Algorithm on Gamma Function and its approximation function derivation

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Abstract:
The paper provides new insight in dealing with gamma function by formulating an approximation function which converts the convoluted integral in the repeated multiplicative format through the application Euler Mascheroni constant.
My paper also formulates the relation between trigonometric functions and gamma function. My paper helps traversing in the dimension of trigonometry with the generality as nuanced as possible.

Results:

Result 1:

$$\frac{\Gamma(t)\cos(t\alpha)}{n|s|^{t}} = \int_0^\infty x^{(nt-1)} e^{-ax^n} \cos(bx^n)\, dx$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Result 2:

$$\frac{\Gamma(t)\sin(t\alpha)}{n|s|^{t}} = \int_0^\infty x^{(nt-1)} e^{-ax^n} \sin(bx^n)\, dx$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Result 3:

$$\frac{e^{\frac{-0.57}{y}} \lim_{n\to\infty} \prod_{z=1}^{n} e^{\frac{1}{zy}} \left(1+\frac{1}{zy}\right)^{n}}{a^{\frac{1}{y}}} = \int_0^\infty e^{-ax^y} dt$$

Where $y \neq 0$

Derivation of the Results 1 & 2

The result written is derived in the following pages starting from the basic definition of gamma function. Deriving a relation between gamma function trigonometry function (sine or cos) using complex number approach.

To start from basic definition of gamma function
$\Gamma(t) = \int_0^\infty u^{t-1} e^{-u}\, du$
Where $t \notin Z$ and $t \neq 0$

Now make the following substitution: $\boldsymbol{u = sx^n}$
Where s is the complex number of the (a+ib)
Differentiating both sides

$$du = (s)(n)x^{n-1}\, dx$$

$n \in R - \{0\}$ *i.e. n cannot be equal to 0.*

After making the substitution:

$$\Gamma(t) = \int_0^{\infty} [(sx^n)^{t-1}]\left(e^{-sx^n}\right)(s)(n)\, x^{(n-1)} dx$$

$$\Gamma(t) = \int_0^{\infty} s^{(t-1+1)}\, x^{(nt-n+n-1)} e^{-sx^n} n dx$$

$$\Gamma(t) = \int_0^{\infty} (ns^t)\, x^{nt-1}\, e^{-sx^n} dx$$

$\frac{\boldsymbol{\Gamma(t)}}{\boldsymbol{n(s^t)}} = \int_{\mathbf{0}}^{\infty} \boldsymbol{x}^{\ (\boldsymbol{nt-1})}\ \boldsymbol{e^{-sx^n} dx}$ ------------------------- *(1)*

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Since we assumed 's' to be a complex number of the form (a+ib), we can rewrite Equation(1) by replacing 's' with its complex conjugate, $\bar{s}$*.* $\bar{s}$ *is of the form (a - ib).*

Replacing 's' by $\bar{s}$

$\frac{\boldsymbol{\Gamma(t)}}{\boldsymbol{n(\bar{s})^t}} = \int_{\mathbf{0}}^{\infty} \boldsymbol{x}^{\ (\boldsymbol{nt-1})}\ \boldsymbol{e^{-\bar{s}x^n} dx}$----------------------- *(2)*

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

- *Equation 1*

$$\frac{\boldsymbol{\Gamma(t)}}{\boldsymbol{n(s^t)}} = \int_{\mathbf{0}}^{\infty} \boldsymbol{x}^{\ (\boldsymbol{nt-1})}\ \boldsymbol{e^{-sx^n} dx}$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

- *Equation 2*

$$\frac{\boldsymbol{\Gamma(t)}}{\boldsymbol{n(\bar{s}^t)}} = \int_{\mathbf{0}}^{\infty} \boldsymbol{x}^{\ (\boldsymbol{nt-1})}\ \boldsymbol{e^{-\bar{s}x^n} dx}$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Where s is the complex number of the (a+ib)

$$s = a + ib$$

$$s = |s|e^{i\alpha}$$

where $|s| = \sqrt{a^2 + b^2}$ $\quad \tan\alpha = \frac{b}{a}$

$\boldsymbol{s^t = |s|^t e^{it\alpha}}$ -------------------------*(3)*

$\bar{s} = a - ib$ $\qquad \bar{s} = |s|^t e^{-i\alpha}$

$(\,\bar{\boldsymbol{s}}\,)^{\boldsymbol{t}} = (|\boldsymbol{s}|)^{\boldsymbol{t}} \boldsymbol{e^{-it\alpha}}$ ----------------------*(4)*

- Substituting the results of (3) and (4) in equations (1) and (2) respectively.

-Substituting the following values

- $s = a + ib$
- $\bar{s} = a - ib$

Modified equation (1)

$$\frac{\Gamma(t)}{n|s|^t e^{it\alpha}} = \int_0^\infty x^{(nt-1)}\ e^{-(a+ib)x^n} dx$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Modified equation (2)

$$\frac{\Gamma(t)}{n|s|^t e^{-it\alpha}} = \int_0^\infty x^{(nt-1)}\ e^{-(a-ib)x^n} dx$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

- *Case(i) :*

Adding equation (1) and (2)

- *Case(ii) :*

Subtracting equation (1) from (2)

Simplifying case (i)

➢ $$\frac{\Gamma(t)}{n|s|^t e^{t(i\alpha)}} + \frac{\Gamma(t)}{n|s|^t e^{t(-i\alpha)}} = \int_0^\infty (x^{(nt-1)} * [\, e^{-(a+ib)x^n} + e^{-(a-ib)x^n}]) dx$$

➢ $$\frac{\Gamma(t)}{n|s|^t}(e^{-i\alpha} + e^{i\alpha}) = \int_0^\infty (x^{(nt-1)}\, e^{-ax^n}[\, e^{-ibx^n} + e^{ibx^n}]) dx$$

Applying Euler's formula, which is given by:-

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

on LHS and RHS

➢ $$\frac{\Gamma(t)\cos(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)}\, e^{-ax^n} \cos(bx^n)\, dx$$

$$\frac{\Gamma(t)\cos(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)}\, e^{-ax^n} \cos(bx^n)\, dx$$ ------------*equation(5)*

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Simplifying case (ii)

➢ $$\frac{\Gamma(t)}{n|s|^t e^{t(i+\alpha)}} - \frac{\Gamma(t)}{n|s|^t e^{t(-i+\alpha)}} = \int_0^\infty (x^{(nt-1)} [\, e^{-(a+ib)x^n} - e^{-(a-ib)x^n}]) dx$$

➢ $$\frac{\Gamma(t)}{n|s|^t}\left(e^{t(-i+\alpha)} - e^{t(i+\alpha)}\right) = \int_0^\infty (x^{(nt-1)}\, e^{-ax^n}[\, e^{-ibx^n} - e^{ibx^n}]) dx$$

Apply Euler's formula which is given by:- $\boldsymbol{\sin y = \frac{e^{iy}-e^{-iy}}{2}}$

on LHS and RHS

- $\frac{\Gamma(t)\sin(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)}\, e^{-ax^n} \sin(bx^n)\, dx$

$\boldsymbol{\frac{\Gamma(t)\ \sin(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)}\, e^{-ax^n} \sin(bx^n)\, dx}$ -------------*(6)*

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Result 1 & 2

- ***Result 1***

$$\boldsymbol{\frac{\Gamma(t)\cos(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)}\, e^{-ax^n} \cos(bx^n)\, dx}$$

Where $t \notin Z^-$ **&** $n \neq 0$ **&** $t \neq 0$

- ***Result 2***

$$\boldsymbol{\frac{\Gamma(t)\sin(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)}\, e^{-ax^n} \sin(bx^n)\, dx}$$

Where $t \notin Z^-$ **&** $n \neq 0$ **&** $t \neq 0$

Where $\boldsymbol{s = a + ib}$

$$\boldsymbol{|s| = \sqrt{a^2 + b^2}}$$

$$\boldsymbol{\alpha = \tan^{-1}\frac{b}{a}}$$

Derivation of Result 3:

$$(\Gamma(s)) = \int_0^\infty t^{s-1}\, e^{-t}\, dt$$

$$e^{-t} = \lim_{n\to\infty}\left(1 + \frac{(-t)}{n}\right)^n$$

$$\Gamma(s) = \int_0^\infty t^{s-1} \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n dt$$

Applying ILATE Sequence

$$\lim_{n\to\infty}\left|\frac{t^s}{s}\left(1-\frac{t}{n}\right)^n\right|_0^n - \lim_{n\to\infty}\int_0^\infty \frac{t^s}{s} n\left(1-\frac{t}{n}\right)^{n-1}\left(\frac{-1}{n}\right) dt$$

$$\Gamma(s) = \lim_{n\to\infty}\int_0^\infty \left(\frac{n}{s.n}\right) \times (t^s).\left(1-\frac{t}{n}\right)^{n-1} dt$$

$$\Gamma(s) = \left(\frac{n}{s.n}\right) \times \frac{(n-1)}{(s+1)\times n} \times \lim_{n\to\infty}\int_0^\infty t^{s+1}\left(1-\frac{t}{n}\right)^{n-2} dt$$

$$\boldsymbol{\Gamma}(\mathrm{s}) = \left(\frac{\mathrm{n}}{\mathrm{s.n}}\right) \times \frac{(\mathrm{n}-1)}{(\mathrm{s}+1)\times\mathrm{n}} \times \frac{(\mathrm{n}-2)}{(\mathrm{s}+1)\times\mathrm{n}} \ldots\ldots \times \frac{1}{(\mathrm{s}+\mathrm{n}-1)\times\mathrm{n}} \times \lim_{n\to\infty} \int_0^\infty \mathrm{t}^{\mathrm{s}+\mathrm{n}-1}\, dt$$

$$(\boldsymbol{\Gamma}(\mathrm{s}))=\left(\frac{n!}{n^{n}\times s.(s+1)\ldots\ldots(s+(n-1))}\right) \times \lim_{n\to\infty} \left|\frac{t^{s+n}}{st^{n}}\right|$$

$$= (\boldsymbol{\Gamma}(\mathrm{s}))=\left(\frac{n!}{n^{n}\times s.(s+1)\ldots\ldots(s+(n-1))}\right) \times \frac{n^{s+n}}{(s+n)}$$

$$= (\boldsymbol{\Gamma}(\mathrm{s}))=\frac{n!\times n^{s}}{s\times s+1\times\ldots\ldots s+n}$$

$$=> (\boldsymbol{\Gamma}(\mathrm{s})) = n^{s} \times \lim_{n\to\infty} \prod_{z=0}^{n} \frac{z}{s+z}$$

$$=> \boldsymbol{\Gamma}(\mathrm{s}) = \frac{n^{s}}{s} \times \lim_{n\to\infty} \left[\prod_{z=1}^{n} \left(\frac{z}{s+z}\right)\right]$$

I took out the care where z=0

Important Result

$$=> (\boldsymbol{\Gamma}(\mathbf{s})) = \frac{\boldsymbol{n^{s}}}{\boldsymbol{s}} \times \mathbf{lim}_{\boldsymbol{n}\to\infty} \left[\prod_{\boldsymbol{z}=1}^{\boldsymbol{n}} \left(\frac{\boldsymbol{z}}{\boldsymbol{s}+\boldsymbol{z}}\right)\right]$$

Further Simplification

$$(\boldsymbol{\Gamma}(\mathrm{s}))=\frac{n^{s}}{s} \times \lim_{n\to\infty} \prod_{z=1}^{n} \left(\frac{z}{z\left(1+\frac{s}{z}\right)}\right)$$

$$(\boldsymbol{\Gamma}(\mathrm{s}))=\frac{n^{s}}{s} \times \lim_{n\to\infty} \prod_{z=1}^{n} \left(1+\frac{s}{z}\right)^{-1}$$

$$(\boldsymbol{\Gamma}(\mathrm{s}))=\frac{e^{s\log n}}{s} \times \lim_{n\to\infty} \prod_{z=1}^{n} \left(1+\frac{s}{z}\right)^{-1}$$

$$(\boldsymbol{\Gamma}(\mathrm{s}))=\frac{e^{0}\times e^{s\log n}}{s} \times \lim_{n\to\infty} \prod_{z=1}^{n} \left(1+\frac{s}{z}\right)^{-1}$$

$$\Gamma \mathrm{S}=\frac{e^{\sum_{k=1}^{\infty}\frac{s}{k}-\sum_{k=1}^{\infty}\frac{s}{k}} \times e^{s\log n}}{s} \times \lim_{n\to\infty} \prod_{z=1}^{n} \left(1+\frac{s}{z}\right)^{-1}$$

$$\left(\sum_{k=1}^{\infty}\frac{s}{k} - \mathbf{s}\log \boldsymbol{n} = \boldsymbol{s\gamma}\right)$$

Where $\boldsymbol{\gamma} = \boldsymbol{euler\ mascheroni\ constant}$

$$(\boldsymbol{\Gamma}(\mathrm{s})) = \frac{e^{\sum_{k=1}^{\infty}\frac{s}{k}-\gamma s}}{s} \times \lim_{n\to\infty}\prod_{z=1}^{n}\left(1+\frac{s}{z}\right)^{-1}$$

$$(\boldsymbol{\Gamma}(\mathbf{s})) = \frac{\boldsymbol{e^{-\gamma s}}}{\boldsymbol{s}} \times \prod_{\boldsymbol{z=1}}^{\boldsymbol{n}}\left[\boldsymbol{e^{\frac{s}{z}}} \times \left(\mathbf{1}+\frac{\boldsymbol{s}}{\boldsymbol{z}}\right)^{-\mathbf{1}}\right] \ldots\ldots.(\mathbf{1})$$

(Using Result1)

$$\frac{(\boldsymbol{\Gamma}(\mathrm{s}))\cos(\alpha s)}{n|p|^{s}} = \int_{0}^{\infty} x^{ns-1}\, e^{-ax^{n}} \cos(bx^{n})dt$$

$$\text{where } \alpha = \arctan\left(\frac{b}{a}\right)$$

$$|\mathrm{p}| = \sqrt{a^2+b^2}$$

Substituting $s = \frac{1}{n}$,

$$\frac{(\boldsymbol{\Gamma}\frac{1}{n})\cos\left(\frac{\alpha}{n}\right)}{n|p|^{1/n}} = \int_{0}^{\infty} e^{-ax^{n}} \cos(bx^{n})\ dx$$

Now $b = 0$;

$$|p| = \sqrt{a^2+b^2} = \sqrt{a^2} = a$$

$$\mathrm{Tan}\,\alpha = \frac{b}{a} \quad \text{since} b = 0 \Rightarrow \alpha = 0$$

$$\cos 0 = 1$$

$$\frac{(\boldsymbol{\Gamma}\frac{\mathbf{1}}{\boldsymbol{n}})}{\boldsymbol{n}(\boldsymbol{a})^{\mathbf{1}/\boldsymbol{n}}} = \int_{\mathbf{0}}^{\infty} \boldsymbol{e^{-ax^{n}}}\ \boldsymbol{dx} \ldots\ldots(\mathbf{2})$$

$$replace\ n \to y$$

$$\frac{(\boldsymbol{\Gamma}\frac{\mathbf{1}}{\boldsymbol{y}})}{\boldsymbol{y}(\boldsymbol{a})^{\mathbf{1}/\boldsymbol{y}}} = \int_{\mathbf{0}}^{\infty} \boldsymbol{e^{-ax^{y}}}\ \boldsymbol{dx} \ldots\ldots(\mathbf{4})$$

now (1) modifies to ...

$$(\boldsymbol{\Gamma}(\boldsymbol{s})) = \frac{\boldsymbol{e^{-\gamma s}}}{\boldsymbol{s}} \times \lim_{n\to\infty}\prod_{\boldsymbol{z=1}}^{\boldsymbol{n}}\left[\boldsymbol{e^{\frac{s}{z}}} \times \left(\mathbf{1}+\frac{\boldsymbol{s}}{\boldsymbol{z}}\right)^{-\mathbf{1}}\right] \ldots\ldots.(\mathbf{1})$$

$$\boldsymbol{subtitute\ s} = \frac{\mathbf{1}}{\boldsymbol{y}}$$

$$(\boldsymbol{\Gamma}\frac{\mathbf{1}}{\boldsymbol{y}}) = \frac{\boldsymbol{e^{-\frac{0.57}{y}}}}{\mathbf{1}/\boldsymbol{y}} \times \lim_{n\to\infty}\prod_{\boldsymbol{z=1}}^{\boldsymbol{n}}\left[\boldsymbol{e^{\frac{1}{zy}}} \times \left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{zy}}\right)^{-\mathbf{1}}\right] \ldots.(\mathbf{3})$$

Substituting (3) → (4)

$$\frac{ye^{\frac{-0.57}{y}}\lim_{n\to\infty}\prod_{z=1}^{n}e^{\frac{1}{zy}}\left(1+\frac{1}{zy}\right)^{n}}{ya^{\frac{1}{y}}}=\int_{0}^{\infty}e^{-ax^{y}}dt$$

Result 3

$$\frac{e^{\frac{-0.57}{y}}\lim_{n\to\infty}\prod_{z=1}^{n}e^{\frac{1}{zy}}\left(1+\frac{1}{zy}\right)^{n}}{a^{\frac{1}{y}}}=\int_{0}^{\infty}e^{-ax^{y}}dt$$

Checking Result 3 and its approximation

(A) check the result for y=1,a=1

$$\int_{0}^{\infty}e^{-x}\,dx=|e^{-x}|_{0}^{\infty}=(+1)$$

Checking result (A)
Y=1, a=1

$$\frac{e^{\frac{-0..57}{1}}}{1^{\frac{1}{1}}}\lim_{n\to\infty}\prod_{z=1}^{n}e^{\frac{1}{z}}\left(1+\frac{1}{z}\right)^{-1}$$

Taking 10 terms

$$e^{-0.57}\times e^{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\right)}\times\left(1+\frac{1}{1}\right)^{-1}\times\left(1+\frac{1}{2}\right)^{-1}\times\left(1+\frac{1}{3}\right)^{-1}\times\left(1+\frac{1}{4}\right)^{-1}\times\left(1+\frac{1}{5}\right)^{-1}\times\left(1+\frac{1}{6}\right)^{-1}\times\left(1+\frac{1}{7}\right)^{-1}\times\left(1+\frac{1}{8}\right)^{-1}\times\left(1+\frac{1}{9}\right)^{-1}\times\left(1+\frac{1}{10}\right)^{-1}$$

$$e^{-0.57}\times e^{1+0.5+0.334+0.25+0.2+0.167+0.142+0.125+0.112+0.1}\times\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}}{\cancel{4}}\times\frac{\cancel{4}}{\cancel{5}}\times\frac{\cancel{5}}{\cancel{6}}\times\frac{\cancel{6}}{\cancel{7}}\times\frac{\cancel{7}}{\cancel{8}}\times\frac{\cancel{8}}{\cancel{9}}\times\frac{\cancel{9}}{\cancel{10}}\times\frac{\cancel{10}}{11}$$

$$e^{-0.57+2.92}\times(1/11)$$

$=\frac{e^{2.35}}{11}=0.9532336\cong \mathbf{1.0000000}$

This approximation will reach the R.H.S. if we increase the terms

(B) Check the result for y=2,a=1

$$\int_{0}^{\infty}e^{-x^{2}}\,dx=\sqrt{\frac{\pi}{2}}=0.8862269$$

Checking result (B)
Y=2, a=1

$$=> \frac{e^{\frac{-0.57}{2}}}{1^{\frac{1}{2}}} \times \lim_{n\to\infty} \prod_{z=1}^{n} e^{\frac{1}{2z}} \left(1+\frac{1}{2z}\right)^{-1}$$

Taking 5 terms

$$e^{\frac{-0.57}{2}} \times e^{\frac{1}{2}} \times e^{\frac{1}{4}} \times e^{\frac{1}{6}} \times e^{\frac{1}{8}} \times e^{\frac{1}{10}} \times \left(1+\frac{1}{2}\right)^{-1} \times \left(1+\frac{1}{4}\right)^{-1} \times \left(1+\frac{1}{6}\right)^{-1} \times \left(1+\frac{1}{8}\right)^{-1} \times \left(1+\frac{1}{10}\right)^{-1}$$

$$e^{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}-\frac{0.57}{2}} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} \times \frac{10}{11}$$

$$= e^{0.5+0.25+0.167+0.125+0.1-0.285} \times \frac{64\times 60}{105\times 99}$$

$$= e^{0.857} \times \frac{3840}{10395}$$

$$= e^{0.857} \times 0.36940$$

$$= 2.35608 \times 0.36940$$

$$\mathbf{=0.87033\ \&}$$

$$\frac{\sqrt{\pi}}{2} = \frac{1.77245}{2} = 0.886$$

$$\mathbf{0.87033 \cong 0.886}$$

This approximation will reach the R.H.S. if we increase the terms

Conclusion

Result 1:

$$\frac{\Gamma(t)\cos(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)} e^{-ax^n} \cos(bx^n)\,dx$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Result 2:

$$\frac{\Gamma(t)\sin(t\alpha)}{n|s|^t} = \int_0^\infty x^{(nt-1)} e^{-ax^n} \sin(bx^n)\,dx$$

Where $t \notin Z^-$ & $n \neq 0$ & $t \neq 0$

Result 3:

$$\frac{e^{\frac{-0.57}{y}} \lim_{n\to\infty} \prod_{z=1}^{n} e^{\frac{1}{zy}} \left(1+\frac{1}{zy}\right)^{n}}{a^{\frac{1}{y}}} = \int_0^\infty e^{-ax^y} dt$$

Where $y \neq 0$

References

- http://www.ijmttjournal.org/archive/ijmtt-v43p514