a square sum graph.

□

**Example 2.18.** *Square sum labeling in duplication of apex vertex of $B_{5,5}$ is shown in the following Figure 6.*

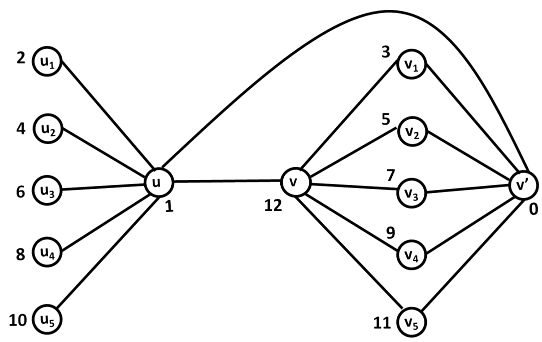


Figure 6:

We define the restricted total graph of $B_{n,n}$ as follows.

**Definition 2.19.** *The restricted total graph of $B_{n,n}$ is a graph with vertex set $= V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v_i, u'_i, v'_i / 1 \le i \le n\}$, where $u$ and $v$ are apex vertices, $u_i$ and $v_i$ are pendent vertices, $w$, $u'_i$ and $v'_i$ are vertices corresponding to* *the edges of $B_{n,n}$ and edge set* $= E(B_{n,n}) \cup \{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \leq i \leq n\}$.

**Theorem 2.20.** *The restricted total graph of $B_{n,n}$ is a square sum graph.*

*Proof.* Let $G$ be a restricted total graph of $B_{n,n}$.
$V(G) = V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v_i, u'_i, v'_i/1 \leq i \leq n\}$, where $u$ and $v$ are apex vertices, $u_i$ and $v_i$ are pendent vertices, $w$, $u'_i$ and $v'_i$ are vertices related to edges.
$E(G) = E(B_{n,n}) \cup \{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \leq i \leq n\}$.
So, $|V(G)| = 4n + 3$ and $|E(G)| = 8n + 3$.
We define a bijection $f : V(G) \to \{0, 1, 2 \ldots, 4n+2\}$ as follows.

$f(u) = 1$,
$f(u_i) = 2(i - 1);\ 1 \leq i \leq n$,
$f(u'_i) = 2n + 2(i - 1);\ 1 \leq i \leq n$.
$f(v) = 4n + 2$,
$f(v_i) = 2i + 1; 1 \leq i \leq n$,
$f(v'_i) = 2n + 2i + 1; 1 \leq i \leq n$.
$f(w) = 4n$.

So, from above defined function $f$, the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Hence restricted total graph of $B_{n,n}$ is a square sum graph. □

**Example 2.21.** *Square sum labeling in restricted total graph of $B_{5,5}$ is shown in the following Figure 7.*

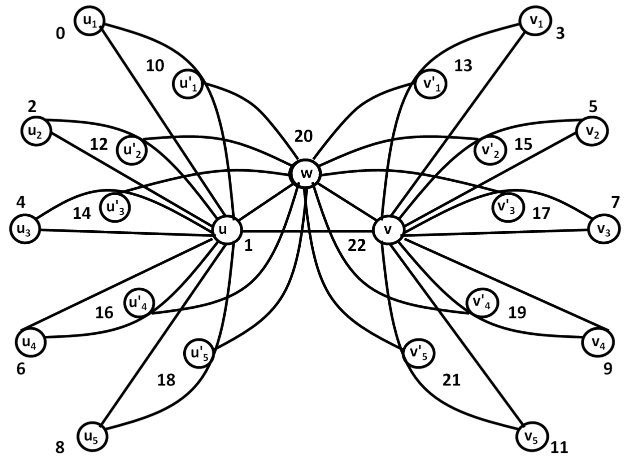

Figure 7:

We define the restricted middle graph of $B_{n,n}$ as follows.

**Definition 2.22.** *The restricted middle graph of $B_{n,n}$ is a graph with vertex set* $= V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v_i, u'_i, v'_i/1 \leq i \leq n\}$, *where $u$ and $v$ are apex vertices, $u_i$ and $v_i$ are pendent vertices, $w$, $u'_i$ and $v'_i$ are vertices corresponding to the edges of $B_{n,n}$ and edge set* $= \{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \leq i \leq n\}$.

**Corollary 2.23.** *The restricted middle graph of bistar $B_{n,n}$ is a square sum graph.*

**Example 2.24.** *Square sum labeling in restricted middle graph of $B_{6,6}$ is shown in the following Figure 8.*

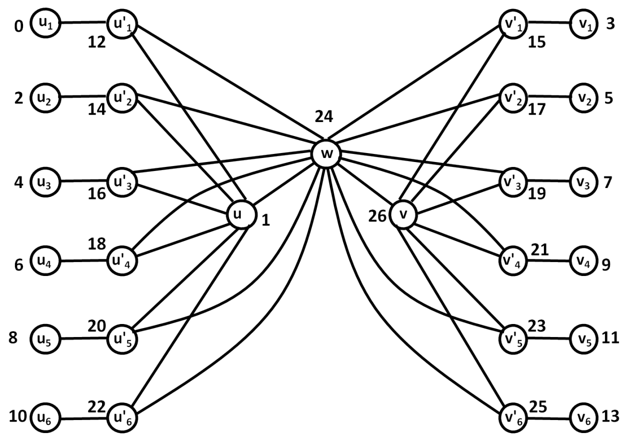

Figure 8:

# 3 Conclusion

It is very interesting to study graphs which admit square sum labeling. Here we discover some bistar related square sum graphs also we define restricted total and restricted middle graph of bistar graph. To investigate equivalent results for different graph families is an open area of research.

# References

[1] A. Rosa, *On certain valuations of the vertices of theory of graphs*, International Symposium, Rome, July 1966, Gordon and Breach, N. Y. and Dunod Paris, (1967), 349 − 355.

[2] David M. Burton , Elementary Number Theory (Sixth edition), Tata McGraw-Hill, (2006).

[3] G. V. Ghodasara and Mitesh J. Patel, *Innovative results on square sum labeling of graphs*, Proceeding of International Conference on Technology and Management(ICTM - 2017), (17, 18 February 2017), HUM 001-HUM 004.

[4] J. A. Bondy and U.S. Murty, *Graph Theory with Applications*, Elsevier Science Publication.

[5] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronics Journal of Combinatorics, 17(2014), #DS6, $1 - 384$.

[6] J. Shiama, *Square sum labeling for some middle and total graphs*, International Journal of Computer Applications, 37(4), (January 2012).

[7] K. A. Germina and R. Sebastian, *Futher results on square sum graph*, International Mathematical Forum, 8(1), (2013), $47 - 57$.

[8] L. W. Beineke and S. M. Hegde, Discuss. Math. Graph Theory $21, 63(2001)$.

[9] P. Selvaraju, P. Balaganesan, J. Renuka and V. Balaj, *Degree splitting graph on graceful, felicitous and elegant labeling*, The International Journal Mathematical Combinatorics, 2(2012) $96 - 102$.

[10] S. K. Vaidya and N. H. Shah, *On Square Divisor Cordial Graphs*, Journal of Scientific Research,6(3), (2014), $445 - 455$.

[11] V. Ajitha, S. Arumugam, and K. A. Germina, *On square sum graphs*, AKCE J Graphs and Combinatorics, 6(1), (2009), $1 - 10$.