A Mathematical Review on Generalized Hyper geometric Differential Equation Including $_3F_2$ Type Function

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ABSTRAT- *The present paper is a review on generalized hypergeometric function. In this paper we have discussed about Frobenius method on various types on differential equation of hypergeometric function such as Confluent hypergeometric function, Gauss hypergeometric function,* $_3F_2$ *type hypergeometric function, generalized hypergeometric function and obtain the solution of these kinds of differential equations.*

Keywords: *Regular points, Confluent hypergeometric function, Gauss hypergeometric function, generalized hypergeometric function, Frobenius method.*

1. INTRODUCTION

The Confluent hypergeometric function is defined by Kummer in 1837 as a solution of second order linear differential equation such that [3]

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw=0 \qquad (1.1)$$

Where a,b are real constants and z is independent variable. $z=0$ is the regular point.

The Gauss hypergeometric function is defined as a solution of second order linear differential equation such that [4]

$$z(1-z)\frac{d^2w}{dz^2}+\{c-(a+b+1)z\}\frac{dw}{dz}-abw=0$$
(1.2)

Where $a,b,c\in C$ and $z=0,1$ are the regular points of (1.2).

Hypergeometric differential equation for $_3F_2$ is defined as

$$z^2(1-z)\frac{d^3w}{dz^3}+\left[(b_1+b_2+1)-(a_1+a_2+a_3+3)z\right]z\frac{d^2w}{dz^2}$$
$$+\left[b_1b_2-(a_1a_2+a_2a_3+a_1a_3+a_1+a_2+a_3+1)z\right]\frac{dw}{dz}$$

$$-a_1a_2a_3w=0 \qquad (1.3)$$

The generalized hypergeometric differential equation is defined as [6]

$$\theta(\theta+b_1-1)(\theta+b_2-1)\ldots(\theta+b_q-1)w$$

$$=z(\theta+a_1)(\theta+a_2)\ldots(\theta+a_p)w \qquad (1.4)$$

Where $\theta=z\frac{d}{dz}$ is the differential operator and a_p,b_q are constant. $z=0,1$ are regular point of (1.4).

2. Classification of hypergeometric function

A. Confluent hypergeometric function

The Confluent hypergeometric function $_1F_1$ is defined as [2]

$$_1F_1\left(\begin{matrix}a\\b\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!} \qquad (2.1)$$

Where $(a)_n$ is known as shifted factorial which is defined as

$$(a)_n = \begin{cases} 1 & n=0 \\ a(a+1)(a+2)\ldots(a+n-1) & n=1,2,3,\ldots \end{cases}$$

(2.2)

Remark: $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ is in the form of gamma function.

B. Gauss hypergeometric function

The Gauss hypergeometric function ${}_2F_1$ is defined as [5]

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{2.3}$$

C. Generalized hypergeometric function

The generalized hypergeometric function ${}_pF_q$ is defined as [1]

$${}_pF_q\left(\begin{matrix} a_1,a_2,a_3,\ldots,a_p \\ b_1,b_2,b_3,\ldots,b_q \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

(2.4)

3. Confluent hypergeometric differential equation

Confluent hypergeometric differential equation is defined by the equation (1.1). By Frobenius method, let

$$w = \sum_{k=0}^{\infty} P_k z^{r+k} \tag{3.1}$$

is the solution of (1.1).

$$\frac{dw}{dz} = \sum_{k=0}^{\infty} P_k (r+k) z^{r+k-1} \tag{3.2}$$

$$\frac{d^2w}{dz^2} = \sum_{k=0}^{\infty} P_k (r+k)(r+k-1) z^{r+k-2} \tag{3.3}$$

Substituting the equations (3.1), (3.2), (3.3) in (1.1) we found

$$z\sum_{k=0}^{\infty} P_k (r+k)(r+k-1) z^{r+k-2}$$

$$+(b-z)\sum_{k=0}^{\infty} P_k (r+k) z^{r+k-1} - a\sum_{k=0}^{\infty} P_k z^{r+k} = 0$$

$$\Rightarrow P_0 r(r-1) z^{r-1} + \sum_{k=1}^{\infty} P_k (r+k)(r+k-1) z^{r+k-1} + bP_0 r z^{r-1}$$

$$+b\sum_{k=1}^{\infty} P_k (r+k) z^{r+k-1} - \sum_{k=0}^{\infty} P_k (r+k) z^{r+k} - a\sum_{k=0}^{\infty} P_k z^{r+k} = 0$$

$$\Rightarrow P_0\left[r(r-1)+rb\right] z^{r-1} + \sum_{k=0}^{\infty}\left[P_{k+1}(r+k+1)(r+k)\right.$$

$$\left.+bP_{k+1}(r+k+1) - P_k(r+k) - aP_k\right] z^{r+k} = 0$$

$$\Rightarrow P_0\left[r(r-1)+rb\right] z^{r-1} + \sum_{k=0}^{\infty}\left[P_{k+1}(r+k+1)(r+k+b)\right.$$

$$\left.-P_k(r+k+a)\right] z^{r+k} = 0 \tag{3.4}$$

Equation (3.4) being identity, we can equate to zero the coefficient of various power of z. Now equating to zero the coefficient of z^{r-1} then $P_0\left[r(r-1)+rb\right]=0$

$$\Rightarrow r=0 \quad \text{or} \quad r=1-b \tag{3.5}$$

Now equating the coefficient of z^{r+k} then we obtain

$$P_{k+1} = \frac{(r+k+a)}{(r+k+1)(r+k+b)} P_k$$

(3.6)

On putting the values of $k=0,1,2,3,\ldots$

$$P_1 = \frac{(r+a)}{(r+1)(r+b)} P_0$$

$$P_2 = \frac{(r+a+1)(r+a)}{(r+2)(r+1)(r+b+1)(r+b)} P_0$$

$$P_3=\frac{(r+a+2)(r+a+1)(r+a)}{(r+3)(r+2)(r+1)(r+b+2)(r+b+1)(r+b)}P_0$$

On putting the values of $P_1, P_2, P_3, \ldots$ in equation (3.1) then

$$w=P_0z^r+\frac{(r+a)}{(r+1)(r+b)}P_0z^{r+1}$$

$$+\frac{(r+a+1)(r+a)}{(r+2)(r+1)(r+b+1)(r+b)}P_0z^{r+2}+\frac{(r+a+2)}{(r+3)(r+2)}$$

$$\times\frac{(r+a+1)(r+a)}{(r+1)(r+b+2)(r+b+1)(r+b)}P_0z^{r+3}+\ldots$$

$$\Rightarrow w=P_0z^r\left[1+\frac{(r+a)}{(r+1)(r+b)}z\right.$$

$$+\frac{(r+a+1)(r+a)}{(r+2)(r+1)(r+b+1)(r+b)}z^2+\frac{(r+a+2)}{(r+3)(r+2)}$$

$$\left.\times\frac{(r+a+1)(r+a)}{(r+1)(r+b+2)(r+b+1)(r+b)}z^3+\ldots\right]$$

(3.7)

From virtue of (3.5), putting $r=0$

$$\Rightarrow w=P_0\left[1+\frac{a}{1.b}z+\frac{a(a+1)}{1.2b(b+1)}z^2\right.$$

$$\left.+\frac{a(a+1)(a+2)}{1.2.3b(b+1)(b+2)}z^3+\ldots\right] \quad (3.8)$$

Setting $P_0=1$ then equation (3.8) becomes

$$w=1+\frac{a}{b}z+\frac{a(a+1)}{1.2b(b+1)}z^2+\frac{a(a+1)(a+2)}{1.2.3b(b+1)(b+2)}z^3+\ldots$$

$$w={}_1F_1(a;b;z) \quad (3.9)$$

From equation (3.5) putting $r=1-b$ in equation (3.7)

$$w=P_0z^{1-b}\left[1+\frac{(a-b+1)}{(2-b).1}z+\frac{(a-b+2)(a-b+1)}{(3-b)(2-b)1.2}z^2\right.$$

$$\left.+\frac{(a-b+3)(a-b+2)(a-b+1)}{1.2.3(4-b)(3-b)(2-b)}z^3+\ldots\right] \quad (3.10)$$

Again setting $P_0=1$ then (3.10) becomes

$$\Rightarrow w=z^{1-b}\,{}_1F_1(a-b+1;2-b;z) \quad (3.11)$$

4. *Gauss hypergeometric differential equation*

Gauss hypergeometric differential equation is defined as equation (1.2). By Frobenius method, let

$$w=\sum_{k=0}^{\infty}P_kz^{r+k}$$

(4.1)

is the solution of (1.2).

$$\frac{dw}{dz}=\sum_{k=0}^{\infty}P_k(r+k)z^{r+k-1}$$

(4.2)

$$\frac{d^2w}{dz^2}=\sum_{k=0}^{\infty}P_k(r+k)(r+k-1)z^{r+k-2}$$

(4.3)

Substituting the equation (4.1), (4.2), (4.3) in equation (1.2)

$$\Rightarrow z(1-z)\sum_{k=0}^{\infty}P_k(r+k)(r+k-1)z^{r+k-2}$$

$$+\{c-(a+b+1)z\}\sum_{k=0}^{\infty}P_k(r+k)z^{r+k-1}-ab\sum_{k=0}^{\infty}P_kz^{r+k}=0$$

$$\Rightarrow P_0r(r-1)z^{r-1}+\sum_{k=1}^{\infty}P_k(r+k)(r+k-1)z^{r+k-1}$$

$$-\sum_{k=0}^{\infty}P_k(r+k)(r+k-1)z^{r+k}+cP_0rz^{r-1}$$

$$+c\sum_{k=1}^{\infty}P_k\left(r+k\right)z^{r+k-1}-\left(a+b+1\right)\sum_{k=0}^{\infty}P_k\left(r+k\right)z^{r+k}$$

$$-ab\sum_{k=0}^{\infty}P_k z^{r+k}=0$$

$$\Rightarrow P_0\left[r\left(r-1\right)+rc\right]z^{r-1}+\sum_{k=0}^{\infty}\Big[P_{k+1}\left\{\left(r+k+1\right)\right.$$

$$\left.\times\left(r+k+c\right)\right\}-P_k\left\{\left(r+k\right)\left(r+k-1\right)\right.$$

$$\left.+\left(a+b+1\right)\left(r+k\right)+ab\right\}\Big]=0$$

(4.4)

Equation (4.4) being identity, we can equate to zero the coefficient of various power of z. Now equating to zero the coefficient of z^{r-1} then $P_0\left[r\left(r-1\right)+rc\right]=0$

$$\Rightarrow r=0 \text{ or } r=1-c \qquad (4.5)$$

Now equating the coefficient of z^{r+k} then we obtain

$$P_{k+1}=\frac{\left(r+k+a\right)\left(r+k+b\right)}{\left(r+k+1\right)\left(r+k+c\right)}P_k \qquad (4.6)$$

On putting the values of $k=0,1,2,3,...$

$$P_1=\frac{\left(r+a\right)\left(r+b\right)}{\left(r+1\right)\left(r+c\right)}P_0$$

$$P_2=\frac{\left(r+a+1\right)\left(r+a\right)\left(r+b+1\right)\left(r+b\right)}{\left(r+2\right)\left(r+1\right)\left(r+c+1\right)\left(r+c\right)}P_0$$

$$P_3=\frac{\left(r+a+2\right)\left(r+a+1\right)\left(r+a\right)}{\left(r+3\right)\left(r+2\right)\left(r+1\right)}$$

$$\times\frac{\left(r+b+2\right)\left(r+b+1\right)\left(r+b\right)}{\left(r+c+2\right)\left(r+c+1\right)\left(r+c\right)}P_0$$

On putting the values of $P_1, P_2, P_3,...$ in equation (4.1) then

$$\Rightarrow w=\left[P_0z^r+\frac{\left(r+a\right)\left(r+b\right)}{\left(r+1\right)\left(r+c\right)}P_0z^{r+1}\right.$$

$$+\frac{\left(r+a+1\right)\left(r+a\right)\left(r+b+1\right)\left(r+b\right)}{\left(r+2\right)\left(r+1\right)\left(r+c+1\right)\left(r+c\right)}P_0z^{r+2}$$

$$+\frac{\left(r+a+2\right)\left(r+a+1\right)\left(r+a\right)}{\left(r+3\right)\left(r+2\right)\left(r+1\right)}$$

$$\left.\times\frac{\left(r+b+2\right)\left(r+b+1\right)\left(r+b\right)}{\left(r+c+2\right)\left(r+c+1\right)\left(r+c\right)}P_0z^{r+3}+...\right]$$

$$\Rightarrow w=P_0z^r\left[1+\frac{\left(r+a\right)\left(r+b\right)}{\left(r+1\right)\left(r+c\right)}z\right.$$

$$+\frac{\left(r+a+1\right)\left(r+a\right)\left(r+b+1\right)\left(r+b\right)}{\left(r+2\right)\left(r+1\right)\left(r+c+1\right)\left(r+c\right)}z^2$$

$$+\frac{\left(r+a+2\right)\left(r+a+1\right)\left(r+a\right)}{\left(r+3\right)\left(r+2\right)\left(r+1\right)}$$

$$\left.\times\frac{\left(r+b+2\right)\left(r+b+1\right)\left(r+b\right)}{\left(r+c+2\right)\left(r+c+1\right)\left(r+c\right)}z^3+...\right] \qquad (4.7)$$

By virtue of (4.5) putting $r=0$

$$\Rightarrow w=P_0\left[1+\frac{a.b}{1.c}z+\frac{a\left(a+1\right)b\left(b+1\right)}{1.2c\left(c+1\right)}z^2\right.$$

$$\left.+\frac{a\left(a+1\right)\left(a+2\right)b\left(b+1\right)\left(b+2\right)}{1.2.3c\left(c+1\right)\left(c+2\right)}z^3+...\right] \qquad (4.8)$$

Setting $P_0=1$, then equation (4.8) becomes

$$\Rightarrow w=\left[1+\frac{a.b}{1.c}z+\frac{a\left(a+1\right)b\left(b+1\right)}{1.2c\left(c+1\right)}z^2\right.$$

$$\left.+\frac{a\left(a+1\right)\left(a+2\right)b\left(b+1\right)\left(b+2\right)}{1.2.3c\left(c+1\right)\left(c+2\right)}z^3+...\right]$$

$$\Rightarrow w = {}_2F_1(a,b;c;z) \qquad (4.9)$$

From equation (4.5), putting $r = 1-c$ in equation (4.7)

$$\Rightarrow w = P_0 z^{1-c}\left[1 + \frac{(a-c+1)(b-c+1)}{1.(2-c)} z + \frac{(a-c+2)(a-c+1)(b-c+2)(b-c+1)}{1.2(2-c)(3-c)} z^2 + \frac{(a-c+3)(a-c+2)(a-c+1)}{1.2.3} \times \frac{(b-c+3)(b-c+2)(b-c+1)}{(4-c)(3-c)(2-c)} z^3 + \ldots\right]$$

(4.10)

Again setting $P_0 = 1$ then (4.10) becomes

$$\Rightarrow w = z^{1-c}\, {}_2F_1(a-c+1, b-c+1; 2-c; z)$$

(4.11)

5. *Differential equation for ${}_3F_2$ type function*

Differential equation for ${}_3F_2$ type function is defined by equation (1.3). By Frobenius method, let

$$w = \sum_{k=0}^{\infty} P_k z^{r+k} \qquad (5.1)$$

is the solution of (1.3)

$$\frac{dw}{dz} = \sum_{k=0}^{\infty} P_k (r+k) z^{r+k-1} \qquad (5.2)$$

$$\frac{d^2w}{dz^2} = \sum_{k=0}^{\infty} P_k (r+k)(r+k-1) z^{r+k-2} \qquad (5.3)$$

$$\frac{d^3w}{dz^3} = \sum_{k=0}^{\infty} P_k (r+k)(r+k-1)(r+k-2) z^{r+k-3}$$

(5.4)

Substituting the equations (5.1), (5.2), (5.3), (5.4) in equations (1.3)

$$\Rightarrow z^2(1-z)\sum_{k=0}^{\infty} P_k (r+k)(r+k-1)(r+k-2) z^{r+k-3}$$

$$+\left[(b_1+b_2+1)-(a_1+a_2+a_3+3)z\right]$$

$$\times z\sum_{k=0}^{\infty} P_k (r+k)(r+k-1) z^{r+k-2}$$

$$+\left[b_1b_2-(a_1a_2+a_2a_3+a_1a_3+a_1+a_2+a_3+1)z\right]$$

$$\times\sum_{k=0}^{\infty} P_k (r+k) z^{r+k-1} - a_1a_2a_3\sum_{k=0}^{\infty} P_k z^{r+k} = 0$$

$$\Rightarrow P_0\left[r(r-1)(r-2)+(b_1+b_2+1)r(r-1)+b_1b_2r\right]$$

$$\times z^{r-1} + \sum_{k=0}^{\infty} P_{k+1}(r+k+1)(r+k)(r+k-1) z^{r+k}$$

$$-\sum_{k=0}^{\infty} P_k (r+k)(r+k-1)(r+k-2) z^{r+k} + (b_1+b_2+1)$$

$$\times\sum_{k=0}^{\infty} P_{k+1}(r+k+1)(r+k) z^{r+k} - (a_1+a_2+a_3+3)$$

$$\times\sum_{k=0}^{\infty} P_k (r+k)(r+k-1) z^{r+k} + b_1b_2\sum_{k=0}^{\infty} P_{k+1}(r+k+1) z^{r+k}$$

$$-\left[(a_1a_2+a_2a_3+a_1a_3+a_1+a_2+a_3+1)\right]$$

$$\times\sum_{k=0}^{\infty} P_k (r+k) z^{r+k} - a_1a_2a_3\sum_{k=0}^{\infty} P_k z^{r+k} = 0$$

$$\Rightarrow P_0\left[r(r-1+b_1)(r-1+b_2)\right]\times z^{r-1}$$

$$+\sum_{k=0}^{\infty}\Big[P_{k+1}\{(r+k-1)(r+k)(r+k+1)$$
$$+(1+b_1+b_2)(r+k)(r+k+1)+b_1b_2(r+k+1)\}$$
$$-P_k\{(r+k-1)(r+k)(r+k-2)$$
$$+(a_1+a_2+a_3+3)(r+k-1)(r+k)$$
$$+(a_1a_2+a_2a_3+a_1a_3+a_1+a_2+a_3+1)$$
$$\times(r+k)+a_1a_2a_3\}\Big]z^{r+k}=0 \qquad (5.5)$$

Equation (5.5) being identity, we can equate to zero the coefficient of various power of z. Now equating to zero the coefficient of z^{r-1} then
$$P_0\left[r(r-1+b_1)(r-1+b_2)\right]=0$$

$$\Rightarrow r=0, \quad r=1-b_1 \quad \text{and} \quad r=1-b_2$$
(5.6)

Now equating the coefficient of z^{r+k} then we obtain

$$P_{k+1}=\frac{(r+k+a_1)(r+k+a_2)(r+k+a_3)}{(r+k+1)(r+k+b_1)(r+k+b_2)}P_k$$
(5.7)

On putting the values of $k=0,1,2,3,...$

$$P_1=\frac{(r+a_1)(r+a_2)(r+a_3)}{(r+1)(r+b_1)(r+b_2)}P_0$$

$$P_2=\frac{(r+a_1+1)(r+a_1)(r+a_2+1)}{(r+2)(r+1)(r+b_1+1)}$$
$$\times\frac{(r+a_2)(r+a_3+1)(r+a_3)}{(r+b_1)(r+b_2+1)(r+b_2)}P_0$$

$$P_3=\frac{(r+a_1+2)(r+a_1+1)(r+a_1)(r+a_2+2)}{(r+3)(r+2)(r+1)(r+b_1+2)(r+b_1+1)}$$
$$\times\frac{(r+a_2+1)(r+a_2)(r+a_3+2)(r+a_3+1)(r+a_3)}{(r+b_1)(r+b_2+2)(r+b_2+1)(r+b_2)}P_0$$

On putting the values of $P_1,P_2,P_3,...$ in equation (5.1) then

$$\Rightarrow w=P_0z^r+\frac{(r+a_1)(r+a_2)(r+a_3)}{(r+1)(r+b_1)(r+b_2)}P_0z^{r+1}$$
$$+\frac{(r+a_1+1)(r+a_1)(r+a_2+1)(r+a_2)}{(r+2)(r+1)(r+b_1+1)(r+b_1)}$$
$$\times\frac{(r+a_3+1)(r+a_3)}{(r+b_2+1)(r+b_2)}P_0z^{r+2}+\frac{(r+a_3+2)}{(r+3)}$$
$$\times\frac{(r+a_3+1)(r+a_3)(r+a_2+2)(r+a_2+1)}{(r+2)(r+1)(r+b_1+2)(r+b_1+1)}$$
$$\times\frac{(r+a_2)(r+a_1+2)(r+a_1+1)}{(r+b_1)(r+b_2+2)(r+b_2+1)}$$
$$\times\frac{(r+a_1)}{(r+b_2)}P_0z^{r+3}+...$$

$$\Rightarrow w=P_0z^r\Bigg[1+\frac{(r+a_1)(r+a_2)(r+a_3)}{(r+1)(r+b_1)(r+b_2)}z$$
$$+\frac{(r+a_1+1)(r+a_1)(r+a_2+1)(r+a_2)}{(r+2)(r+1)(r+b_1+1)(r+b_1)}$$
$$\times\frac{(r+a_3+1)(r+a_3)}{(r+b_2+1)(r+b_2)}z^2+\frac{(r+a_3+2)}{(r+3)}$$
$$\times\frac{(r+a_3+1)(r+a_3)(r+a_2+2)(r+a_2+1)}{(r+2)(r+1)(r+b_1+2)(r+b_1+1)}$$
$$\times\frac{(r+a_2)(r+a_1+2)(r+a_1+1)}{(r+b_1)(r+b_2+2)(r+b_2+1)}$$

$$\times\frac{(r+a_1)}{(r+b_2)}z^3+\dots\Bigg] \qquad (5.8)$$

From virtue of (5.6), putting $r=0$

$$\Rightarrow w=P_0\Bigg[1+\frac{a_1a_2a_3}{1.b_1b_2}z+\frac{(a_1+1)a_1}{1.2}$$

$$\times\frac{(a_2+1)a_2(a_3+1)a_3}{(b_1+1)b_1(b_2+1)b_2}z^2+\frac{(a_1+2)}{1.2.3}$$

$$\times\frac{(a_1+1)a_1(a_2+2)(a_2+1)a_2}{(b_1+2)(b_1+1)b_1}$$

$$\times\frac{(a_3+2)(a_3+1)a_3}{(b_2+2)(b_2+1)b_2}z^3+\dots\Bigg] \qquad (5.9)$$

Setting $P_0=1$, then equation (5.9) becomes

$$\Rightarrow w=\Bigg[1+\frac{a_1a_2a_3}{1.b_1b_2}z+\frac{(a_1+1)a_1}{1.2}$$

$$\times\frac{(a_2+1)a_2(a_3+1)a_3}{(b_1+1)b_1(b_2+1)b_2}z^2+\frac{(a_1+2)}{1.2.3}$$

$$\times\frac{(a_1+1)a_1(a_2+2)(a_2+1)a_2}{(b_1+2)(b_1+1)b_1}$$

$$\times\frac{(a_3+2)(a_3+1)a_3}{(b_2+2)(b_2+1)b_2}z^3+\dots\Bigg]$$

$$\Rightarrow w={}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix};z\right)$$

(5.10)

From equation (5.6), putting $r=1-b_1$ in equation (5.8)

$$\Rightarrow w=P_0z^{1-b_1}\Bigg[1+\frac{(a_1-b_1+1)(a_2-b_1+1)(a_3-b_1+1)}{1.(2-b_1)(b_2-b_1+1)}z$$

$$+\frac{(a_1-b_1+2)(a_1-b_1+1)(a_2-b_1+2)}{1.2(3-b_1)(2-b_1)}$$

$$\times\frac{(a_2-b_1+1)(a_3-b_1+2)(a_3-b_1+1)}{(b_2-b_1+2)(b_2-b_1+1)}z^2+\dots\Bigg]$$

(5.11)

Again setting $P_0=1$ then (5.11) becomes

$$w=z^{1-b_1}\,{}_3F_2\left(\begin{matrix}a_1-b_1+1,a_2-b_1+1,a_3-b_1+1\\ 2-b_1,b_2-b_1+1\end{matrix};z\right)$$

(5.12)

From equation (5.6), putting $r=1-b_2$ in equation (5.8)

$$\Rightarrow w=P_0z^{1-b_2}\Bigg[1+\frac{(a_1-b_2+1)(a_2-b_2+1)(a_3-b_2+1)}{1.(2-b_2)(b_1-b_2+1)}z$$

$$+\frac{(a_1-b_2+2)(a_1-b_2+1)(a_2-b_2+2)}{1.2(3-b_2)(2-b_2)}$$

$$\times\frac{(a_2-b_2+1)(a_3-b_2+2)(a_3-b_2+1)}{(b_1-b_2+2)(b_1-b_2+1)}z^2+\dots\Bigg]$$

(5.13)

Again setting $P_0=1$ then (5.13) becomes

$$w=z^{1-b_2}\,{}_3F_2\left(\begin{matrix}a_1-b_2+1,a_2-b_2+1,a_3-b_2+1\\ 2-b_2,b_1-b_2+1\end{matrix};z\right)$$

(5.14)

6. CONCLUSION:

By Frobenius method, equation (1.1) gives (3.9) and (3.11) which is known as confluent hypergeometric function. Furthermore, equation (1.2) gives (4.9) and (4.11) which is known as Gauss hypergeometric function. By virtue of (1.4) differential equation is obtained for ${}_3F_2$ type function and (5.10), (5.12) and (5.14) are the solutions of (1.3).

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