On δ - lower semi precontinuous functions

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Abstract : In this paper, the concept of δ - lower semi precontinuous functions is to be introduced. Some characterization theorems and their basic properties are also to be investigated. It is note that δ - lower semi precontinuous functions play an important role in defining δ - preinduced fuzzy supra topological spaces. The connection between the δ - preinduced fuzzy supra topology and its corresponding topological space is to be studied. The relationship between δ - preinduced fuzzy supra topological spaces and that of the induced fuzzy topological spaces due to Lon owen are to be investigated. Lastly some applications are to be shown.

Key Words : δ - lower semi precontinuous function, induced fuzzy topological spaces, δ - induced fuzzy topological spaces, δ - preinduced fuzzy topological spaces.

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1. Introduction

The concept of induced fuzzy topological space (IFTS) was introduced by Weiss [16]. Lowen [8] called these spaces as topologically generated space. The concept of fuzzy supra topological space was introduced by Abd El-Monsef and Ramadan [1]. The notion of lower semi continuous functions [16] plays an important role in defining the above concept. The concept of lower semi precontinuous functions was introduced by Mukherjee [11]. Mukherjee and Halder [12] introduced the concepts of δ - lower semi continuous functions and δ - induced fuzzy topological spaces. The aim of this chapter is to introduce a new class of functions, called δ - lower semi precontinuous functions as a generalization of δ - precontinuous functions. Also the purpose of this chapter is to introduce the notion of δ - preinduced fuzzy supra topological spaces. The notion of δ - lower semi precontinuous functions plays an important role in defining δ - preinduced fuzzy supra topological spaces.

In section 3, I would introduce the concept of δ - lower semi precontinuous functions. Some of their characterization theorems and basic properties are also to be studied. In section 4, δ - preinduced fuzzy supra topological spaces is to be defined. I would also study some connections between the properties of a topological space (X, T) and that of the corresponding δ - preinduced fuzzy supra topological space (X, D(T)). In section 5, I introduce and study δ - p - initial spaces.

2. Preliminaries

In this section I would like to mention some known results and definitions for ready references.

Definition 2.1. [15] A set A of a topological space (X, T) is called regular open if A = intcl(A).

Definition 2.2. [9] A set A of a topological space (X, T) is called preopen if $A \subset$ intcl(A).

Definition 2.3. [15] A set A of a topological space (X, T) is called δ - open if for each point $x \in A$, there exists a regular open set W such that $x \in W \subseteq A$. The complement of a δ - open set is said to be δ - closed.

Definition 2.4. [5] A set A of X is called δ - preopen if $A \subset$ intδcl(A). The complement of a δ - preopen set is said to be δ - preclosed.

Definition 2.5. [6] The strong r - cut and weak r - cut of α are defined by $\sigma_r(\alpha) = \{x \in X : \alpha(x) > r\}$ and $W_r(\alpha) = \{x \in X : \alpha(x) \geq r\}$ where $\alpha \in I^X$ and $r \in I = [0,1]$.

Definition 2.6. [16] A function $f : (X, T) \to (R, U)$ is said to be lower semi continuous (LSC) at a point x_0 of X if and only if for every $\varepsilon > 0$ there exists an open neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

Definition 2.7. [16] Let (X, T) be a topological space. The collection W(T) of all lower semi continuous (LSC) functions $f : (X, T) \to I$ (I is the unit closed interval) forms a fuzzy topology on X. Then (X, W(T)) is known as an induced fuzzy topological (IFT) space.

Definition 2.8. [11] A function $f : (X, T) \to (R, U)$ is said to be lower semi precontinuous (LSPC) at a point x_0 of X if and only if for every $\varepsilon > 0$ there exists an preopen neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

Definition 2.9. [12] A function $f : (X, T) \to (R, U)$ is said to be δ - lower semi continuous at a point x_0 of X if and only if for every $\varepsilon > 0$, there exists a δ - open neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

Definition 2.10. [12] Let (X, T) be a topological space. The collection W(T) of all δ - lower semi continuous (LSC) functions $f : (X, T) \to I$ (I is the unit closed interval) forms a fuzzy topology on X. Then (X, W(T)) is known as an δ - induced fuzzy topological space.

3. *δ - lower (upper) semi precontinuous functions*

In this section, I would introduce a new class of functions as a generalization of δ - precontinuous functions. A function $f : (X, T_1) \to (Y, T_2)$ from a topological space (X, T_1) to another topological space (Y, T_2) is called δ - precontinuous if and only if the inverse image of each every open subset in Y is δ - preopen in X. By X, we mean a topological space (X, T).

Definition 3.1. A function $f : X \to R$, where R is the real line is said to be δ - lower semi precontinuous (δLSPC) at x_0 of X if and only if for each $\varepsilon > 0$, there exists a δ - preopen neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

A function $f : X \to R$ is said to be δLSPC if it is so at each point of X.

Definition 3.2. A function $f : X \to R$, where R is the real line is said to be δ - upper semi precontinuous (δUSPC) at x_0 of X if and only if for each $\varepsilon > 0$, there exists a δ - preopen neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) < f(x_0) + \varepsilon$.

A function $f : X \to R$ is said to be δUSPC if it is so at each point of X.

Theorem 3.3. The necessary and sufficient condition for a real valued function f to be δLSPC is that for all $a \in R$, the set $\{x \in X : f(x) > a\}$ is δ - preopen.

Proof : Let f be δLSPC function and let $x_0 \in X$ and $N(x_0)$ be the δ - preopen nbd. of a point x_0 of X. Then $f(x_0)$ is a real number so that $f(x_0) - \varepsilon$ is a fixed real number in R for a point x_0 of X. By definition $f(x) > f(x_0) - \varepsilon = a$ (say). Hence the set of all points x for which $f(x) > a$ is δ - preopen.

Conversely, let $x \in X$ be such that $f(x) > a$ is δ - preopen and x_0 be any point in X. Let us choose a real number ε such that $f(x_0) - \varepsilon = a \in R$. From the given condition the set $\{x \in X : f(x) > f(x_0) - \varepsilon\}$ is δ - preopen which implies that if $\{x \in A \subseteq X\}$ is δ - preopen, the condition $f(x) > f(x_0) - \varepsilon$ is true which shows that f is δLSPC.

Theorem 3.4. The necessary and sufficient condition for a real valued function f to be δUSPC is that for all $a \in R$, the set $\{x \in X : f(x) \le a\})$ is δ - preclosed.

Proof : From Theorem 3.3., it follows that the necessary and sufficient condition for a real valued function f to be δLSPC is that for all $a \in R$, the set $\{x \in X : f(x) \le a\}$ is δ - preclosed, being the complement of δ - preopen.

Hence the theorem.

Theorem 3.5. A function f from a topological space (X, T) onto a space (R, σ_1), where $\sigma_1 = \{(r, \infty) : r \in R\}$ is δLSPC if and only if the inverse image of any open subset of (R, σ_1) is δ - preopen in (X, T).

Proof : Let $f : (X, T) \to (R, \sigma_1)$ be a δLSPC function and $N(x_0)$ be any δ - preopen nbd. of a point x_0 of X. Then for any point x in $N(x_0)$, $f(x) > f(x_0) - \varepsilon$. It is clear that the set $\{x \in X : f(x) > f(x_0) - \varepsilon\} = N(x_0)$, which is δ - preopen. Taking $f(x_0) - \varepsilon = r$, we have $\{x \in X : f(x) > r\}$, which is same as the set $\{x \in f^{-1}(r, \infty) : f(x) > r\}$. Thus for any open set (r, ∞) in (R, σ_1), $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ is δ - preopen.

Conversely, let the inverse image of an open subset (r, ∞) is δ - preopen in (X, T). Then $f^{-1}(r, \infty)$ consists of those points of X whose images are > r, i.e., $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ is δ - preopen. Thus by **Theorem 3.3.**, f is δLSPC function.

Theorem 3.6. The characteristic function of a δ - preopen set is δLSPC function.

Proof : Let A be a δ - preopen set. The characteristic function of A is defined by

$$\mu_A = \begin{cases} 1, & x \in A \\ 0, & x \in X - A. \end{cases}$$

We have to show that μ_A is δLSPC function. We wish to show that $\{x : \mu_A(x) \le r\}$ is δ - preclosed, for each $r \in R$. For $r < 0$, the set $\{x : \mu_A(x) \le r\} = \Phi$, which is δ - preclosed. For $0 \le r < 1$, the set $\{x : \mu_A(x) \le r\} = X - A$, which is again δ - preclosed, being complement of δ - preopen subset of A. For $r \ge 1$, the set $\{x : \mu_A(x) \le r\} = X$, which is δ - preclosed. Hence μ_A is δLSPC function.

Hence the theorem.

Theorem 3.7. If $\{f_j : j \in I\}$ is an arbitrary family of δLSPC functions, then the function g defined by $g(x) = \mathrm{Sup}(f_j(x))$ is δLSPC.

Proof : Let $r \in R$ and $g(x) < r$, then $f_j < r$, for all $j \in I$. Now $\{x \in X : g(x) \le r\} = \cap\{x \in X : f_j(x) \le r\}$. But each f_j being δ - lower semi precontinuous, by **Theorem 3.4.**, each set $\{x \in X : f_j(x) \le r\}$ is δ - preclosed in X. Now we know that an arbitrary intersection of δ - preclosed sets is δ - preclosed. Therefore g is δLSPC.

Theorem 3.8. If $f_1, f_2, f_3, \ldots\ldots\ldots, f_n$ are δLSPC functions, then the function h defined by $h(x) = \inf(f_j(x))$ is not δLSPC, where $j = 1$ to n.

Proof : The theorem follows from the fact that an arbitrary intersection of δ - preopen subsets may not be δ - preopen.

Result 3.9. Every δ-lower semi continuous (δ-LSC) function is δ-lower semi precontinuous (δ-LSPC) function.

Proof : Since every δ-open subset is δ-preopen, the result follows immediately.

The converse of the above result is not true, which can be shown from the fllowing example.

Example 3.10. Let X = {a, b, c} and Y = {0, 1, 2}. Let τ_1={X, φ, {x, b}} and τ_2={Y, φ, {1}, {2}, {1, 2}} be the topologies on X and Y respectively. A function $f : X \to Y$ is defined by f(a) = 1, f(b) = 2, f(c) = 0. Now $f^{-1}(0) = \{c\}$, $f^{-1}(2) = \{b\}$, $f^{-1}(1, 2) = \{a, b\}$. We observe that {a}, {b}, {a, b}, {a, b}, {a, c}, {b, c} are preopen subsets in (X, τ_1) [11]. Hence each of them is δ-preopen set in (X, τ_1). We fix r = 1, then the set $\{x \in X : f(x) > 1\}$ is δ - preopen in X (since the inverse image of the open subset {2} in Y is δ - preopen in X), thus by **Theorem 3.3.,** f is δ-LSPC. But f is not δ-LSC since $\{b\} \notin \tau_1$.

4. *δ - preinduced fuzzy supra topological spaces*

The notion of induced fuzzy topology W(T) was due to Weiss [16]. It is the collection of all semicontinuous functions from a topological space (X, T) to the unit closed interval I = [0, 1]. If $A \in T$, then $I_A \in W(T)$. Abd El-Monsef and Ramadan [1], introduced the concept of fuzzy supra topology as follows :

A family $T^* \subset I^X$ is called a fuzzy supra topology on X, if $0, 1 \in T^*$ and T* is closed under arbitrary union.

In this section, the notion of δ - preinduced fuzzy supra topology (δPIFST) is to be introduced. Its properties and the concept of fuzzy supra δ - precontinuity in δPIFST spaces are to be studied.

Theorem 4.1. Let (X, T) be a topological space. The family of all δ - lower semi precontinuous functions from the topological space (X, τ) to the unit closed interval I = [0, 1] forms a fuzzy supra topology on X.

Proof : Let D(T) be the collection of all δLSPC functions from (X, T) to I. We will now prove that D(T) forms a fuzzy supra topology on X.

(i) Since X is open, it is δ - preopen and thus 1_X is δLSPC, i.e., $1_X \in D(T)$ (by Theorem 3.6.).

(ii) Φ is δ - preopen, thus $1_\Phi \in D(T)$.

(iii) Let $\{f_j : j \in I\}$ be an arbitrary family of δLSPC functions, then $\mathrm{Sup}(f_j(x)) \in D(T)$ [by **Theorem 3.6.**].

Thus D(T) forms a fuzzy supra topology.

Definition 4.2. The fuzzy supra topology obtained as above is called δ - preinduced fuzzy supra topology (δPIFST) and the space (X, D(T)) is called the δ - preinduced fuzzy supra topological (δPIFST) space. The members of D(T) are called fuzzy supra δ-open subsets.

Example 4.3. Let X = {a, b, c} and τ = {X, φ, {a, b}} be a topology on X. Besides the members of τ, it follows that {a}, {b}, {a, c}, {b, c} are also preopen subsets, hence δ - preopen susets. Thus 1_φ, 1_X, $1_{\{a\}}$, $1_{\{b\}}$,$1_{\{a,\ b\}}$, $1_{\{a,\ c\}}$, $1_{\{b,\ c\}}$ are all δLSPC. Then the collection of all these functions form a δPIFST on X, $P(\tau) = \{1_\varphi, 1_X, 1_{\{a\}}, 1_{\{b\}}, 1_{\{a, b\}}, 1_{\{a, c\}}, 1_{\{b, c\}}$.

Lemma 4.4. If A is δ - preopen in (X, T), then $\mu_A \in D(T)$.

Proof : By the **Theorem 3.6.**, the lemma follows immediately.

Lemma 4.5. If W(T) is an induced fuzzy topology and D(T) is a δ - preinduced fuzzy supra topology on X, then $W(T) \subset D(T)$.

Proof : If $\alpha \in W(T)$, i.e., α is lower semi continuous function. Since every lower semi continuous function is δ - lower semi precontinuous, $\alpha \in D(T)$. Hence $W(T) \subset D(T)$.

Theorem 4.6. A fuzzy subset A in a δPIFST space (X, D(T)) is fuzzy supra open if and only if for each $r \in I$, strong r - cut $\sigma_r(A) = \{x \in X : A(x) > r\}$ is δ - preopen in the topological space (X, T).

Proof : A fuzzy subset A is fuzzy supra open in (X, D(T)) if $A \in D(T)$. Now $A \in D(T)$ if and only if A is δLSPC. Again A is δLSPC if and only if for each $r \in I$, $\{x \in X : A(x) > r\}$ is δ - preopen in (X, T), i.e., $\sigma_r(A)$ is the δ - preopen in (X, T).

Corollary 4.7. A fuzzy subset A in a δPIFST space (X, D(T)) is fuzzy supra closed if and only if for each $s \in I$, the weak s - cut $W_s(A) = \{x \in X : A(x) \geq s\}$ is δ - preclosed in the topological space (X, T).

Definition 4.8. A function $f : (X, D(T_1)) \to (X, D(T_2))$ from a δPIFST space to another δPIFST space is said to be fuzzy supra δ - precontinuous if $f^{-1}(A) \in D(T_1)$ for every $A \in D(T_2)$.

Definition 4.9. A function $f : X \to Y$ from a topological space (X, T_1) to another topological space (Y, T_2) is said to be δ - preirresolute if and only if the inverse image of each δ - preopen subset in Y is δ - preopen in X.

Theorem 4.10. A function $f : (X, D(T_1)) \to (X, D(T_2))$ is fuzzy supra δ - precontinuous if and only if $f : (X, T_1) \to (Y, T_2)$ is δ - preirresolute.

Proof : Let f be fuzzy supra δ - precontinuous and A is δ - preopen in (Y, T_2). Then

$$\begin{aligned} f^{-1}(A) &= \{ x \in X : \mu_A(f(x)) = 1\} \\ &= \{x \in X : f^{-1}(\mu_A(x)) > r, 0 < r < 1\} \\ &= \sigma_r(f^{-1}(\mu_A)). \end{aligned}$$

Since A is δ - preopen, by **Lemma 4.4.**, $\mu_A \in D(T)$, thus, $f^{-1}(\mu_A) \in D(T_1)$. Hence by **Theorem 4.6.**, $\sigma_r(f^{-1}(\mu_A))$ is δ - preopen in (X, D_1) which implies f is δ - preirresolute.

Conversely, let us consider $f : (X, D_1) \to (Y, D_2)$ to be δ - preirresolute and $B \in D(T_2)$. Now, for any $s > 0$,

$$\begin{aligned} \sigma_s(f^{-1}(B)) &= \{ x \in X : f^{-1}(B)(x)) > s, 0 < s < 1\} \\ &= \{ x \in X : B(f(x)) > s\} \\ &= (Bf)^{-1}(s, \infty) \\ &= f^{-1}(B^{-1}(s, \infty)). \end{aligned}$$

Since $B \in D(T_2)$, B is δLSPC, therefore, $B^{-1}(s, \infty)$ is δ - preopen in (Y, T_2). Hence by hypothesis, $f^{-1}(B^{-1}(s, \infty))$ is δ - preopen in (X, T_1), i.e., $\sigma_s(f^{-1}(B))$ is δ - preopen in (X, τ_1), which implies $f^{-1}(B) \in D(T_1)$.

Hence f is fuzzy supra δ - precontinuous.

5. *δ - p - initial spaces*

The notion of initial fuzzy topology was first introduced by Lowen [8] and was further studied by Chang Ming [10]. Mukherjee [11] introduced and studied the concept of p - initial topology. In this section, the concepts δ - p - initial topology are to be introduced in a δPIFST and some of its properties are to be investigated. The relationship between the topologies of δ - p - initial space and an p - initial space is also to be studied.

Definition 5.1. Let (X, D(T)) be a δPIFST spaces. The family $\sigma_r(A) : A \in D(T), r \in I\}$ of δ - preopen subsets of X forms a subbase of some topology of X, called δ

- p -initial topology of D(T) and is denoted by $i_{\delta p}(T')$ where T' = D(T). Then (X, $i_{\delta p}(T')$) is known as δ - p - initial topological space or δ - p - initial space.

Example 5.2. Let X = {a, b, c} and τ = {X, φ}, {a, b} be a topology on X. Here preopen sets, hence δ-preopen sets of (X, τ) are {X, φ, {a}, {b}, {a, b}, {a, c}, {b, c} }. $i_{\delta p}(T)$ = {X, {a}, {b}, {a, b}.

Lemma 5.3. i(W)(T) ⊂ $i_{\delta p}(D(T))$, where $i_{\delta p}(D(T))$ is the δ - p - initial topology and i(W) is the initial topology on X.

Proof : Straight forward.

Theorem 5.4. Let (X, T_1) and (Y, T_2) be two topological spaces and $i_{\delta p}(D(T_1)$ and $i_{\delta p}(D(T_2)$ be the δ - p - initial topological spaces on X and Y respectively. Then

(a) If f : (X, T_1) → (Y, T_2) is δ - precontinuous, then f : (X, $i_{\delta p}(D(T_1))$) → (Y, $i_{\delta p}(D(T_2))$) is δ - continuous.

(b) If f : (X, T_1) → (Y, T_2) is δ - preirresolute, then f : (X, $i_{\delta p}(D(T_1))$) → (Y, $i_{\delta p}(D(T_2))$) is δ - continuous.

Proof : Straight forward.

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