

A Short Review on Prime Number Theorem

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Abstract: Prime number theorem is a well known theorem in Mathematics, specially in Number Theory which describes the asymptotic distribution of prime numbers. One of the remarkable discovery regarding this topic is Riemann Hypothesis. Since a long period several renowned mathematicians are trying to prove or disprove this hypothesis and to reduce the error bound of the asymptotic distribution of prime counting function.

Keywords: Prime Number Theorem, Riemann Hypothesis, asymptotic approximation, prime counting function, Scale invariant infinitesimals.

1. Introduction :

Analytic number theory is a branch of number theory that uses methods from mathematical analysis to solve problems about natural numbers [1, 2]. The modern study of analytic number theory may be said to have begun in the eighteenth century with Euler's proof of the divergence of the series of inverse prime $\sum \frac{1}{p} = \infty$ and latter with Dirichlet's introduction of Dirichlet L-functions in the first half of nineteenth century to give the first proof of Dirichlet's theorem on arithmetic progressions [1]. Dirichlet's theorem on arithmetic progressions was one of the major achievements of 19th century mathematics. One of the major milestones in this subject is the Prime Number Theorem (PNT).

Analytic number theory can be classified into two classes depending on the nature of the problems, one is multiplicative number theory and the second is additive number theory . Multiplicative number theory deals with the distribution of prime numbers, such as estimating the number of primes in an interval and includes the prime number theorem and Dirichlet's theorem on primes in arithmetic progressions. Additive number theory is concerned with the additive structure of integers such as Goldbach's conjecture which states that every even number greater than 2 is the sum of two primes. One of the main results in additive number theory is the solution of Waring's problem. Developments within analytic number theory are

often refinements of earlier techniques which reduce the error terms and widen their applicability. The biggest technical change after 1950 has been the development of Sieve methods as a tool, particularly in multiplicative problems. These are combinatorial in nature and quite varied. The external branch of combinatorial theory has in return been greatly influenced by the value placed in analytic number theory on quantitative upper and lower bounds. Another recent development is probabilistic number theory. Which uses tools from probability theory as estimate the distribution of number theoretic functions, such as how many prime divisors a number has. In this paper we mainly focus on prime number theorem.

2. Statement of Prime Number Theorem :

Let $\Pi(x)$ be the prime counting function that gives the number of primes less than or equal to x , for any real number x . For example, $\Pi(8) = 3$ because there are three prime numbers (3, 5 and 7) less than or equal to 8. The prime number theorem then states that the limit of the quotient of the two functions $\Pi(x)$ and $\frac{x}{\log x}$ as x approaches infinity is 1, which is expressed by the formula $\lim_{x \rightarrow \infty} \frac{\Pi(x)}{\frac{x}{\log x}} = 1$, known as the asymptotic law of distribution of prime numbers. Using asymptotic notation this result can be restated as $\Pi(x) \sim \frac{x}{\log x}$.

This notation (and the theorem) does not say anything about the limit of difference of the two functions as x approaches infinity. Indeed, the behaviour of this difference is very complicated and related to the Riemann Hypothesis. Instead, the theorem states that $\frac{x}{\log x}$ approaches $\Pi(x)$ in the sense that the relative error of this approximation approaches 0 as x approaches infinity. Accordingly to the RH, the relative correction (error) should be given by $\frac{\Pi(x)}{\frac{x}{\log x}} = 1 + O(x^{-\frac{1}{2}+\epsilon})$, for any $\epsilon > 0$. So far no proof of the PNT could retrieve and substantiate the RH correction term, although all the current experimental searches on primes are known to agree with the RH value [2].

The prime number theorem is equivalent to the statement that the n th prime number p_n is approximately equal to $n \ln n$, again with the relative error of this approximation approaches 0 as n approaches infinity.

3. Theoretical development of Prime Number Theorem :

Let $\Pi(x)$ be the number of primes $p \leq x$. It was discovered experimentally by Gauss about 1793 [3] and by Legendre in 1798 that as $\Pi(x) \sim \frac{x}{\log x}$.

This statement is the prime number theorem. Actually Gauss used the equivalent formulation $\Pi(x) \sim \int_2^x \frac{dt}{\log t}$.

In 1850 Chebyshev [5] proved a result for weaker than the prime number theorem – that for certain constants $0 < K_1 < 1 < K_2$, $K_1 < \frac{\Pi(x)}{x/\log x} < K_2$.

An elementary proof of Chebyshev’s theorem is given in Andrews [6]. Chebyshev’s introduced the functions $\theta(x) = \sum_{p \leq x, p \text{ prime}} \log p$, (Chebyshev’s theta function) and $\Psi(x) = \sum_{p^n \leq x, p \text{ prime}} \log p$ (Chebyshev’s psi function). Note that $\Psi(x) = \sum_{n=1}^{\infty} \theta(x^{1/n})$ where the sum is finite for each x since $\theta(x^{1/n}) = 0$ if $x < 2^n$. Chebyshev proved that prime number theorem is equivalent to either of the relations $\theta(x) \sim x$, $\Psi(x) \sim x$.

Chebyshev also showed that if $\lim_{x \rightarrow \infty} \frac{\theta(x)}{x}$ exists, then it must be 1, which then implies the prime number theorem. He was, however, unable to establish the existence of the limit.

Like Gauss, Riemann formulate his estimate of $\Pi(x)$ in terms of the logarithmic integral $Li(x) = (pv) \int_0^x \frac{dt}{\log t}$, $x > 1$. In his famous 1859 paper [7] he related the relative error in the asymptotic approximation $\Pi(x) \sim Li(x)$ to the distribution of the complex zeros of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$ (1)

The Riemann zeta function was actually introduced by Euler as early as 1737. Riemann did not prove the prime number theorem in his 1859 paper. He found an explicit analytic expression for $\Pi(x)$. He did comment that $\Pi(x)$ is about $Li(x)$ and that

$\Pi(x) = Li(x) + O(x^{1/2})$. This would imply $\frac{\Pi(x)}{Li(x)} = 1 + O(x^{-\frac{1}{2} \log x}) = 1 + O(1)$ which again gives the prime number theorem. In 1896 the prime number theorem was finally proved by Jacques Hadamard [8] and also by Poussin [9]. The first part of the proof is to show that $\zeta(s) \neq 0$ if $\text{Re } s = 1$. As a general principle, finding zero-free regions for the zeta function in the critical strip leads to better estimates of the error in $\Pi(x) \sim Li(x)$. In his paper [7] Riemann asserts that $\Pi(x) < Li(x)$. This is known to be true for all $x \leq 10^8$ but was proved false in general in 1914 by Littlewood [10]. Littlewood showed that $\Pi(x) - Li(x)$ changes sign infinitely often. Littlewood also showed that there is a constant $k > 0$ such that $\frac{(\Pi(x) - Li(x)) \log x}{x^{1/2} \log \log x}$ is greater than k and less than $-k$ for arbitrarily large x . Littlewood’s methods gives no information on where the first sign change occurs. In 1933 Skewes [11] shows that there is atleast one sign change at x for some $x < 10^{10^{34}}$. Skewes’s proof required the Riemann hypothesis. In 1955 [12] he obtained a bound with out using Riemann hypothesis. This new bound was $10^{10^{964}}$. In 1966 Shermann Lehmann [13] showed that between 1.53×10^{1165} and 1.65×10^{1165} there are more than 10^{500} successive integers x for which $\Pi(x) > Li(x)$. In 1987 Riele [14] showed that between 6.62×10^{370} and 6.69×10^{370} there are more than 10^{180} successive integers x for which $\Pi(x) > Li(x)$.

Ramanujan estimates $\Pi(x)$ by $\Pi(x) \approx \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n})$ (2)
(Ramanujan’s 2nd letter to Hardy in 1913, see [14], page 53)

Where $\mu(n)$ is the Mobius function. This expression was obtained by Riemann in 1859, except Riemann has additional terms, arising from the complex zeros of $\zeta(s)$. Littlewood points out that $\Pi(x) - Li(x) + \frac{1}{2} Li(x^{1/2}) \neq O(\frac{x^{\frac{1}{2}}}{\log x})$

It follows that $\Pi(x) - Li(x) + \frac{1}{2} Li(x^{\frac{1}{2}}) \neq O(Li(x^{1/2}))$ (3)

Thus it is clear that equation (2) can not be interpreted as an asymptotic series for $\Pi(x)$. Ramanujan says to truncate the series at the first term less than one. This gives an excellent approximation to $\Pi(x)$, but it is empirical. The

actual expression obtained by Ramanujan is $\Pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n})$, $J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$ where ρ runs over the complex roots of the zeta function. The first term here is actually finite for each x since $J(x) = \sum_{n=1}^{\infty} \frac{1}{n} \Pi(x^{1/n})$ is 0 for $x < 2$. The complete proof of Riemann's formula (in a different form) was given by Von Mangoldt [16] in 1895. In connection with equation (2) we now have $\Pi(x) - \sum_{n=1}^N \frac{\mu(n)}{n} Li\left(x^{\frac{1}{n}}\right) = \sum_{n=1}^N \sum_{\rho} Li(x^{\rho/n}) +$ "other terms", where the omitted terms are not particularly significant. The terms in the double sum are Riemann's "periodic" terms. Individually they are quite large, but there must be a large amount of cancellation to account for the fact that equation (2) gives a very close estimate of $\Pi(x)$.

Riemann's formula for $\Pi(x)$ shows that the error term in this approximation can be expressed in terms of zeros of the zeta function. In his 1859 paper, Riemann conjectured that all the "Non-trivial" zeros of ζ lie on the line $R(s) = 1/2$ but never provided a proof of this statement. This famous and long-standing conjecture is known as Riemann Hypothesis. Based on some deep results derived on 1859 by Riemann on the relationship of PNT and the complex zeros of the Riemann zeta function, the first proof of PNT was given independently by J Hadamard and de la Vallee Poussin on 1896 using methods of advanced theory of complex analysis. The first elementary proof of the PNT without using complex analysis was obtained by A.Selberg and P.Erdos on 1949. The proof does not use advanced tools such as complex analysis-in fact their argument is a complicated one. Their proof must some how show that there is no zero on the real line $R(S)=1$, and indeed their combinatorics masks a subtle complex analysis proof beneath the surface for a careful examination of the argument.

Recently we have presented a new proof of Prime Number Theorem [17]. We call it elementary because the proof does not require any advanced techniques from the analytic number theory and complex analysis. The proof of the PNT is derived on the scale invariant, non-archimedean model \mathbf{R} of real number system R , involving non-trivial infinitesimals and infinities. The model \mathbf{R} is realized as a completion of the field of rational numbers Q under a new non-archimedean absolute value $||\cdot||$,

which treats arbitrarily small and large numbers separately from any finite number [18]. The so-called scale invariant infinitesimals are modelled as p-adic integers X_i with $|X_i| < 1$, $|\cdot|_p$ being the p-adic absolute value and is given by the adelic formula $X = X_p \prod_{q>p} (1 + X_q)$. The infinitesimals considered here are said to be active as the definition involves an asymptotic limit of the form $x \rightarrow 0^+$. As a consequence the value of a scale invariant infinitesimals X would undergo infinitely slow variations over p-adic local fields Q_p as a scale free variable x^{-1} , called the internal time variable, approaches ∞ through the sequence of primes p . In our paper [17] we have showed that these p-adic infinitesimals living in \mathbf{R} , have an influence over the structure of the ordinary real number system R there by extending it into an associated infinite dimensional Euclidean space R , so that a finite real number r gets an infinitely small correction term given by $r_{cor} = r + \varepsilon(x)||X||$, where $\varepsilon(x^{-1}) = \frac{\log x^{-1}}{x^{-1}}$ is the inverse of the asymptotic PNT formula of the prime counting function $\Pi(x^{-1}) = \sum_{p < x^{-1}} 1$. The proof of the PNT in this formalism is accomplished by proving that the value of $||X||$ of a scale free infinitesimal actually corresponds to the prime counting function $\Pi(x^{-1})$ as the internal time x^{-1} approaches infinity through larger and larger scales denoted by primes p . The error term in this proof respects Riemann hypothesis.

4. Some Consequences of Riemann hypothesis:

The Riemann Hypothesis is one of the hardest and most famous problems in mathematics. Its original formulation, which comes from the theory of complex functions, asserts that all non-real zeros of the Riemann zeta function have real part equal to one-half. Because of its technical formulation, it is not easy to talk about the Riemann hypothesis without assuming knowledge of the complex function theory, but we can exploit its connections to other branches of mathematics. One of the most important is the light it sheds on the distribution of prime numbers. And there are also some elementary conjectures that turn out to be equivalent to the Riemann hypothesis. Some of the consequences are listed below.

- i) **Large prime gap conjecture** – The Prime number theorem implies that an average of the gap between the prime p and its successor is $\log p$.

However, some gaps between primes may be much larger than the average. Cramer proved that, assuming the Riemann hypothesis, every gap is $O(\sqrt{p} \log p)$. This is a case when even the best bound that can currently be proved using the Riemann hypothesis is far weaker than what seems to be true. Cramer's conjecture implies that every gap is $O((\log p)^2)$ which, while larger than the average gap, is far smaller than the bound implied by the Riemann hypothesis. Numerical evidence supports Cramer's Conjecture (1999).

ii) Distribution of Prime numbers – Riemann's explicit formula for the number of primes less than a given number in terms of a sum over zeros of Riemann zeta function says that the magnitude of the oscillations of primes around their expected position is controlled by the real parts of the zeros of the zeta function. In particular the error term in the prime number theorem is closely related to the position of zeros; for example, the supremum of real parts of the zeros is the infimum of numbers β such that the error is $O(x^\beta)$.

Von Koch (1901) proved that the Riemann hypothesis is equivalent to the "best possible" bound for the error of the Prime number theorem. A precise version of Koch's result, due to Schornfeld (1976), says that the Riemann hypothesis is equivalent to $\Pi(x) - Li(x) < \frac{1}{8\pi} \sqrt{x} \log x$, for all $x \geq 2657$.

iii) Growth of Arithmetic function – The Riemann hypothesis implies strong bounds on the growth of many other arithmetic functions, in addition to the prime counting function above. One example involves the Mobius function μ . The statement that the equation $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ is valid for every s with real part greater than $\frac{1}{2}$, with the sum on the right hand side converging, is equivalent to the Riemann hypothesis. From this we can also conclude that if the Mertens function is defined by $M(x) = \sum_{n \leq x} \mu(n)$ then the claim that $M(x) = O(x^{\frac{1}{2}+\epsilon})$ for every positive ϵ is equivalent to the Riemann hypothesis.

5. Criticism of Riemann hypothesis :

There are several arguments for and against Riemann hypothesis. Some authors such as Riemann (1859) or Bombieri (2000), express an opinion that they expect that it is true. The few

authors express serious doubt about it. Ivic (2008) lists some reasons for being sceptical and Littlewood (1962) flatly states that it would be false and that there is no evidence whatever for it and no imaginable reason for it to be true.

Some of the arguments for (or against) Riemann hypothesis are listed by Sarkar (2008), Conrey (2003) and Ivic (2008) and include the following reasons.

- Several analogues of Riemann hypothesis have already been proved. The proof of the Riemann hypothesis for varieties over finite fields by Deligne (1974) is strong theoretical reason in favour of the Riemann hypothesis. This provides some evidence for the more general conjecture that all zeta functions associated with automorphic forms satisfy Riemann hypothesis, which includes the classical Riemann hypothesis as a special case. The Riemann hypothesis for the Gross zeta function was proved by Sheats (1998). In contrast to these positive examples, however, some Epstein zeta functions do not satisfy the Riemann hypothesis, even though they have an infinite number of zeros on the critical line (Titchmarsh 1986).
- The numerical verification that many zeros lie on the line seems at first sight to be strong evidence for it. However analytic number theory has had many conjectures supported by large amounts of numerical evidence that turn out to be false. Denjoy's probabilistic argument for the Riemann hypothesis (Edwards 1974) based on observation that if $\mu(x)$ is a random sequence of "1"s and "-1"s then, for every $\epsilon > 0$, the partial sums $M(x) = \sum_{n \leq x} \mu(n)$ satisfy the bound $M(x) = O(x^{\frac{1}{2}+\epsilon})$ with probability 1. The Riemann hypothesis is equivalent to this bound for the Mobius function μ and the Mertens function M derived in the same way from it. In other words, the Riemann hypothesis is in some sense equivalent to saying that $\mu(x)$ behaves like a random sequence of coin tosses. When $\mu(x)$ is non-zero its sign gives the parity of the number of prime factors of x , so informally the Riemann hypothesis says that the parity of the number of prime factors of an integer behaves randomly.

6. Conclusion:

The prime number theorem has a long and interesting history. Subsequent research has

provided wider and wider subregions of the critical strip without zeros of $\zeta(s)$ (and thus improved approximations to the number of primes up to x), without coming anywhere near to proving the Riemann hypothesis. This remains as an

outstanding open problem of mathematics. In this paper we have mentioned just a few of the many historical issues related to the PNT including our work in this field.

References :

1. Apostol, Tom. M , Introduction to analytic number theory, Undergraduate Texts in Mathematics, NewYork-Heidelberg: Springer-Verlag, (1976), 7.
2. H.M.Edwards, Riemann's Zeta function, Dover Publications, New York, (2001).
3. C.F.Gauss, Werke, 2nd ed., Koniglichen Gesellschaft der Wissenschaften, Gottingen, (1876)
4. G.H.Hardy, Prime numbers, British Association Report, British Association, Manchester, Vol.2 (1915) ,pp-350-354
5. P.L.Chebysev, La totalite des nombres premiers inferieurs a uni limite done, J.Math.Pures Appl. 17 (1852) ,pp-341-365
6. George E.Andrews, Number theory, W.B.Saunders Co., Philadelphia, 1971 (Reprinted by Dover, NewYork,1994)
7. B.Riemann,Ueber die Anzahl der primzahlen under liner gegebenen Grosse, Monatsberichte der Berliner Akademic (1859), 671-680
8. J.Hadamard, Sur la distribution des zeros de la fonction $\zeta(s)$ et ses consequences arithmetiques, Bull.Soc.Math.France 14 (1896), 199-220
9. Charles-Jean de la V Poussin, Recherches analytiques sur la theorie des numbers; Premiere partie; la fonction $\zeta(s)$ de Riemann et les numbers
10. J.E.Littlewood, Sur la distribution des numbers premiers, C.R.Acad.Sci, Paris 158 (1914), 263-266
11. S.Skewes, on the difference $\Pi(x) - Li(x)$, Jour.London.Math.Soc.8 (1933),277-283
12. S.Skewes, on the difference $\Pi(x) - Li(x)$, Proc.London.Math.Soc.5 (1955),48-70
13. A.S.Lehman, on the difference $\Pi(x) - Li(x)$, Acta Arithmetica 11 (1966),397-410
14. Human J.J.te Riele, On the sign of the difference $\Pi(x) - Li(x)$, Mathematics of computation 48(1987), 323-328.
15. B.C.Bredt and Robert A.Rankin, Ramanujan Letters and Commentary, Amer.Math.Soc. and London Math.Soc,Providence, Rhode Island, 1995
16. H.Von Mangoldt, Zu Riemann's Abhandlung ' Ueber die Anzahl der primzahlen unter einer gegebenen Grosse', J.Reine Angrew.Math.114 (1895), 255-305
17. D.P.Datta and A.Ray Chaudhuri, Scale free analysis and prime number theorem, Fractals, 18,(2010), 171-184
18. A. Ray Chaudhuri, Scale Invariant Limit and Emergence of Complexity: Applications to Traffic Flow, International Journal of Mathematics Trends and Technology, Vol. 19, Issue 2 (2015) 102 – 107. ISSN: 2231-5373