# On detection number of cycle related graphs

Sanma.G.R[1],T. Nicholas[2].

[1.]Research scholar, Research Department of Mathematics, St. Jude's College, Thoothoor – 629176, Kanyakumari District, Tamil Nadu, India.

[2] Associate Professor and Head, Research Department of Mathematics, St. Jude's College, Thoothoor – 629176, Kanyakumari District, Tamil Nadu, India

***Abstract***-- *Let G be a connected graph of order $n \geq 3$ and let k-labeling c: E(G) → {1, 2, 3, . . . ,k } of the edges of G, (where adjacent edges may be colored the same). For each vertex v of G, the color code of v with respect to c is the k-tuple $c(v) = (a_1, a_2, . . . , a_k)$ where $a_i$ is the number of edges incident with v that are colored i $(1 \leq i \leq k)$. The k-labeling c is detectable if every two adjacent vertices of G have distinct codes. The minimum positive integer k for which G has a detectable k-labeling is the detection number det(G) of G. In this paper we obtain the detection number of some known graphs such as $P_n$ x $P_m$, circular halin graph of level two, wheel, crown graph etc.*

**Keywords:** *detection number, helm graph, gear graph, wheel, fan, friendship graph.*

**MSC AMS classification 2010: 05C15, 05C70.**

## 1. Introduction

Let G be a finite, connected, undirected and simple graph with order $n \geq 3$. Let c: E(G) → {1, 2, . . . , k} be a labeling of the edges of G, where k is a positive integer. The color code of a vertex v of G is the ordered k-tuple $c(v) = (a_1, a_2, . . . , a_k)$ where $a_i$ is the number of edges incident with v that are labeled i for i ϵ {1, 2, . . . , k}. The labeling c is called a detectable coloring of G if any pair of adjacent vertices of G have distinct color codes. The detection number or detectable chromatic number of G, denoted det(G) is the minimum positive integer k for which G has detectable k- coloring.

In order to distinguish the vertices of a connected graph and to distinguish adjacent vertices of a graph, with the minimum number of colors, the concept of detection number was introduced by Karonski et al.[6] (2004). To differentiate the vertices of a connected graph[4], the concept of detective coloring was originated. In [1] ( G. Chartrand, H. Escuadro, F. Okamoto and P.Zhang) the detective number of stars, double stars, cycles, paths, complete graphs and complete bipartite graphs are determined. Also in [3] (H. Escuardo, Ping Zhang) they establish a formulae for the detection number of path in terms of its order. Furthermore it is shown that for integers $n \geq 3$ and $k \geq 2$, there exists a unicyclic graph G of order n having det(G) = k if and only if $d_u(n) \leq k \leq D_u(n)$ where $d_u(n)$ and $D_u(n)$ are the minimum and maximum detection number among all unicyclic graphs. In [5] (Frederic Havet, Nagarajan Paramaguru, Rathinasamy Sampathkumar) showed that it is NP-complete to decide if the detection number of a cubic graph is 2. They also show that the detection number of every bipartite graph of minimum degree at least 3 is atmost 2. Also they gave some sufficient conditions for a cubic graph to have detection number 3. In this paper we find the detection number of helm graph, gear graph, wheel, fan, friendship graph.

We make use of the following result which is already proved in [3] [7].

***Observation 1.1:*** [3] Every connected graph of order 3 or more has detection number at least 2.

***Observation 1.2****:* [7] For every connected graph G, the following two conditions are equivalent:

**(i) det(G) = 1.**

**(ii) G has no adjacent vertices with the same degree.**

***Observation 1.3*****: [5] For $n \geq 3$ ,$det(K_n) = 3$.**

## 2.Main Results.

*Theorem 2.1:* For a wheel $G = W_D$, det(G) = $\begin{cases} 3 \text{ where D is odd,} \\ 2 \text{ where D is even.} \end{cases}$

*Proof.* Let x be the central vertex.

Let $v_1, v_2. . . . , v_D$ be the vertices on the cycle taken in the clockwise direction.

Let $e_1$, $e_2$, $e_3$, . . . . ,$e_D$ be the corresponding edges connecting x from $v_1$, $v_2$. . . . , $v_D$ respectively.

Let $c_1$, $c_2$, . . . ,$c_D$ be the edges on the cycle where $c_i = v_i v_{i+1}$.

*Case1*. D is odd.

To prove det(G) = 3 first we have to show det(G) ≤ 3.

Let c: E(G) → {1, 2, . . . , k} where k is a positive integer such that

$$c(e)=\begin{cases} 1 \text{ for } e_i, i = 1,3,5,\ldots,(D-2) \text{ and for all } c_i, \\ 2 \text{ if } e_i, i = 2,4,\ldots,(D-1), \\ 3 \quad \text{for } e_D . \end{cases}$$

The vertices $v_1$, $v_2$. . . . , $v_D$ have degree 3 and the central vertex x have degree D.

Here $c(x) = \left(\frac{D-1}{2}, \frac{D-1}{2}, 1\right)$ where $D \geq 3$ and the edges incident to the adjacent vertices receive any of the labels 113, 111, 112 resulting in the codes (2, 0,1), (3, 0, 0), (2, 1, 0) which is not equal to c(x).

If $c(v_i) = (2,1,0), i = 2, 4,\ldots,(D-1)$, then the edges incident to the adjacent vertices receive any of the labels 111, 113, 123 or 11223 or 1112223, . . . . resulting in the codes (3, 0, 0), (2, 0, 1), (1, 1, 1) or (2, 2, 1) or (3, 3, 1) . . . which is not equal to $c(v_i)$ for i = 2, 4, 6, . . . , (D-1).

If $c(v_D) = (2,0,1)$, then the edges incident to the adjacent vertices receive any of the labels 112, 111, 123 or 11223 or 1112223, . . . . resulting in the codes (2, 1, 0), (3, 0, 0), (1, 1, 1) or (2, 2, 1) or (3, 3, 1) . . . which is not equal to $c(v_D)$.

If $c(v_i) = (3,0,0), i = 1,3,\ldots,(D-2)$, then the edges incident to the adjacent vertices receive any of the labels 112, 113, 123 or 11223 or 1112223, . . . . resulting in the codes (2, 1, 0), (2, 0, 1), (1, 1, 1) or (2, 2, 1) or (3, 3, 1) . . . which is not equal to $c(v_i)$ for i = 1, 3, 5, . . . , (D-2).

By the above labeling the codes of all the vertices become

$$c(v) = \begin{cases} (3,0,0) \text{ for } v_i, i = 1,3,\ldots,(D-2), \\ (2,1,0) \text{ for } v_i, i = 2, 4,\ldots,(D-1), \\ (2,0,1) \quad \text{for } v_D, \\ \left(\frac{D-1}{2}, \frac{D-1}{2}, 1\right) \text{ for x.} \end{cases}$$

Therefore all the adjacent vertices have different codes. Hence det(G) ≤ 3.---------------(1).

Claim: $c(e_D) \neq 1$ or 2.

If $c(e_D) = 1$, then the adjacent vertices x, $v_D$ have the labels 1111222, 111 resulting in the codes (4, 3, 0), (3, 0, 0) repectively. That is the adjacent vertices $v_1$, $v_D$ have the same codings. Hence $c(e_D) \neq 1$.

If $c(e_D) = 2$, then the adjacent vertices x, $v_D$ have the labels 1112222, 112 resulting in the codes (3, 4, 0), (2, 1, 0) repectively. That is the adjacent vertices $v_D$, $v_{D-1}$ have the same codings. Hence $c(e_D) \neq 2$. Therefore edge $c(e_D) \neq 1$ and $c(e_D) \neq 2$.

Hence det(G) ≮ 3 ……………… (2).

From (1) and (2) it is clear that det(G) = 3. ■

Example.

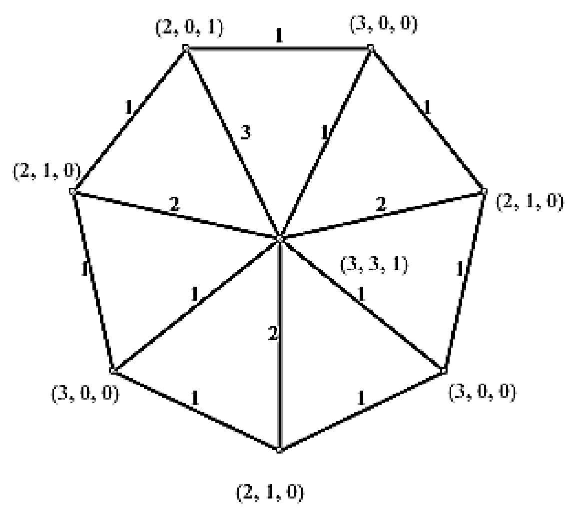


Fig 1.Detective labeling in $W_7$.

*Case 2*. D is even.

To prove det(G) = 2 first we have to show det(G) ≤ 2.

Let c: E(G) → {1, 2, . . . , k} where k is a positive integer such that

$$c(e) = \begin{cases} 1 \text{ for } e_i,\ i = 1,3,5,\ldots,(D-1) \text{ and for all } c_i, \\ 2 \quad \text{if } e_i,\ i = 2,4,\ \ldots\ldots, D. \end{cases}$$

The vertices $v_1$, $v_2$. . . . , $v_D$ have degree 3 and the central vertex have degree D.

Here $c(x) = \left(\frac{D}{2}, \frac{D}{2}\right)$ where $D > 3$ *and* the edges incident to the adjacent vertices receive any of the labels 111, 112 resulting in the codes (3, 0), (2, 1) which is not equal to c(x).

If $c(v_i) = (2,1), i = 2, 4,\ldots,D$, then the edges incident to the adjacent vertices receive any of the labels 111, 1122, or 111222 or 11112222, . . . . resulting in the codes (3, 0), (2, 2) or (3, 3) or (4, 4) . . . which is not equal to $c(v_i)$ for i = 2, 4, 6, . . . , D..

If $c(v_i) = (3,0), i = 1,3,\ldots,(D-1)$, then the edges incident to the adjacent vertices receive any of the labels 112, 1122 or 111222 or 11112222, . . . . resulting in the codes (2, 1), (2, 2) or (3, 3) or (4, 4) . . . which is not equal to $c(v_i)$ for i = 1, 3, 5, . . . , (D-1).

By the above labeling the codes of all the vertices becomes

$$c(v) = \begin{cases} (3,0) \text{ for } v_i, \ i = 1,3,\ldots,(D-1), \\ (2,1) \quad \text{for } v_i, \ i = 2,\ 4,\ldots,D, \\ \left(\frac{D}{2},\ \frac{D}{2}\right) \quad \text{for } x. \end{cases}$$

Therefore all the adjacent vertices have different codes. Therefore $\det(H_1(1, D)) \leq 2$.

Therefore by Observation 1.1 and $\det(H_1(1, D)) \leq 2$.

Hence det(G) = 2.

■

Example.

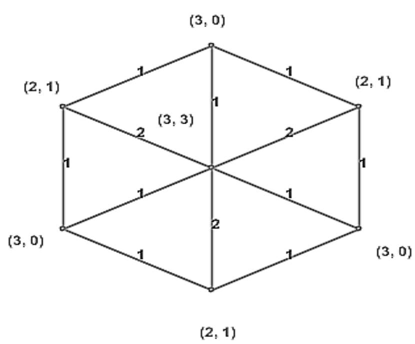


Fig 2. Detective labeling in $W_6$.

A helm graph is a graph obtained from the wheel $W_n = C_n + K_1$ by attaching a pendant edge at each vertex of $C_n$. It is denoted by $H_n$.

*Theorem 2.2.*

For a helm graph G where $n \neq 4$, det(G) = 1.

*Proof.*

Helm is a connected graph (graph without $K_2$ component) in which all the adjacent vertices have different degrees for $n \neq 4$. Hence by Observation 1.3.we get det(G) = 1. ■

*Gear graph:* A gear graph denoted by G(n), $n \geq 4$ is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of the wheel graph $W_n$. Then G(n) has 2n+1 vertices and 3n edges.

*Theorem 2.3*

For a gear graph G, det(G) = 1 for $n \geq 4$.

*Proof.*

Gear graph is a connected graph (graph without $K_2$ component) for $n \neq 3$ in which all the adjacent vertices have different degrees. Hence by Observation 1.3.we get det(G) = 1■

*Friendship graph:* A friendship graph is the one-point union of n copies of the cycle $C_3$ denoted by $C_3^{(n)}$. It has 2n+1 vertices and 3n edges. Let x be the common vertex identified. Let $v_t$ and $v_t'$ be the vertices of the $t^{th}$ cycle $C_3$ for t = 1, 2, . . . ,n adjacent to x in the clockwise direction. Let $e_t$ and $e_t'$ be the edges connecting x from $v_t$ and x from $v_t'$ respectively for t = 1, 2, 3, . . , n. Let $f_i$ be the edge connecting $v_t$ and $v_t'$ for t = 1, 2, . . . , n.

*Theorem 2.4.*

For a Friendship graph G, det(G) = 2.

*Proof.*

To prove det(G) = 2 first we have to show that $\det(G) \leq 2$.

Let the labeling c: E(G) → {1, 2, 3, . . . , k} where k is a positive integer such that

$$c(e) = \begin{cases} 1 \text{ for } e_t \text{ and } f_t, t = 1,2,3,\ldots,n. \\ 2 \text{ for } e_t', t = 1,2,\ldots,n. \end{cases}$$

Here, the vertex x has degree 2n and the remaining vertices have degree two each.

If c(x) = (n, n), then the edges incident to the adjacent vertices receive the labels 12 and 11 results in the codings (1, 1) and (2, 1) respectively which is not equal to c(x).

If $c(v_t) = (2, 0)$ for $t = 1,2,\ldots,n$ then the edges incident to the adjacent vertices receive the labels 12, 1122 or 111222 or 11112222 . . . results in the codings (1, 1), (2, 2) or (3, 3) or (4, 4) . . . respectively which is not equal to $c(v_t)$.

If $c(v_t') = (1, 1)$ for $t = 1,2,\ldots,n$ then the edges incident to the adjacent vertices receive the labels 11, 1122 or 111222 or 11112222 . . . results in the codings (2, 0), (2, 2) or (3, 3) or (4, 4) . . . respectively which is not equal to $c(v_t')$.

Hence by the above labeling, the corresponding codes of the vertices are

$$c(v) = \begin{cases} (n, n) & \text{for } x, \\ (1,1) \text{ for } v_t', t = 1, 2, \dots, n, \\ (2,0) \text{ for } v_t, t = 1, 2, \dots, n. \end{cases}$$

That is , adjacent vertices have distinct codes and therefore $det(G) \leq 2$.

Therefore by Observation 1.1 and $det(H_1(2, D)) \leq 2$.

Hence $det(G) = 2$.

■

Example.

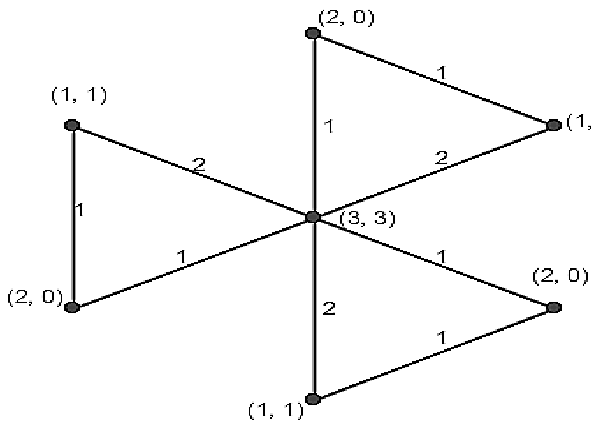


Fig 3.detection number of $C_3^{(3)}$.

Fan $F_t = P_t + K_1$, where the vertex of $K_1$ is called the apex vertex x and the vertices $v_1, v_2, v_3, \dots, v_t$ are the path vertices.Let $e_1, e_2, e_3, \dots, e_n$ be the edges from the apex vertex x to $v_1, v_2, \dots, v_t$ respectively.Let $h_i$ be the edge from $v_i$ to $v_{i+1}$ for i = 1, 2, 3, . . . , (t-1).

*Theorem 2.5.*

For a fan detection number of $F_2 = 3$ and $F_n = 2$ for n = 3,4,5, . . . .

*Proof.*

*Case 1.* $F_2$ is a complete graph on three vertices .Hence by Observation 1.3 $det(F_2) = 3$.

To prove $det(G) = 2$ for $F_n$ n = 3, 4, 5, . . . first we have to show that $det(G) \leq 2$.

*Case 2.*n is even.

Let c: $E(G) \rightarrow \{1, 2, \dots, k\}$ where k is a positive integer such that

$$c(e) = \begin{cases} 1 & \text{for } e_i, i = 1, 3, 5, \dots, (n-1), \\ 2 & \text{for } e_i, i = 2, 4, 6, \dots, n \text{ and } h_j, j = 1, 2, \dots, (n-1). \end{cases}$$

The vertex x has degree n and vertices $v_1$, $v_n$ have degree 2. Remaining vertices have degree 3.

Here $c(x) = (\frac{n}{2}, \frac{n}{2})$ and the edges incident to the adjacent vertices receive the labels 12, 222, 122, 22 results in the codings (1, 1), (0, 3), (1, 2), (0, 2) respectively which is different from c(x).

If $c(v_1) = (1, 1)$, then the edges incident to the adjacent vertices receive the labels 222, 1122 or 111222 or 11112222 . . . results in the codings (0, 3), (2, 2) or (3, 3) or (4, 4) . . . which is not equal to $c(v_1)$.

If $c(v_i) = (0, 3)$ for $i = 2, 4, \dots, (n-2)$, then the edges incident to the adjacent vertices receive the labels 12, 122, 1122 or 111222 or 11112222 . . . . results in the codings (1, 1), (1, 2), (2, 2) or (3, 3) or (4, 4) . . .which is different from $c(v_i)$ for $i = 2, 4, \dots, (n-2)$.

If $c(v_i) = (1, 2)$ for $i = 3, 5, \dots, (n-1)$, then the edges incident to the adjacent vertices receive the labels 22, 222, 1122 or 111222 or 11112222 . . . . results in the codings (0, 2), (0, 3), (2, 2) or (3, 3) or (4, 4) . . .which is different from $c(v_i)$ for $i = 3, 5, \dots, (n-1)$.

If $c(v_n) = (0, 2)$, then the edges incident to the adjacent vertices receive the labels 222, 1122 or 111222 or 11112222 . . . . results in the codings (0, 3), (2, 2) or (3, 3) or (4, 4) . . .which is different from $c(v_n)$ .

$$c(v) = \begin{cases} \left(\frac{n}{2}, \frac{n}{2}\right) & \text{for } x, \\ (1,1) & \text{for } v_1, \\ (0,3) & \text{for } v_i, i = 2, 4, \dots, (n-2), \\ (1,2) & \text{for } v_i, i = 3, 5, \dots, (n-1), \\ (0,2) & \text{for } v_n. \end{cases}$$

That is , adjacent vertices have distinct codes and therefore $det(G) \leq 2$.

Example.

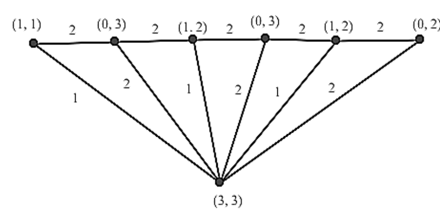


Fig 4. Detective labeling in $F_6$

*Case 3.*n is odd.

Let c: E(G) → {1, 2, . . . , k} where k is a positive integar such that

$$c(e) = \begin{cases} 1 \text{ for } e_i, i = 1, 3, 5, \ldots, n, \\ 2 \text{ for } e_i, i = 2, 4, 6, \ldots, (n-1) \text{ and } h_j, j = 1, 2, \ldots, (n-1). \end{cases}$$

The vertex x has degree n and vertices $v_1$, $v_n$ have degree 2. Remaining vertices have degree 3.

If c(x) = $(\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor)$, then the edges incident to the adjacent vertices receive the labels 12, 222, 122 results in the codings (1, 1), (0, 3), (1, 2) respectively which is different from c(x).

If $c(v_1)$ = (1, 1), then the edges incident to the adjacent vertices receive the labels 222, 11122 or 1111222 or 111112222 . . . results in the codings (0, 3), (3, 2) or (4, 3) or (5, 4) . . . which is not equal to $c(v_1)$.

If $c(v_i)$ = (0, 3) for $i = 2, 4, \ldots, (n-1)$, then the edges incident to the adjacent vertices receive the labels 12, 122, 11122 or 1111222 or 111112222 . . . . results in the codings (1, 1), (1, 2), (3, 2) or (4, 3) or (5, 4) . . .which is different from $c(v_i)$ for $i = 2, 4, \ldots, (n-1)$.

If $c(v_i)$ = (1, 2) for $i = 3, 5, \ldots, (n-2)$, then the edges incident to the adjacent vertices receive the labels 222, 11122 or 1111222 or 111112222 . . . . results in the codings (0, 3), (3, 2) or (4, 3) or (5, 4) . . .which is different from $c(v_i)$ for $i = 3, 5, \ldots, (n-$ 2.

$$c(v) = \begin{cases} (\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor) & \text{for } x, \\ (1,1) & \text{for } v_i, i = 1, n. \\ (0,3) & \text{for } v_i, i = 2, 4, \ldots, (n-1), \\ (1,2) & \text{for } v_i, i = 3, 5, \ldots, (n-2). \end{cases}$$

That is , adjacent vertices have distinct codes and therefore det(G) ≤ 2.

Therefore by Observation 1.1 and det($H_1$(2, D)) ≤ 2 we get det(G) = 2. ■

Example.

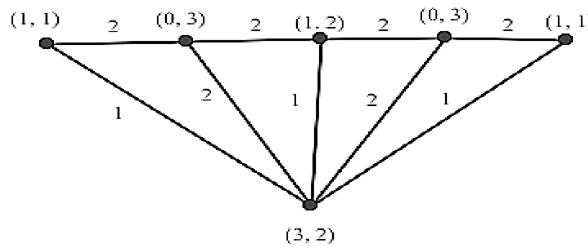


Fig. 5. Detective labeling in $F_5$

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