A Mathematical Model to Study Stability of Biological Interaction: Competition

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Abstract

In this paper we shall construct a mathematical model to study the impact of competition on the growth of two species in a given geographical region and for a given interval of time. In our paper we shall also discuss the stability of autonomous dynamical system representing the growth of two species and study the relation between the population density of competitively superior species and competitively inferior species competing for the same resources in a given region and in a given interval of time

Key words: *Autonomous system, Biological interaction, Competition, Competition parameter, Asymptotically stable, Stable, Unstable*

1 Introduction

In a natural ecosystem no living organism is in complete isolation, each and every organism must interact with its immediate environment and other organisms of the same biological community and vice versa is known as biological interaction. These interaction. These interactions are fundamental to the functioning of the ecosystem as a whole.

Biological interaction can be classified on the basis of occurrence between organisms of same or different species or the basis of strength, duration and direction and direction of effects of interaction or the mechanism of interaction between the organisms. The interaction between two populations can be

beneficial or detrimental or neutral (when the organisms in the populations are neither harmed or benefited). These interactions are classified into six types viz. mutualism, predation, parasitism, commensalism, amensalism and competition.

Competition is an interaction between organisms or species in which both the species are harmed. Competition may occur due to limited supply of food, water or territory which is common to both as a resource. According to evolutionary theory, this competition within and between species for resources is important in natural selection

1.1 TYPES OF COMPETITION

1. On the basis of mechanism- By mechanism, competition is typically classified into two types viz. Interference and exploitative competition. • Interference competition- Interference competition occurs directly between individuals via aggression etc. when individuals interfere with foraging or breeding etc. of others, or by opposing their physical establishment in an ecological niche. During interference competition, organisms interact directly by fighting for limited resources. • Exploitative competition- When competition occurs due to a common limiting resource which acts as an intermediate, it is called exploitative competition. For instance, use of common resources by an individual depletes the amount available to others, i.e they compete indirectly for common resources.

2. By size asymmetry- Competition can be Completely symmetric- all individuals receive the same amount of resources, irrespective of their size. Perfectly size symmetric- all individuals exploit the same amount of resources per unit biomass. Absolutely size-asymmetric- the largest individuals exploit all the available resources. The degree of size asymmetry has major eff ects on the structure and diversity of ecological communities. For example, in ecological community of plants(size asymmetric) competition for light has stronger eff ects on diversity compared with competition for soil resources.

3. By species- competition by species is classified into two types, intraspecific and interspecific competition. • Intraspecific competition- this kind of interactions occurs when individuals of the samespecies compete for similar resources in an ecosystem. As the population size increases resources become more limiting and as a result some individuals die or do not breed. Individuals can compete for food, water, space, light, mates or any other resource which is required for survival or reproduction. The resource must be limited for competition to occur; if every member of the species can obtain a sufficient amount of every resource then individuals do not compete and the population grows exponentially. •

Interspecific competition- competition for any resource between individuals of two different species is known as interspecific competition.For details we suggest [4]

Definitions:

• Competitively superior species- A species which is able to use resources more efficiently, or able to reproduce more and leave larger number of progeny which have higher chances to survive as compared to a different species when there is competition between them.

• Competitively inferior species- A species which lags behind in resource utilisation and reproductive capability as compared to the competitively superior species.

• Competitively analogous species- If both the competing species have equivalent resource utilisation and reproductive capability.

In [2] Volterra's Prey-Predator model, non-linear autonomous system was used to study biological interaction: predation for two animals rabbit and fox. In [1] R.B. Ogunrinde and J. Sunday apply second order differential equation to study the same interaction between rats and cats under the condition that food is abundant for rats and cat. For other field studies and mathematical models on biological interactions in ecosystems we suggest [1], [2], [3], [6], [7].

2 Non-Linear Autonomous System

In both the physical and biological system the importance of steady state in a system lies in its degree of permanence which is defined by the stability of system. For a non-linear autonomous system stability of critical points plays a very important role in the existence of system under given situation.[5]

Consider the non-linear autonomous system;

$$(A)\ \dots \quad \begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y \end{cases}$$

A point (x_0, y_0) is said to be a critical point of non-linear autonomous system (A), if

$$a_1x_0 + b_1y_0 = 0$$
$$a_2x_0 + b_2y_0 = 0$$

Also, (0,0) is the critical point of non-linear autonomous system (A) if $W(t) = a_1b_2 - a_2b_1 \neq 0$

Let $x = Ae^{mt}$ and $y = Be^{mt}$ be a non-trivial solution of non-linear autonomous system (A). Then the auxiliary equation for this autonomous system of differential equation is given by $m^2 - (a_1 + b_2)m + a_1b_2 - a_2b_1 = 0$ which is a quadratic equation in m and so there are two roots m_1 and m_2 the stability of dynamical system induced by the autonomous differential equation (A) depends on the nature of roots which are broadly classified into five categories which represents the type of critical point, general solution and its stability.

Category I: When the roots are real and equal that is, $m_1 = m_2 = m$ then the critical point is a *node*and the general solution is given by;

$$x = c_1Ae^{mt} + c_2(A_1 + At)e^{mt}$$
$$x = c_1Be^{mt} + c_2(B_1 + Bt)e^{mt}$$

Where A, B, A_1, B_2 are definite constants and c_1, c_2 are arbitrary constant. If $m < 0$ then the critical point is asymptotically stable and if $m > 0$ then the critical point is unstable.

Category II: When the roots are real, distinct and of same sign then the critical point is a *node*and the general solution is given by;

$$x = c_1A_1e^{m_1t} + c_2A_2e^{m_2t}$$
$$y = c_1B_1e^{m_1t} + c_2B_2e^{m_2t}$$

In this category the critical point is asymptotically stable if both the roots are negative and unstable if both are positive.

Category III: When the roots are real, distinct and are of opposite sign then the critical point is a saddle point and in this case the critical point is unstable.

Category IV: When the roots are complex conjugates that is roots are $a \pm ib$ with $a \neq 0$ then the critical point is a spiral and the general solution is given by;

$$x = e^{at}[c_1(A_1cosbt - A_2sinbt) + c_2(A_1sinbt + A_2cosbt)]$$
$$y = e^{at}[c_1(B_1cosbt - B_2sinbt) + c_2(B_1sinbt + B_2cosbt)]$$

If a < 0 then the critical point is asymptotically stable and if a > 0 then the critical point is unstable.

Category V: When the roots are purely imaginary then the critical point is a center which is stable and the general solution is given by;

$$x = c_1(A_1cosbt - A_2sinbt) + c_2(A_1sinbt + A_2cosbt)$$
$$y = c_1(B_1cosbt - B_2sinbt) + c_2(B_1sinbt + B_2cosbt)$$

3 Mathematical Model for Competition

Let P and Q be two species in a region R such that P is competitively superior then Q and Q is competitively inferior then P competing for same resources. Let us define the following,

1. R denotes the bounded geographical region.
2. T denotes the period of competition.
3. t denotes time at any instant.
4. x denotes the population density of species P.
5. y denotes the population density of species Q.
6. N denotes the population density of each species at $t = 0$.
7. λ denotes the competition parameter which is a positive constant quantity.
8. a denotes the physical constant responsible for growth of both the species.

Assumptions

• Resources is the region R is constant in a given period of time [0,T].

• There is no resource partitioning and other ecological adjustments.

• If a > 0 then, death rate is less then birth rate and other factors responsible for growth of population are also favourable except competition.

• If a < 0 then, death rate is greater then birth rate and other factors responsible for growth of population are not favourable except competition.

• Only the species P and Q are competing for the same resources and no other species share or compete for the same resource.

• Immigration for both the species P and Q are not allowed.

Let us define an autonomous dynamical system representing the growth rate of population of both the species with respect to time t where the competition plays an important role in the growth of both the species and a competitively superior species must flourish in a faster rate as compare to competitively inferior species.
Further the resources in a given geographical area is assumed to be constant and therefore the growth of competitively inferior species in such a geographical region is assumed to be small and λ being the growth factor of population P with respect to population Q and vice versa we easily see that rate of growth of population of P with respect to Q depends on λ and rate of growth of population of Q with respect to P depends on −λ. There are three possible cases for the growth rate of competitively superior species.

Case I: When rate of growth of competitively superior species depends positively on competition parameter λ and population density of competitively inferior species.
Let us construct an autonomous dynamical system,

$$\begin{cases} \dfrac{dx}{dt} = ax + \lambda y \\ \dfrac{dy}{dt} = -\lambda x + ay \end{cases}$$

Since a and λ are both positive constant in a given interval of time $W(t) = a^2 + \lambda^2 \neq 0$

And it follows that (0,0) is the only critical point. Considering the auxiliary equation of (A),

$$m^2 - 2am + a^2 + \lambda^2 = 0$$
$$m = a \pm i\lambda$$

With non-zero real part and so the trajectory is a spiral about critical point (0,0). Then the solution to autonomous differential equation is given by;

$$x = e^{at}[c_1(A_1 cos\lambda t - A_2 sin\lambda t) + c_2(A_1 sin\lambda t + A_2 cos\lambda t)]$$
$$y = e^{at}[c_1(B_1 cos\lambda t - B_2 sin\lambda t) + c_2(B_1 sin\lambda t + B_2 cos\lambda t)]$$

Also the differential equation of trajectory is given by;

$$\frac{dy}{dx} = \frac{-\lambda x + ay}{ax + \lambda y}$$

Which is a homogenous differential equation of degree 0. And therefore equation of trajectory is given by;

$$y = x \tan\left[\frac{\lambda}{2a} \ln\left(\frac{1}{c(x^2 + y^2)}\right)\right]$$

Where c is a arbitrary constant.

Initially at $t = 0$, $x = y = N$ and so $\ln\frac{1}{c} = \frac{a\pi}{2\lambda} + \ln 2N^2$.

Thus, the equation of trajectory is given by

$$y = x \tan\left[\frac{\lambda}{2a} \ln\left(\frac{1}{(x^2+y^2)}\right) + \frac{\pi}{4} + \ln 2N^2\right]$$

Physical Interpretation

If $a < 0$ then the dynamical system is stable and under this condition physical constant for growth of both the species is negative. This is equivalent to the situation when birth rate is comparatively less then death rate and in this situation the species Q which is competitively inferior is wiped out from the region R. Further, the species P which is competitively superior exists in the region R, this is due to the fact that competition parameter λ acts positively on population P and therefore growth rate of population P is not diminished.

If a > 0 then the dynamical system is unstable. In this case due to the competition between two species both of them flourish though the rate of growth of competitively superior species is faster then competitively inferior species the point (x, y) at any time t recedes from the critical point (0,0). But, R is bounded and resources are definite for a given period of time T therefore there exists a time $T > 0$ when all the resources gets exhausted by both the species and gradually the competition parameter decreases to zero that is, there is no more competition between the species P and Q and in that case either both the species adapt themselves towards other resources or they migrate from one geographical region to the other. In either way both the species grows exponentially independent of one another or both of them become extinct from the region R.

The observation that can be made from the equation of trajectory for this case is that, population density of Q decreases with increase of competition parameter λ for a given population density of P. This represents the competitive superiority of the species P over the species Q.

Case II: When rate of growth of competitively superior species depends only on physical factors that is the competition parameter λ = 0 and competitively superior species grows naturally without any help from nearby competitively inferior species.

Let us construct an autonomous dynamical system ,

$$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dy}{dt} = -\lambda x + ay \end{cases}$$

Then $W(t) = a^2 \neq 0$

Therefore, (0,0) is the only critical point. And the auxiliary equation is $m^2 - 2am + a^2 = 0$ this implies $m = -a, -a$. Thus, the roots of auxiliary equation are real and equal with $m = -a < 0$. It follows that in this case the system is a node and as t increases the trajectory approaches asymptotically stable and the solution is given by;

$$x = c_1 A e^{at} + c_2 (A_1 + A_2 t) e^{at}$$

$$y = c_1 B e^{at} + c_2 (B_1 + B_2 t) e^{at}$$

Where A, B, A_1, B_2, A_1, B_2 are definite constants and c_1, c_2 are absolute constants.

The differential equation of trajectory for this dynamical system,

$$\frac{dy}{dx} = \frac{-\lambda x + ay}{ax}$$

Which is a homogenous differential equation of degree 0. And therefore equation of trajectory is given by;

$$x = c^{-1} e^{-\frac{ay}{\lambda x}}$$

Where c is an arbitrary constant.

Initially at $t = 0$, $x = y = N$ and so $c = N^{-1} e^{-\frac{a}{\lambda}}$

Thus, the equation of trajectory under given initial condition is given by;

$$x = N^{-1} e^{\frac{a}{\lambda}\left(\frac{x-y}{x}\right)}$$

Physical Interpretation:

If a > 0 then roots of auxiliary equation is negative and therefore the critical point is stable. In this case population P grows naturally and being a competitively superior species over Q, the competition parameter λ acts negatively on the rate of growth of population Q, due to this, population density of Q decreases with the increase of population density of the species Q.

If a < 0 then roots of auxiliary equation is positive and therefore the critical point is unstable. In this case population density of the species P decreases gradually as physical factor a responsible for natural growth of population is not helping population P and due to this population density of the species P is diminished in due period of time T. Species P being competitively superior over the species Q, we can observe form the equation of trajectory that, population density of Q diminish faster then population density of the species P.

Case III: When rate of growth of competitively superior species depends negatively on competition parameter and population density of competitively inferior species.

Let us construct an autonomous dynamical system,

$$\begin{cases} \dfrac{dx}{dt} = ax - \lambda y \\ \dfrac{dy}{dt} = -\lambda x + ay \end{cases}$$

Then $W(t) = a^2 - \lambda^2$

If λ = a then we can see from the above system of equation that population density of both the species remains constant throughout the period of competition which is possible only when birth rate and death rate are equal in both the species further P and Q partition the resources inthe given geographical region R and neither one is competitively superior or inferior with respect to the other which is against our hypothesis. So we must have, $\lambda \neq a$ it follows that (0,0) is the only critical point in this case. And the auxiliary equation is $m^2 - 2am + a^2 - \lambda^2 = 0$ this implies $m = a \pm \lambda$ that is roots are real and distinct. There are two possible cases for critical point.

Case A: If $0 < \lambda < a$ then roots are of same sign and critical points is an unstable node. That is at any instant, point (x, y) has the tendency to recede from $(0,0)$. And the general solution is given by;

$$x = c_1 A_1 e^{(a+\lambda)t} + c_2 A_2 e^{(a-\lambda)t}$$

$$y = c_1 B_1 e^{(a+\lambda)t} + c_2 B_2 e^{(a-\lambda)t}$$

Where A_1, A_2, B_1, B_2 are definite constants and c_1, c_2 are absolute constants.

The differential equation of the trajectory for this system is given by;

$$\frac{dy}{dx} = \frac{ax - \lambda y}{-\lambda x + ay}$$

Which is a homogenous differential equation of degree 0. And therefore the equation of trajectory is given by;

$$(y - x)^{(a-\lambda)} = c(y + x)^{(a+\lambda)},$$

Where c is arbitrary constant,

Initially at $t = 0$, $x = y = N$ and so $c = 0$

Then the equation of trajectory under given condition is $x = y$.

Physical Interpretation: Thus phase portrait of this trajectory is a straight line passing through origin which represents that growth rate of both the species P and Q is same provided value of physical constant responsible for growth of both the species must be greater than competition parameter. This is not true in general as competition parameter and physical constant for growth, are independent of one another. Such situation exists in nature when the species P and Q are competitively analogous that is, competition has negative impact on both the species in equal amount but the rate of growth of population of both the species P and Q are parallel to one another. But as far as our assumptions are concerned, resources in the geographical region R are constant and resource partitioning is not allowed, this will either drive the species P and Q getting diminished or they tend to migrate from region R. And in either of the cases both the species P and Q fail to exist in the geographical region R.

Case B: If $0 < a < \lambda$ then, roots are of same sign and in this case critical point is a saddle point which is obviously unstable further the equation of trajectory does not satisfy the initial condition. Thus this case is not possible

4 Conclusions

In this paper, we shall construct an autonomous linear dynamical system to study and analyse the competition between two species in a bounded geographical region for given period of time. The problem is subdivided into three different cases based on the dependence of growth rate of competitively superior species on competition parameter. We analyses each case on the basis of competition parameter and physical constant responsible for growth. Finally, we interpret the results physically.

Construction of this model is based on two important assumptions that, no immigration of the species chosen to study competition in the region and there is no resource partitioning between the species under consideration. We may incorporate the immigration factor to the autonomous differential equation so that we can study competition between two species, where population density of species under consideration may vary with time, though the variation in population density may be continuous with time or discrete.

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