Fixed Point Theorems for occasionally weakly compatible self maps in Generalized Fuzzy Metric Spaces

Sukanya K P[#1], Dr.Sr.MagieJose[*2]

[#]Research Scholar, Department of Mathematics, St.Mary's College, Thrissur, Kerala, India
[*]Associate Professor, Department of Mathematics, St.Mary's College, Thrissur , Kerala ,India

Abstract: *In this paper we are proving common fixed Point theorems for occasionally weakly compatible self maps in Generalized fuzzy metric spaces.*

Keywords —*Fuzzy Metric Space,G-Metric Space,Q-Fuzzy Metric Space, weakly compatible and occasionally weakly compatible self mappings*

I. INTRODUCTION

In Mathematics,the concept of Fuzzy set was introduced by L A Zadeh[2].It is a new way to represent vaguenessin our daily life.In 1975 Kramosil and Michalek[10] introduced the concept of fuzzy metric spaces which opened anew way for further development of analysis in suchspaces.George and Veeramani modified the concept of fuzzy metricspace.After that several fixed point theorems have been proved in fuzzy metric spaces.

A.Al-Thagafi and NaseerShahzad[11] introduced the concept of occasionally weakly compatible maps. They showed that occasionally weaklycompatible map is weakly compatible but converse is not true.G.Jungck and B.E Rhoades[9] also proved fixed pointtheorems for occasionally weakly compatible mappings. C T Aage and J N Salunke[4] proved fixed point theorems infuzzy metric spaces for occasionally weakly compatible self maps.

In 2006, Mustafa.Z and B.Sims[7] presented a definitionof G-metric space.After that several fixed point results have been proved in G-metric spaces.Later,GuangpengSun and Kai Yang[5] introduced the notion of Q-fuzzy metric space, which can be considered as a generalization offuzzy metric space. They also prove fixed point theorems in Q-fuzzy metric spaces.

In this paper we prove commonfixed point theorems for four mappings under occasionally weakly compatible condition in generalized fuzzy metricspace.To prove our results we use the concept of occasionally weakly compatible maps due to Al-Thagafiand NaseerShahzad[11]. While proving the results we take Q-fuzzymetric spaces which is ageneralization of fuzzy metric spacedue to Guangpeng Sun and Kai Yang[5].The results presented in this paper generalize and improve some known resultsdue to C.T Aage and J.N Salunnke[4].

II. PRELIMINARY NOTES

Definition2.1[2]: A fuzzy set A in X is a function with domain X and values in [0,1]

Definition2.2[6]:A binary operation *[0,1]× [0,1] →[0,1] is a continuous t-norm if * satisfies the following conditions

- *is commutative and associative
- * is continuous
- a*1=a for all a∊[0,1]
- a*b ≤ c*d whenever a≤c and b≤d for all a,b,c,d∊[0,1]

Definition2. 3[1]: A 3-tuple$(X,M,*)$ is called a fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions

- $M(x,y,t) > 0$
- $M(x,y,t) = 1$,for all $t > 0$ if and only if $x = y$
- $M(x,y,t) = M(y,x,t)$
- $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$
- $M(x,y,.)$: $(0,\infty) \rightarrow (0,1]$ is continuous for all $x,y,z \epsilon X$ and $s,t > 0$.

M(x,y,t) denotes the degree of nearness between x and y with respect to t.

Definition2. 4[7]: Let X be a nonempty set and let$G: X\times X\times X \rightarrow R+$ be a function satisfying the following

- $G(x,y,z) = 0$ if $x = y = z$
- $0 < G(x,x,y)$for all $x,y \epsilon X$ with $x \neq y$
- $G(x,x,y) \leq G(x,y,z)$for all x,y,z in X with $z \neq y$
- $G(x,y,z) = G(x,z,y) = G(y,z,x) =$ ……..(symmetry in all three variables)
- $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all x,y,z, a ∊ X (Rectangle inequality)

Then the function is called a generalizedmetric , or more specifically a G-metric on X and the pair (X,G) is a G-metric space.

Example:Let (X,d) be a metric space.Then $G: X\times X\times X \rightarrow R+$defined by $G(x,y,z) = d(x,y)+d(y,z)+d(x,z)$.Then (X,G) is a G-metric space.

Definition2.5[5]: A 3-tuple(X,Q,*) is called a Q-fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and Q is a fuzzy set in $X^3\times(0,\infty)$ satisfying the following conditions for each x,y,z,a ϵ X and t,s>0

- $Q(x,x,y,t) > 0$ and $Q(x,x,y,t) \le Q(x,y,z,t)$ for all $x, y, z \in X$ with $z \ne y$
- $Q(x,y,z,t) = 1$, for all t>0 if and only if $x = y = z$
- $Q(x,y,z,t) = Q(p(x,y,z),t)$ (symmetry) ,where p is a permutation function
- $Q(x,a,a,t) * Q(a,y,z,s) \le Q(x,y,z,t+s)$
- $Q(x,y,z,.): (0,\infty) \to [0,1]$ is continuous .

Q-fuzzy metric space can be considered as a generalization of fuzzy metric space.

Example: Let X is a non empty set and G is the G-metric on X.The t- norm is

a*b = ab for all a,bϵ [0,1].For each t>0

$Q(x,y,z,t) = \frac{t}{t+G(x,y,z)}$

Then (X,Q,*) is a fuzzy Q-metric.

Lemma2.6[5]: If (X,Q,*) be a Q-fuzzy metric space , then $Q(x,y,z,t)$ is non-decreasing with respect to t for all $x,y,z \in X$

Definition 2.7[5]: Let (X,Q,*) be a Q- fuzzy metric space. A sequence$\langle x_n \rangle$ in X converges to a point xϵ X if and only if$Q(x_m,x_n,x,t)\to 1$ as $n\to\infty, m\to\infty$, for each t >0.It is called a Cauchy sequence if for each 0<ϵ<1 and t>0,there exists n_0 ϵN such that$Q(x_m,x_n,x_l,t) > 1-\epsilon$ for each $l,m,n \ge n_0$. A Q-fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition2.8[5]: Let f and g be self mappings on a Q- fuzzy metric space (X,Q,*).Then the mappings are said to be compatible if $\lim_{n\to\infty} Q(fgx_n, gfx_n, gfx_n, t) = 1$, for all t>0 whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

Definition2.9[4]: Let X be a set. Let f and g be self maps on X. A point x in X is called a coincidence point of f and g if and only if $fx = gx$. In this case $w = fx = gx$ is called point of coincidence of f and g.

Definition2.10[4]: A pair of self mappings(f,g) is said to be weakly compatible if they commute at the coincidence points,that is if $fu = gu$ for some uϵX,then $fgu = gfu$.

Definition2.11[9]: Two self maps f and g of a set are occasionally weakly compatible(owc) iffthere is a point x in X which is a coincidence point of f and g at which f and g commute.

Lemma2.12[4]: Let X be a set,f,g owcself maps of X.If fand g have a unique point of coincidence,$w = fx = gx$, then w is the unique common fixed point of f and g.

III.THEOREMS

Theorem3.1: Let (X,Q,*) be a complete Q–fuzzy metric space and let A,B,S and T be self mappings of X.Let the pair {A,S} and {B,T} be occasionally weakly compatible(owc).If there exist a kϵ(0,1) such that

$$Q(Ax,By,Bz,kt) \ge \varphi(\{Q(Sx,Ty,Tz,t), Q(Sx,By,Tz,t), Q(By,Ty,Tz,t), Q(Ax,Ty,Tz,t), Q(Ax,Ty,Bz,t)\} \quad \text{—— (1)}$$

For all x,y,z ϵX and $\varphi:[0,1]^5\to[0,1]$ such that $\varphi(t,t,1,t,t) > t$ for $0 < t < 1$. Then there exists a unique common fixed point of A,B,S and T

Proof: The pair of self mappings$\{A,S\}$ and $\{B,T\}$ be occasionally weakly compatible(owc). So there are points $x, \epsilon X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$.If not by inequality (1) we have

$$Q(Ax,By,By,kt) \ge \varphi(\{Q(Sx,Ty,Ty,t),Q(Sx,By,Ty,t), Q(By,Ty,Ty,t), (Ax,Ty,Ty,t),Q(Ax,Ty,By,t)\})$$

$$=\varphi(\{Q(Ax,By,By,t),Q(Ax,By,By,t),Q(By,By,By,t), Q(Ax,By,By,t),Q(Ax,By,By,t)\})$$

$$=\varphi(\{Q(Ax,By,By,t),Q(Ax,By,By,t),1, Q(Ax,By,By,t),Q(Ax,By,By,t)\})$$

$$> Q(Ax,By,By,t)$$

Which is a contradiction. Therefore $Ax = By$, ie$Ax = Sx = By = Ty$.

Let there exist another point z such that$Az = Sz$.Then by inequality (1) we have $z = Sz = By = Ty$. Therefore $Ax = Az$.ie$w = Ax = Sx$is the unique point of coincidence of A and S. Then by Lemma2.12 w is the common fixed point of A and S. Similarly there is a unique point z ϵ X such that $z = Bz = Tz$.

Assume that$w \ne z$. We have

$$Q(w,z,z,kt) = Q(Aw,Bz,Bz,kt)$$

$$\ge\varphi(\{Q(Sw,Tz,Tz,t),Q(Sw,Bz,Tz,t),(Bz,Tz,Tz,t), Q(Aw,Tz,Tz,t),Q(Aw,Tz,Bz,t)\})$$

$$=\varphi(\{Q(w,z,z,t),Q(w,z,z,t),Q(z,z,z,t), Q(w,z,z,t),Q(w,z,z,t)\})$$

$$=\varphi(\{Q(w,z,z,t),Q(w,z,z,t),1, Q(w,z,z,t),Q(w,z,z,t)\}$$

$$> Q(w,z,z,t)$$

Hence $z = w$ and z is a common fixed point of $A, B, S$and T.

To prove uniqueness, let z' be another common fixed point of ,,S and$T$.

Let $z \neq z'$ .We have

$Q(z', z, z, kt) = Q(Az', Bz, Bz, kt)$

$\geq \varphi(\{Q(Sz', Tz, Tz, t), Q(Sz', Bz, Tz, t), Q(Bz, Tz, Tz, t), Q(Az', Tz, Tz, t), Q(Az', Tz, Bz, t)\})$

$= \varphi(\{Q(z', z, z, t), Q(z', z, z, t), Q(z, z, z, t), Q(z', z, z, t), Q(z', z, z, t)\})$

$= \varphi(\{Q(z', z, z, t), Q(z', z, z, t), 1, Q(z', z, z, t), Q(z', z, z, t)\})$

$> Q(z', z, z, t)$

Which is a contradiction.Hence $z = z'$, is a unique common fixed point of $A, B, S$and .

Theorem 3.2: Let(X,Q,*) be a complete Q –fuzzy metric space and let A,B,S and T be self mappings of X.Let the pair {A,S} and {B,T} be occasionally weakly compatible(owc).If there exist a kϵ(0,1) such that for every x,y,z ϵX and t > 0

$$Q(Ax, By, Bz, kt) \geq \min\{Q(Sx, Ty, Tz, t), Q(Sx, By, Tz, t), Q(By, Ty, Tz, t), Q(Ax, Ty, Tz, t) + Q(Ax, Ty, Bz, t)/2\} \quad \text{\_\_\_\_(2)}$$

Then there exists a unique common fixed point of A,B,S and T

Proof:The pair of self mappings{A,S} and {B,T} be occasionally weakly compatible(owc). So there exist points x,y ϵ X such that$Ax = Sx$and $By = Ty$. First claim that $Ax = By$.If not by inequality (2)

$$Q(Ax, By, By, kt) \geq \min\{Q(Sx, Ty, Ty, t), Q(Sx, By, Ty, t), Q(By, Ty, Ty, t) Q(Ax, Ty, Ty, t) + Q(Ax, Ty, By, t)/2\}$$

$$= \min\{Q(Ax, By, By, t), Q(Ax, By, By, t), Q(By, By, By, t), Q(Ax, By, By, t) + Q(Ax, By, By, t)/2\}$$

$$= Q(Ax, By, By, t)$$

which is a contradiction.Therefore $Ax = By$,ie $Ax = Sx = By = Ty$.

Let there exist another point z such that $Az = Sz$.Then by inequality (2) we have $Az = Sz = By = Ty$. Therefore $Ax = Az. ie\ w = Ax = Sx$is the unique point of coincidence of A and S.

Then by Lemma2.12 w is the common fixed point of A and S. Similarly there is a unique point z ϵ X such that $z = Bz = Tz$.

Assume that w ≠ z . We have

$$Q(w, z, z, kt) = Q(Aw, Bz, Bz, t)$$

$$\geq \min\{Q(Sw, Tz, Tz, t), Q(Sw, Bz, Tz, t), Q(Bz, Tz, Tz, t), Q(Aw, Tz, Tz, t) + Q(Aw, Tz, Bz, t)/2\}$$

$$= \min\{Q(w, z, z, t), Q(w, z, z, t), Q(z, z, z, t), Q(w, z, z, t) + Q(w, z, z, t)/2\}$$

$$= Q(w, z, z, t)$$

Hence z = w and z is a common fixed point of A,B,S and T.

To prove uniqueness, let z' be another common fixed point of A,B,S and T.

If z ≠z' We have

$$Q(z', z, z, kt) = Q(Az', Bz, Bz, t)$$

$$\geq \min\{Q(Sz', Tz, Tz, t), Q(Sz', Bz, Tz, t), Q(Bz, Tz, Tz, t), Q(Az', Tz, Tz, t) + Q(Az', Tz, Bz, t)/2\}$$

$$= \min\{Q(z', z, z, t), Q(z', z, z, t), Q(z, z, z, t), Q(z', z, z, t) + Q(z', z, z, t)/2\}$$

$$= Q(z', z, z, t)$$

Which is a contradiction.Hence z = z',iez is a unique common fixed point of A,B,S and T.

Example

Consider X = [1,∞)and (X,Q, *) is a complete Q- fuzzy metric space defined by

$Q(x, y, z, t) = \frac{t}{t+G(x,y,z)}$

Here (X,G) is a G-metric space defined by $G(x, y, z) = d(x, y) + d(y, z) + d(x, z)$,d is usual metricon X. A, B, S and Tare self mappings on X given by $Ax = x$; $Bx = x^2$,$Sx = x^3$,$Tx = x^4$. Clearly the pairs {A, S }and{B, T }are occasionally weakly compatibleφ: $[0,1]^5 \rightarrow [0,1]$such that$\varphi(x, y, z, s, t) = \max\{x, y, z, s, t\}$for 0 < t < 1 Here A,B, S and T satisfies condition (1) oftheorem 3.1.Hence the four mappings have unique common fixed point. 1 is the common fixed point of A, B, S and T

IV. CONCLUSION

Fixed point theory has many applications in severalbranches of science such as game theory,nonlinearprogramming,economics,theory of differential equations,etc.In this paper we prove common fixed point theoremsfor four occasionally weakly compatible self maps ingeneralized fuzzy metric space.To prove the results weuse the concept of occasionally weakly compatible maps.While proving the results we take Q-fuzzy metric spaces,which is a generalization of fuzzy metric space.Our resultspresented in this paper generalize and improve someknown results in fuzzy metric space.

ACKNOWLEDGMENT

The first author would like to express sincere gratitude to St Mary's College, Thrissur and University of Calicut for giving facilitiesand support during the preparation ofthis work.

REFERENCES

[1] George.A and P.Veeramani,1994, On some resultsin Fuzzy metric spaces.Fuzzy Sets Systems,64,395-399.

[2] L.A.Zadeh,Fuzzy Sets,1965 Fuzzy Sets.Informationand control 8,338-353

[3] B.C.Dhage,1992, Generalized metric spacesand mappings with fixed point.Bull. Calcutta Math.Soc.,84(4),329-336

[4] C.T.Aage and J.N.Salunke,2010, On fixed point theoremsin fuzzy metric spaces. Int.J. Open ProblemsCompt. Math.Vol.3,No.2,123-131

[5] Guangpeng Sun and Kai Yang,2010, Generalized fuzzy metric spaces with properties. Research Journal of Applied Sciences, Engineering and Technology 2(7),673-678

[6] B.Schweizer and A.Sklar, Probabilistic Metric Spaces,NorthHolland,New York,1983

[7] Mustafa.Z and B.Sims,2006,A new approach to generalized metric spaces,J.Nonlinear Convex Anal.7:289-297

[8] MagieJose,A study of fuzzy normed linearspaces, Ph.Dthesis,Madras University,Chennai,India

[9] G.Jungck and B.E.Rhoades Fixed Point Theorems for Occasionally Weakly compatibleMappings, Fixed Point Theorey,Volume7,No.2,2006,287-296.

[10]I.Kramosil and J.Michalek, Fuzzymetric andstatisticalmetric spaces,Kybernetika,11(1975),326-334

[11] A.Al-Thagafi and NaseerShahzad, Generalized I-Nonexpansiveselfmapsand Invariant Approximations, Acta Mathematic a Sinica, English Series, May, 2008, Vol.24, No.5, pp.867876

[12]S.C. Ghosh; M. Srivastava,2017, Fixed Point Theorem on Complete N - Metric Spaces, International Journal of Mathematics Trends and Technology (IJMTT), Volume-47,Number-1

[13] AntimaSindersiya, AkleshPariya, Nirmala Gupta, V. H. Badshah, 2017, Common Fixed Point Theorem in G-Metric Spaces, International Journal of Mathematics Trends and Technology (IJMTT),Volume-47 , Number-2