Compactness in Multiset Topology

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Abstract — *The purpose of this paper is to introduce the concept of compactness in multiset topological space. We investigate some basic results in compact multiset topological space similar to the results in compact topological space. Furthermore we redefine the concept of functions between two multisets.*

Keywords — *Multisets, M-topology, compactness, continuity.*

I. Introduction

The notion of a multiset is well established both in mathematics and computer science. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, known as multiset (mset for short), is obtained.

Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. So, the only possible relation between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not always like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc.

A wide application of msets can be found in various branches of mathematics. Algebraic structures for multiset space have been constructed by Ibrahim et al.in[9]. Application of mset theory in decision making can be seen in [17]. In 2012, Girish and Sunil [5] introduced multiset topologies induced by multiset relations. The same authors further studied the notions of open sets, closed sets, basis, sub-basis, closure, interior, continuity and related properties in M-topologiacal spaces in [6]. In 2015 S. A. El-Sheikh, R. A-K. Omar and M. Raafat introduce notion of separation axioms on multiset topological space [18]. In 2015, El-Sheikh et al. [4] introduce some types of generalized open msets and their properties. In [12] J. Mahanta and D. Das have worked on semi compactness of M-topological spaces. But seimi compactness implies compactness.

This paper deals with the notion of compactness in M-topology. A definition of compactness in M-topology is introduced along with several examples. Relations among compactness, closedness and Hausdroffness are studied. A notion of function between two multisets is introduced and continuity of such functions under the M-topological context is defined. Behaviour of compactness under such continuous mappings is also studied.

II. Preliminaries

DEFINITION 2.1 [5]: An mset M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to \mathrm{N}$ where N represents the set of non-negative integers.

Here C_M (x) is the number of occurrences of the element x in the multiset M. Let M be a mset from the set $\mathrm{X} = \{\mathrm{x}_1,...\mathrm{x}_n\}$ with x appearing n times in M. It is denoted by $\mathrm{x} \in^{\mathrm{n}} \mathrm{M}$ or $\mathrm{n/x} \in \mathrm{M}$. A mset M drawn from the set X is denoted by $M = \{k_i / x_i \mid i = 1,...,m\}, m \le n$, where $x_i \in^{k_i} M$ i.e. C_M ($\mathrm{x_i}$) = $\mathrm{k_i}$. However those elements which are not included in the mset M have zero count.

DEFINITION 2.2 [5]: Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X \mid C_M(x) > 0\}$ i.e. M^* is an ordinary set and it is also called root set.

DEFINITION 2.3 [5]: A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times.

The mset space $[X]^\infty$ is the set of all msets over domain X such that there is no limit on the number of occurrences of an element in an mset. If $X = \{x_1,...,x_k\}$ then

$$[X]^m = \{\{m_1/x_1,...,m_k/x_k\} \mid m_i \in \{0,...,m\}, i = 1,...,k\}.$$

DEFINITION 2.4 [5]: Let $M, N \in [X]^m$. Then:

1. $M = N$ if C_M (x) = C_N (x), for all x $\in$ X.
2. $M \subseteq N$ (i.e. M is a submset of N) if C_M (x) ≤ C_N (x) for all x $\in$ X.
3. $P = M \bigcup N$ if C_P (x) = max{ C_M (x), C_N (x) }, for all x $\in$ X.
4. $P = M \bigcap N$ if C_P (x) = min{ C_M (x), C_N (x) }, for all x $\in$ X.
5. $P = M \oplus N$ if C_P (x) = min{ C_M (x) + C_N (x), m }, for all x $\in$ X.

6. $P = M \ominus N$ if C_P (x) = max{ C_M (x) - C_N (x), 0 }, for all x $\in$ X.

Where $\oplus$ and $\ominus$ represents mset addition and mset substraction respectively.

DEFINITION 2.5 [5]: Let $M \in [X]^m$. Then the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that $C_{M^c}(x) = m - C_M(x), \forall x \in X$. And if $N \subseteq M$ then the complement N^c of N in M is $M \ominus N$.

DEFINITION 2.6 [5]: Let M be a mset drawn from the set X and if $C_M(x) = 0$ for all $x \in X$ then M is called empty set and denoted by φ i.e. $C_\varphi(x) = 0$ for all $x \in X$.

DEFINITION 2.7 [5]: A submset N of M is a whole submset of M if $C_N(x) = C_M(x)$, for all $x \in N^*$.

DEFINITION 2.8 [5]: A submset N of M is a partial whole submset of M if $C_N(x) = C_M(x)$, for at least one $x \in N^*$.

DEFINITION 2.9 [5]: A submset N of M is a full submset of M if $M^* = N^*$.

REMARK 2.1 [5]: φ is a whole submset of every mset but it is neither a full submset nor a partial whole submset of any nonempty mset M.

DEFINITION 2.10 [5]: Let $M \in [X]^m$ be a mset. The power whole mset of M denoted by $PW(M)$ is defined as the set of all the whole submsets of M. i.e., for constructing power whole submsets of M, every element of M with its full multiplicity behaves like an element in a classical set. The cardinality of $PW(M)$ is 2^n where n is the cardinality of M^*.

DEFINITION 2.11 [5]: Let $M \in [X]^m$ be a mset. The power full mset of M denoted by $PF(M)$ is defined as the set of all the full submsets of M. The cardinality of $PF(M)$ is the product of the counts of the elements in M.

DEFINITION 2.12 [5]: Let $M \in [X]^m$ be a mset. The power mset $P(M)$ of M is the collection of all the submsets of M. We have $N \in P(M)$ iff $N \subseteq M$. If $N = \varphi$, then $N \in^1 P(M)$. If $N \neq \varphi$, then $N \in^k P(M)$ where

$$k = \prod_z \binom{|[M]_z|}{|[N]_z|},$$

the product $\prod_z$ is taken over distinct elements z of N^* and $|[M]_z| = m$ iff $z \in^m M$, $|[N]_z| = n$ iff $z \in^n N$, and

$$\binom{|[M]_z|}{|[N]_z|} = \binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

The Power set of a mset is the support set of the power mset and is denoted by $P^*(M)$. The following theorem shows the cardinality of the power set of a mset.

DEFINITION 2.13 [5]: Let $P(M)$ be a power mset drawn from the mset $M = \{m_1/x_1, \ldots, m_n/x_n\}$ and $P^*(M)$ be the power set of a mset M. Then, $Card(P^*(M)) = \prod_{i=1}^{n}(1+m_i)$.

DEFINITION 2.14 [5]: The maximum mset is defined as Z wher

$$C_Z(x) = \max\{C_M(x) \mid x \in^k M, M \in [X]^m\}$$

DEFINITION 2.15 [5]: Let $[X]^m$ be an mset space and $\{M_i \mid i \in I\}$ be a collection of msets drawn from $[X]^m$. Then the following operation operations are possible under an arbitrary collection of msets.

1. $\bigcup_{i\in I} M_i = \left\{ C_{\bigcup_{i\in I} M_i}(x)/x \mid C_{\bigcup_{i\in I} M_i}(x) = \max\left\{ C_{M_i}(x) \mid x \in X, i \in I \right\} \right\}$

2. $\bigcap_{i\in I} M_i = \left\{ C_{\bigcap_{i\in I} M_i}(x)/x \mid C_{\bigcap_{i\in I} M_i}(x) = \min\left\{ C_{M_i}(x) \mid x \in X, i \in I \right\} \right\}$

3. $\bigoplus_{i\in I} M_i = \left\{ C_{\bigoplus_{i\in I} M_i}(x)/x \mid C_{\bigoplus_{i\in I} M_i}(x) = \min\left\{ \sum_{i\in I} C_{M_i}(x), m \right\}, x \in X \right\}$

4. $M^c = Z \ominus M = \left\{ C_{M^c}(x)/x \mid C_{M^c}(x) = C_Z(x) - C_M(x), x \in X \right\}$

This is called mset complement.

DEFINITION 2.16 [5]: Let M_1 and M_2 be two msets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as

$$M_1 \times M_2 = \{(m/x, n/y)/mn \mid x \in^m M_1, y \in^n M_2\}$$

Here the pair (x, y) is repeated mn times in $M_1 \times M_2$.

DEFINITION 2.17 [5]: A sub mset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count, the product of $C_1(x,y)$ and $C_2(x,y)$. m/x related to n/x is denoted by $(m/x)R(n/y)$. Where $C_1(x,y) = C_M(x)$ and $C_2(x,y) = C_M(y)$.

The domain and range of the mset relation R on M is defined as follows:

$$DomR = \{x \in^r M \mid \exists y \in^s M s.t.(r/x)R(s/y)\}$$

$$C_{DomR(x)} = \sup\{C_1(x,y) \mid x \in^r M\}.$$

$$RanR = \{y \in^s M \mid \exists x \in^r M s.t.(r/x)R(s/y)\}$$

$$C_{RanR(x)} = \sup\{C_2(x,y) \mid y \in^s M\}.$$

DEFINITION 2.18 [5]: A mset relation f is called an mset function if for every element m/x in Dom f, there is exactly one n/x in Ran f s.t. $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x,y)$ and $C_2(x,y)$.

DEFINITION 2.19 [5]: Let $M \in [X]^m$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology on M if τ satisfies the following properties:

1. The mset M and φ are in τ.
2. The mset union of the elements of any subcollection of τ is in τ.
3. The mset intersection of elements of any finite subcollection of τ is in τ.

And the ordered pair (M, τ) is called an M-topological space. Each element in τ is called open mset.

DEFINITION 2.20 [5]: Let (M, τ) be a M-topological space and N be a submset of M. The collection $\tau_N = \{U^* \mid U^* = N \cap U, U \in \tau\}$ is a M-topology on N, called the subspace M-topology.

DEFINITION 2.21 [5]: If M is an mset, then the M-basis for an M- topology on M in $[X]^m$ is a collection **B** of submsets of M (called M-basis elements) such that:

1. for each $x \in^m M$, for some m > 0, there is at least one M-basis element $B \in$ **B** such that $x \in^m B$.
2. if $x \in^m B_1 \cap B_2$, where $B_1, B_2 \in$ **B.** Then $\exists$ $B_3 \in$ **B** such that $x \in^m B_3 \subseteq B_1 \cap B_2$.

REMARK 2.2 [5]: If a collection **B** satisfies the conditions of M-basis, then the M-topology τ generated by **B** can be defined as follows. A submset U of M is said to be open mset in M if foe each $x \in^k U$, there is an M-basis element $B \in$ **B** such that $x \in^k B \subseteq U$. Note that each M-basis element is itself an element of τ.

DEFINITION 2.22 [5]: A sub collection S of τ is called a sub M-basis for τ, when the collection of all finite intersections of members of S is an M-basis for τ.

DEFINITION 2.23 [5]: The M-topology generated by the sub M-basis S is defined to be the collection τ of all unions of finite intersections of elements of S.

THEOREM 2.1 [5] Let $M \in [X]^m$ and **B** be an M-basis for an M-topology τ on M. Then τ equals the collection of all mset unions of elements of the M-basis **B**.

THEOREM 2.2 [5] If **B** is an M-basis for the for the M-topology τ on M in $[X]^m$ then the collection $\mathbf{B}_N = \{B \cap N \mid B \in \mathbf{B}\}$ is an M-basis for the subspace M-topology on a submset N of M.

DEFINITION 2.24 [5]: A submset N of a M-topological space M in $[X]^m$ is said to be closed if the mset $M \ominus N$ is open i.e. N^c in M is open.

THEOREM 2.3 [5]: Let (M, τ) be a M-topological space. Then the followings hold:

1. The mset M and the empty mset φ are closed msets.
2. Arbitrary mset intersection of closed msets is a closed mset.
3. Finite mset union of closed msets is a closed mset.

THEOREM 2.4 [5]: Let N be a subspace of an M-topologiacl space M in $[X]^m$. Then an mset A is closed in N ifff it equals the intersection of a closed mset of M with N.

DEFINITION 2.25 [18]: A mset M is called M-singleton and denoted by {k/x} if C_M:X→**N** is such that $C_M(x) = k$ and $C_M(y) = 0$, for all $y \in$ X-{x}.

Note that if $x \in^k M$ then {k/x} is called whole M-singleton submset of M and {m/x} is called M-singleton where 0 < m < k.

DEFINITION 2.26 [18]: Let (M, τ) be a M-topological space. If for every two M-singletons $\{k_1/x_1\}$, $\{k_2/x_2\} \subseteq M$ such that $x_1 \neq x_2$, there is $G, H \in \tau$ such that $\{k_1/x_1\} \subseteq G$, $\{k_2/x_2\} \subseteq H$ and $G \cap H = \varphi$ then (M, τ) is called M-T_2-space.

DEFINITION 2.27 [18]: Let M and N bed two M-topological spaces. The mset function $f: M \to N$ is said to be continuous if for each open submset V of N, the mset $f^{-1}(V)$ is an open submset in M, where $f^{-1}(V)$ is the mset of all points m/x in M for which $f(m/x) \in^n V$ for some n.

III. COMPACT M-TOPOLOGY

A. Cover

DEFINITION 3.1: Let $M \in [X]^m$ and (M, τ) be a M-topological space. A collection C of sub-msets of M is said to be a cover of M if $M \subseteq \bigcup_{N \in C} N$.

EXAMPLE 3.1: Let M={2/x,1/y,3/z} and τ ={ φ, {2/x},{2/x,1/y},M}. Take C={{1/x,3/z},{2/x,1/y}}. Clearly C is a cover of M.

B. Sub-Cover

DEFINITION 3.2: Let $M \in [X]^m$ and (M, τ) be a M-topological space. And let C be a cover of M. If there is some C^* ($\subseteq C$) covering M then we say C^* is a sub-cover of C covering M.

EXAMPLE 3.2: Let M={2/x,1/y,3/z} and τ ={ φ, {2/x},{2/x,1/y},M}. Take C={{1/x,3/z},{2/x},{1/y,1/x},{2/x,1/y}}. Clearly C is a cover of M. And C^*={{2/x},{1/x,1/y},{1/x,3/z}} is a sub-cover of C covering M.

C. Open Cover

DEFINITION 3.3: Let $M \in [X]^m$ and (M, τ) be a M-topological space. Then $C \subseteq \tau$ is called an open cover of M if C covers M.

EXAMPLE 3.3: Let M={1/x,2/y,3/z} and τ ={ φ ,{1/x},{1/y},{1/x,1/y},{3/z},{1/x,3/z},{1/y, 3/z},{1/x,1/y,3/z},{1/x,2/y},M}. Then (M, τ) is a M-topological space. Consider C={{1/x},{1/x,2/y},{3/z},{1/x,3/z},{1/x,1/y}} then C is an open cover of M. Again C^*={{1/x}, {1/x,2/y},{3/z}}($\subseteq C$) is a sub-cover of C covering M.

EXAMPLE 3.4: $M \in [\mathbf{R}]^m$ and M^*=**R**, where **R** is the set of real numbers. And let τ be any given topology on **R**. Let us consider a whole sub-multiset U_M of M such that U_M^*=$U \in \tau$. Now we show that the collection

$$\beta = \bigcup_{U_M^* = U \in \tau} PF(U_M)$$

is a M-basis.

1. Let $x \in^m M$. Then $x \in U$ for some $U \in \tau$ and hence $x \in^m U_M \in \beta$.

2. Let B_1, $B_2 \in \beta$ and $x \in^r B_1 \cap B_2$. Clearly $B_1 \in PF(U_M)$ and $B_2 \in PF(U_M)$ for some U, $V \in \tau$. It is clear that $(B_1 \cap B_2)^*=(U_M \cap V_M)^*=U \cap V$. Hence $x \in^r B_3$= $B_1 \cap B_2 \in PF((U \cap V)_M) \subseteq \beta$. Hence β is a M-basis and will generate a M-topology induced from the given topology τ on **R**, denoted by $\tau_{WPF(\tau)}$. And if C={$U^\alpha \in \tau$ | $\alpha \in \Delta$} is any open cover of **R** then C^M={$U_M^\alpha \in \beta$ | $\alpha \in \Delta$} is an open cover of M.

EXAMPLE 3.5: Similarly we can show that if (X,τ_X) is any topological space. And if $M \in [X]^m$ with M^*=X. Then

$$\beta = \bigcup_{U_M^* = U \in \tau_X} PF(U_M)$$

where U_M is a whole sub-multiset of M is a M-basis and hence will generate a M-topology induced from the given topology τ_X on X, denoted by $\tau_{WPF(\tau_X)}$. And if C={ $U^\alpha \in \tau$ | $\alpha \in \Delta$} is any open cover of X then C^M={$U_M^\alpha \in \beta$ | $\alpha \in \Delta$} is an open cover of M.

D. Compact M-topology

DEFINITION 3.4: Let $M \in [X]^m$ and let τ be a M-topology on M. Then M is said to compact if for any open cover C of M there is a finite sub-cover of C covering M.

It is clear from the definition that any M-topological space M whose support set is finite space is compact.

A sub-multiset N of M-topological space (M, τ_M) is said to be compact if it is compact in the subspace topology induced from τ_M .

EXAMPLE 3.6: Let us consider $M \in [\mathbf{R}]^m$ and M^*=[0,1], where **R** is the set of real numbers. And consider the usual topology $\tau_{[0,1]}$ on [0,1]. Then $\beta_{[0,1]}$ ={$U_{a,b}$|$U_{a,b}$= (a,b) $\cap$ [0,1]} forms a basis of $\tau_{[0,1]}$. Now consider $U_{a,b}^M$ the whole sub-multiset of M such that $(U_{a,b}^M)^*$ = $U_{a,b}$. Then β ={ $U_{a,b}^M$ | $U_{a,b} \in \beta_{[0,1]}$} is a M-basis, since

1. $x \in^m M \Rightarrow x \in U_{a,b}$ for some $U_{a,b} \in \beta_{[0,1]} \Rightarrow x \in^m U_{a,b}^M \in \beta$.

2. let $x \in^r B_1 \cap B_2$, where B_1=$U_{a,b}^M$ and B_2=$U_{c,d}^M$. Then it obvious that $u=max\{a,c\} < v=min\{b,d\}$ and $u<1$ otherwise $B_1 \cap B_2 = \varphi$. And it is clear that $U_{a,b}^M \cap U_{c,d}^M = U_{u,v}^M$. Take B_3=$U_{u,v}^M$. Then $x \in^r B_3$= $B_1 \cap B_2 \in \beta$. Hence β will generate a topology, denoted by $\tau_{W(\tau_{[0,1]})}$. Now if C_1={$U_\lambda \in \tau_{W(\tau_{[0,1]})}$ | $\lambda \in \Lambda$ } be any open cover of M then clearly C^*={$U_\lambda^* \in \tau_{[0,1]}$|$U_\lambda \in C_1$ } is an open cover of [0,1] and hence has a finite sub-cover C^{**}={$U_{\lambda_i}^*$ | i=1, 2, 3, . . . , n} covering [0,1] , since ([0,1] , $\tau_{[0,1]}$) is compact. Then C_2={U_{λ_i} | i=1, 2, 3, . . . , n} is also a finite sub-cover of C_1 covering M. Hence M is compact.

PROPOSITION 3.1: Let (X, τ_X) be a topological space such that X is an infinite set. And let $M \in [X]^m$ s.t. $M^* = X$ and there are infinitely many $y \in M^*$ s.t. $C_M(y)$ >1. Then $(M, \tau_{WPF(\tau_X)})$ is not a compact M-topological space.

PROOF: We know $M \in \tau_{WPF(\tau_X)}$. For $x \in X$ let us define U_M^x as a full submset of M such that

$$C_{U_M^x}(y) = \begin{cases} 1, & y \neq x \\ C_M(y), & y = x \end{cases}$$

Since $(U_M^x)^* = M^* = \tau_X$ so $U_M^x \in \tau_{WPF(\tau_X)}$, for all $x \in X$. Therefore $C = \{U_M^x \mid x \in M^* = X\}$ is an open cover of M . We will now show that it has no finite sub-cover. Suppose it has a finite sub-cover and let $C^* = \{U_M^{x_i} \in C \mid i =,...,n\}$ is the finite sub-cover of C covering M. Let us choose a $y \in M^* = X$ s.t. $y \notin \{x_1,...,x_n\}$ and $C_M(y) > 1$. Then $C_{U_M^{x_i}}(y) = 1, \forall i = 1,...,n.$ And therefore

$$C_M(y) = C_{\bigcup_{i=1}^{n} U_M^{x_i}}(y) = 1,$$

a contradiction since $C_M(y) > 1$. Hence $(M, \tau_{WPF(\tau_X)})$ is not a compact.

THEOREM 3.1: A subset N of a M-topological space M is compact if and only if every covering of N by open msets in M contains a finite covering of N.

PROOF: If N is compact, and $\{U_\lambda \mid \lambda \in \Lambda\}$ is a collection of open msets of M covering N, then $\{U_\lambda \cap N \mid \lambda \in \Lambda\}$ is a collection of relatively open msets in N covering N. A finite subcollection $\{U_{\lambda_i} \cap N \mid i = 1,...,n\}$ covers N by definition, and hence the collection $\{U_{\lambda_i} \mid i = 1,...,n\}$ covers N. Conversely, if $\{V_\lambda \mid \lambda \in \Lambda\}$ is any collection of relatively open msets covering N, thenfor each λ there is an open mset U_λ in M such that. $U_\lambda \cap N = V_\lambda$.The collection $\{U_\lambda \mid \lambda \in \Lambda\}$ covers N, so has a finite sub-cover $\{U_{\lambda_i} \mid i = 1,...,n\}$ covering N. Then $\{V_{\lambda_i} = U_{\lambda_i} \cap N \mid i = 1,...,n\}$ covers N.

THEOREM 3.2: Let $M \in [X]^m$ and (M, τ_M) be a M-topological space. And let C be any family of closed sets possessing finite intersection property. Then M is compact iff

$$C_{\bigcap_{V \in C} V}(x) \neq C_\varphi(x)$$

for some $x \in M^*$.

The proof of this theorem is very easy and similar to **THEOREM 4.12[19]** so I skip this proof.

THEOREM 3.3: Let $M \in [X]^m$ and τ be a M-topology on M. Then M is compact iff every basic open cover has a finite sub-cover overing M.

PROOF: If M is compact then the case is trivial.

Conversely, let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of M and $\{B_\mu \mid \mu \in \Delta\}$ be a base for τ. Each U_λ is the union of certain B_μ's is clearly a basic open cover of M. By hypothesis, this class of B_μ's has a finite sub-cover. For each set in this finite sub-cover we can select a U_λ which contains it. The class of U_λ's which arise in this way is evidently a finite sub-cover of the original open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ of M and hence M is compact.

Now we will prove that every compact sub-mset of mset remains compact under continuous transformation. But before doing this we redefine the function between msets.

DEFINITION 3.5: Let $M \in [X]^m$ and $N \in [Y]^{m'}$. A function $f: M \to N$ is a relation s.t. for each $m/x \in M$ $\exists$ a unique r/y with $\{r/y\} \subseteq N$ s.t. $f(m/x) = r/y$ i.e. for each $m/x \in M$ $\exists$ a unique $y \in N^*$ such that $f(m/x) = r/y$, where $1 \le r \le C_N(y)$. We write a function as

$$f = \{(m/x, r/y)mr \mid m/x \in M, f(m/x) = r/y\}$$

Define range of

$$f = R_f = \bigcup_{m/x \in M} \{f(m/x)\} = f(M),$$

hence $R_f \subseteq N$.

Let $f: M \to N$ be a function,where $M \in [X]^m$ and $N \in [Y]^{m'}$. And let $A \subseteq M$. Let us denote $(A)_W$ to be the whole submset of M with $(A)_W^* = A^*$.

We define the restriction of f upon A by the restriction of f on $(A)_W$ i.e. $f|_A : A \to N$ is defined by $f|_A(k/x) = f(m/x)$, where $k/x \in A$ and $m/x \in (A)_W$, i.e. $k \le m = C_M(x)$. Hence we define $f(A) = f((A)_W)$

Let $M \in [X]^m$, $N \in [Y]^{m'}$, $N_1 \subseteq N$ and $f: M \to N$ be a function. Define

$$f^{-1}(N_1) = \{m/x \in M \mid \{f(m/x)\} \subseteq N_1\}.$$

Hence $f^{-1}(N_1)$ is either a whole submset of M or an emoty set.

Let $f: M \to N$ be a function from $M \in [X]^m$ to $N \in [Y]^{m'}$ then it can be easily shown that if A, B be two submsets of M s.t. $A \subseteq B$ then $f(A) \subseteq f(B)$ and if A, B be two submsets of N s.t. $A \subseteq B$ then $f^{-1}(A) \subseteq f^{-1}(B)$.

THEOREM 3.4: Let $f: M \to N$ be a function from a mset $M \in [X]^m$ to a mset $N \in [Y]^{m'}$. Then for submsets A, B of M and C, D of N

1. $f((A \cup B)_W) = f((A)_W) \cup f((B)_W)$.
2. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

PROOF: It can be easily shown that

1. $(A \cup B)_W = (A)_W \cup (B)_W$.

2. $(A\bigcap B)_W = (A)_W \bigcap (B)_W$.

Now

$r/y \in f((A\bigcup B)_W)$

$\Rightarrow \exists\ m/x \in (A\bigcup B)_W$ s.t. $f(m/x) = r/y$, where $m = C_M(x)$

$\Rightarrow \exists\ m/x \in (A)_W$ or $\exists\ m/x \in (B)_W$ s.t. $f(m/x) = r/y$ (since $(A\bigcup B)_W = (A)_W \bigcup (B)_W$.)

$\Rightarrow \exists\ m/x \in (A)_W$ s.t. $f(m/x) = r/y$ or $\exists\ m/x \in (B)_W$ s.t. $f(m/x) = r/y$

$\Rightarrow \{r/y\} \subseteq f((A)_W)$ or $\{r/y\} \subseteq f((B)_W)$

$\Rightarrow \{r/y\} \subseteq f((A)_W) \bigcup f((B)_W)$

$\Rightarrow f((A\bigcup B)_W) \subseteq f((A)_W) \bigcup f((B)_W)$

Again

$r/y \in f((A)_W) \bigcup f((B)_W)$.

$\Rightarrow r/y \in f((A)_W)$ or $r/y \in f((B)_W)$

$\Rightarrow \exists\ m_1/x_1 \in (A)_W$ s.t. $f(m_1/x_1) = r/y$ or $\exists\ m_2/x_2 \in (B)_W$ s.t. $f(m_2/x_2) = r/y$

$\Rightarrow \exists\ m/x \in (A)_W \bigcup (B)_W$ s.t. $f(m/x) = r/y$

$\Rightarrow \{r/y\} \subseteq f((A)_W \bigcup (B)_W) = f((A\bigcup B)_W)$ (since $(A\bigcup B)_W = (A)_W \bigcup (B)_W$.)

$\Rightarrow f((A)_W) \bigcup f((B)_W) \subseteq f((A\bigcup B)_W)$

Hence $f((A\bigcup B)_W) = f((A)_W) \bigcup f((B)_W)$.

Therefore A, B are two submsets of a mset M, then

$f(A\bigcup B) = f((A\bigcup B)_W)$

$= f((A)_W) \bigcup f((B)_W) = f(A) \bigcup f(B)$.

Similarly it can be shown that

$f(A\bigcap B) = f((A\bigcap B)_W)$

$= f((A)_W) \bigcap f((B)_W) = f(A) \bigcap f(B)$.

Proof 2. Is similar.

THEOREM 3.5: Let $f: M \to N$ be a continuous function from the mset $M \in [X]^m$ to the mset $N \in [Y]^{m'}$, where M and N be two M-topological spaces. And let C be compact submset in M. Then $f(C)$ is compact in N.

PROOF: Let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of $f(C)$. Then $f(C) \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$. Now we show that $(C)_W \subseteq f^{-1}(f(C))$. Let

$m/x \in (C)_W$

$\Rightarrow \{f(m/x)\} \subseteq f((C)_W)$

$\Rightarrow m/x \in f^{-1}(f(C)_W)$

$\Rightarrow (C)_W \subseteq f^{-1}(f(C))$.

Therefore

$(C)_W \subseteq f^{-1}(f(C))$

$\subseteq f^{-1}(\bigcup_{\lambda\in\Lambda} U_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda)$.

Each $f^{-1}(U_\lambda)$ is an open whole submset of M. Now $C \subseteq \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda)$. Therefore $\{f^{-1}(U_\lambda) \mid \lambda \in \Lambda\}$ is an open cover of C and hence has a finite sub-cover $\{f^{-1}(U_{\lambda_i}) \mid i = 1,\ldots,n\}$ covering . Therefore

$C \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\lambda_i})$

$\Rightarrow f(C) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(U_{\lambda_i})) \subseteq \bigcup_{i=1}^{n} U_{\lambda_i}$.

Hence $\{U_\lambda \mid \lambda \in \Lambda\}$ has a finite sub-cover $\{U_{\lambda_i} \mid i = 1,\ldots,n\}$ covering $f(C)$. As $\{U_\lambda \mid \lambda \in \Lambda\}$ is arbitrary so $f(C)$ is compact.

THEOREM 3.6: Let $M \in [X]^m$ and τ be a M-topology on M. Then every closed whole submset of M is compact.

The proof is trivial so I skip it.

REMARK: If we don't consider closed whole submset then the above theorem may be false. Let us consider $M \in [X]^m$ where $X = \mathbf{R}$ (set of reals) s.t. $M^* = [0,1]$ and w> 1 and $C_M(x) > 1\ \forall\, x \in M^*$. Now suppose the usual topology $\tau_{[0,1]}$ on [0,1] is given. Let $\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}$ be a sub-multiset of M such that $C_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}(x) = C_M(x) - 1$, $\forall x \in \left(\frac{1}{4},\frac{1}{2}\right)$ and $C_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}(\frac{1}{4}) = C_M(\frac{1}{4})$, and $C_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}(\frac{1}{2}) = C_M(\frac{1}{2})$, and consider

$\beta_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}} = \left\{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \bigcap U \mid U \in \tau_{W(\tau_{[0,1]})}\right\}$.

Now we will show that

$$\beta = \tau_{W(\tau_{[0,1]})} \bigcup \beta_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}$$

Form a M-basis.

1. Let $x \in^m M$. Then it is clear that $\exists\ U \in \tau_{W(\tau_{[0,1]})}$ such that $x \in^m U \in \beta$.

2. Let $x \in^r B_1 \bigcap B_2$ where $B_1, B_2 \in \beta$.

CaseI: If $B_1, B_2 \in \tau_{W(\tau_{[0,1]})}$. Then $x \in^r B_3 = B_1 \cap B_2 \in \tau_{W(\tau_{[0,1]})} \subseteq \beta$.

CaseII: If $B_1, B_2 \in \beta_{\left[\frac{1}{4}\frac{1}{2}\right]_{M-1}}$. Then $B_1 = \left[\frac{1}{4}\frac{1}{2}\right]_{M-1} \cap U_1, B_2 = \left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_2$ s.t. $U_1, U_2 \in \tau_{W(\tau_{[0,1]})}$. Then

$$r = \min\left\{C_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}(x), C_{U_1}(x), C_{U_2}(x)\right\}$$

$$= C_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}(x)$$

Now take $B_3 = \left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_3$, where $U_3 = U_1 \cap U_2 \in \tau_{W(\tau_{[0,1]})}$. Then clearly $x \in^r B_3 = B_1 \cap B_2 \in \beta_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}} \subseteq \beta$.

CaseIII: If $B_1 \in \tau_{W(\tau_{[0,1]})}$ and $B_2 \in \beta_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}$. Then clearly $B_3 = B_1 \cap B_2 \in \beta_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}} \subseteq \beta$.

Hence β is a basis and will generate a M-topology. Let $C = \{U_\lambda \in \beta \mid \lambda \in \Lambda\}$ be an open cover of M containing basic open msets. If $C^* = \{U_\lambda \in C \mid U_\lambda \in \tau_{W(\tau_{[0,1]})}\}$ covers M, then C^* and hence C has a finite sub-cover covering M because $(M, \tau_{W(\tau_{[0,1]})})$ is compact, otherwise $\exists x \in^m M$ s.t. $m / x \notin U_\lambda, \forall U_\lambda \in C^*$. Therefore $\exists U_{\lambda_0} \in \beta_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}$ s.t. $x \in^m U_{\lambda_0} = \left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_0$ where $U_0 \in \tau_{W(\tau_{[0,1]})}$. Now since

$$C_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}(y) = C_M(y) - 1 < C_M(y) \forall y \in \left(\frac{1}{4},\frac{1}{2}\right),$$

so $x = \frac{1}{4}$ or $x = \frac{1}{2}$. Hence $\exists U_1, U_2 \in \tau_{W(\tau_{[0,1]})}$ s.t. $\frac{1}{4} \in^m \left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_1$ and $\frac{1}{2} \in^n \left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_2$ and $\left\{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_1, \left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_2\right\} \subseteq C$.

Now consider $C^{**} = C^* \cup \{U_1, U_2\}$. Then C^{**} is an open cover of M s.t. $C^{**} \subseteq \tau_{W(\tau_{[0,1]})}$. Again we know that $(M, \tau_{W(\tau_{[0,1]})})$ is compact, so C^{**} has a finite sub-cover C^{***} covering M. Let us take $C^{\#} = \left(C^{***} \setminus \{U_1, U_2\}\right) \cup \left\{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_1, \left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \cap U_2\right\}$ then clearly $C^{\#}$ is a finite sub-cover of C covering M. Now consider the sets $\left[0,\frac{1}{4}\right)_M \left(= U^M_{-1,\frac{1}{4}}\right), \left(\frac{1}{2},1\right]_M \left(= U^M_{\frac{1}{2},2}\right) \in \tau_{W(\tau_{[0,1]})}$ where $\left[0,\frac{1}{4}\right)^*_M = \left[0,\frac{1}{4}\right)$ and $\left(\frac{1}{2},1\right]^*_M = \left(\frac{1}{2},1\right]$ and $\left[\frac{1}{4},\frac{1}{2}\right]_{M-1} \in \beta_{\left[\frac{1}{4},\frac{1}{2}\right]_{M-1}}$. Then $N = \left[0,\frac{1}{4}\right)_M \cup \left(\frac{1}{2},1\right]_M \cup \left[\frac{1}{4},\frac{1}{2}\right]_{M-1}$ is open and hence N^c is closed. And clearly $(N^c)^* = \left(\frac{1}{4},\frac{1}{2}\right)$ and $C_{N^c}(x) = 1$, $\forall x \in N^c$. Now consider

$$G = \left\{\left(\frac{1}{4}+\frac{1}{n},\frac{1}{2}-\frac{1}{n}\right)_M \in \tau_{W(\tau_{[0,1]})} \mid \left(\frac{1}{4}+\frac{1}{n},\frac{1}{2}-\frac{1}{n}\right)^*_M = \left(\frac{1}{4}+\frac{1}{n},\frac{1}{2}-\frac{1}{n}\right), n \in \mathrm{N}, n \geq 8\right\}$$

then G is an open cover of N^c having no finite sub-cover covering N^c otherwise

$$G^* = \left\{\left(\frac{1}{4}+\frac{1}{n},\frac{1}{2}-\frac{1}{n}\right) \mid n \in \mathrm{N}, n \geq 8\right\}$$

will have a finite sub-cover covering the interval $\left(\frac{1}{4},\frac{1}{2}\right)$, a contradiction. Hence N^c is not compact.

THEOREM 3.7: Let $M \in [X]^m$ and (M, τ_M) be a M-T_2-space, $\{k/x\} \subseteq M$ and F be a compact submset of M s.t $\{k/x\} \bigcap F = \varphi$. Then $\exists$ open msets U, V s.t. $\{k/x\} \subseteq U, F \subseteq V$ and $U \bigcap V = \varphi$.

PROOF: Since M is T_2 so for $y \in^n F$ $\exists$ open sets U_y, V_y such that $\{k/x\} \subseteq U_y$ and $\{n/y\} \subseteq V_y$ and $U_y \bigcap V_y = \varphi$. Then the collection $C = \{V_y \mid y \in^n F\}$ is an open cover of F in (M, τ_M). Since F is compact so $\exists n \in \mathrm{N}$ such that $\{V_{y_i} \mid i = 1,..., n\} \subseteq C$ and $F \subseteq \bigcup_{i=1}^{n} V_{y_i} = V$. Take $\bigcap_{i=1}^{n} U_{y_i} = U$. Then U is an open set such that $\{k/x\} \subseteq U$ and clearly $C_{U \cap V}(x) = C_{\varphi}(x)$. Hence $\exists$ open sets U, V such that $\{k/x\} \subseteq U, F \subseteq V$ and $U \bigcap V = \varphi$.

IV.CONCLUSION

In this paper a concept of compactness is introduced in M-topological space and some of its properties are studied. There is a huge scope of future work in studying other topological concepts in this setting.

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