β-Dual of Vector-Valued Double Sequence Spaces of Maddox

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Abstract

The β-dual of a vector-valued double sequence space is defined and studied we show that if an X-valued sequence space E is a BK-space having AK property, then the dual space of E and its β-dual are isometrically isomorphic. We also give characterizations of β-dual of vector-valued sequence spaces of Maddox ℓ^2 (X, λ, p), ℓ^2_∞(X, λ, p), c^2_0 (X, λ, p) and c^2(X, λ, p).

Introduction

Let (X, ||.||) be a Banach space and p = (p_{ij}) a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write X = (x_{ij}) with x_{ij} in X for all i,j ∈ N. The X-valued sequence spaces of Maddox are defined as –

$$c^2_0(X, \lambda, p) = \left\{ x = (x_{ij}) : \lim_{i+j\to\infty} \| \lambda_{ij} x_{ij} \|^{p_{ij}} = 0 \right\}$$

$$c^2(X, \lambda, p) = \left\{ x = (x_{ij}) : \lim_{i+j\to\infty} \| \lambda_{ij} x_{ij} - a \|^{p_{ij}} = 0 \right\} \text{ for some } a \in x$$

$$\ell^2_\infty(X, \lambda, p) = \left\{ x = (x_{ij}) : \sup_{i,j} \| \lambda_{ij} x_{ij} \|^{p_{ij}} < \infty \right\}$$

$$\ell^2(X, \lambda, p) = \left\{ x = (x_{ij}) : \sum_{2\le i+j\le N}\sum \| \lambda_{ij} x_{ij} \|^{p_{ij}} < \infty \right\}$$

When X = K, the scalar held of X, the corresponding spaces are written as c_0(p), c(p), ℓ_∞(p), ℓ (p), respectively. All of these spaces are known as the sequence Maddox. These spaces were introduced and studied by Simons [7] and Maddox [3, 4, 5]. The space ℓ (p) was first defined by Nakano [6] and is known as the Nakano sequence space. Grosse-Erdmann [1] has investigated the structure of the spaces c_0(p), c(p), ℓ (p) and ℓ_∞(p) and has given characterization of β-dual of scalar-valued sequence spaces of Maddox.

In [8], Wu and Bu gave characterizations of Köthe dual of the vector-valued sequence space ℓ_p[X], where ℓ_p[X], 1 < p < ∞, is defined by

$$\ell_p[X] = \left\{ x = (x_k) : \sum_{k=1}^{\infty} | f(x_k) \|^p < \infty \text{ for each } f \in X' \right\}, \qquad (1.2)$$

In this paper, the β-dual of a vector-valued sequence space is defined and studied and we give characterizations of β-dual of vector-valued sequence spaces of Maddox ℓ^2 (X, λ, p), ℓ^2_∞(X, λ, p), c^2_0 (X, λ, p) and c^2 (x, λ, p). Some results, obtained in this paper, are generalizations of some in [1, 3].

2. Notation and Definitions

Let (X, ||.||) be a Banach space. Let W(X) and ϕ(X) denote the space of all sequences in X and the space of all finite sequences in X, respectively. A sequence space in X is a linear subspace of W(X). Let E be an X-valued sequence space. For x ∈ E and i,j ∈ N we write that x_{ij} stand for the i,jth term of x. For x ∈ X and i,j ∈ N, we let

$e^{(i,j)}(x)$ be the sequence $\begin{vmatrix} 0 & 0 & 0 \ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots\ldots 0 \\ . & . & . \ \ldots\ldots\ldots\ldots\ldots 0 \\ . & . & . \ \ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 0 \ldots x \ldots x \ldots 0 \end{vmatrix}$ with x in the i.jth position and let e(x) be the sequence $\begin{vmatrix} x, & x, & x, & \ldots 0 \\ x, & x, & x, & \ldots 0 \\ . & . & . & \ldots 0 \end{vmatrix}$. For a fixed scalar sequence u = (u_{ij}) the sequence space E_u^2 is defined as

$$E_u^2 = \left\{x = (x_{ij}) \in W^2(X) : (u_{ij}\, x_{ij}) \in E^2\right\} \qquad (2.1)$$

An X-valued sequence space E is said to be normal if $(y_{ij}) \in E^2$ whenever $\|y_{ij}\| \le \|x_{ij}\|$ for all $i,j \in N$ and $(x_{ij}) \in E$. Suppose that the X-valued sequence space E is endowed with linear topology τ. The E is called a K-space if, for each $i,j \in N$, the i,jth coordinate mapping $p_{i,j} : E \to X$, defined by $p_{ij}(x) = x_{ij}$, is continuous on E. In addition, if (E, τ) is a Frecher (Banach) space then E is called an FK-(BK)-space. Now, suppose that E contains $\phi(X)$, then E is said to have property AK if $\sum_{2 \le i+j \le N} e^{(ij)}(x_{ij}) \to x$ in E as $N \to \infty$ for every $x = (x_{ij}) \in E^2$.

The spaces $c_0(p)$ and c(p) are FK-spaces. In c_0^2 (X, λ, p), we consider the function g(x) = $\sup_{ij} \| x_{ij} \|^{p_{ij}/M}$, where M = max[1.$\sup_{ij} p_{ij}$], as a paranorm on c_0^2 (X, λ, p) and it is known that c_0^2 (X, λ, p) is an FK-space having property AK under the paranorm p defined as above in ℓ^2 (X, λ, p) we consider it as a paranormed sequence space with the paranorm given by $\|x_{ij}\| = \left(\sum_{i+j \ge N} \| \lambda_{ij} x_{ij} \|^{p_{ij}} \right)^{1/M}$. It is known that ℓ^2 (X, λ, p) is an FK-space under the paranorm defined as above.

For an X-valued sequence space E, define its Köthe dual with respect to the dual pair (X, X') (see [2]) as follows :

$$E^x|(X, X')| = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} | f_k(x_k) | < \infty \ \ \forall x = (x_k) \in E \right\} \qquad (2.2)$$

In this paper, we denote $E^X|(X, X')|$ by E^α and it is called the α-dual of E.

For a sequence space E, the β-dual of E is defined by

$$E^\beta = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges } \forall (x_k) \in E \right\} \qquad (2.3)$$

It is easy to see that $E^\alpha \le E^\beta$.

For the sake of completeness we introduce some further sequence spaces that will be considered as β-dual of the vector-valued sequence spaces of Maddox :

$$M_0^2(X, \lambda, p) = \left\{ x = (x_{ij}) : \sum\sum_{i+j \ge N} \| x_{ij} \|^{M^{-1/p_{ij}}} < \infty \text{ for some } M \in N \right\};$$

$$M_\infty^2(X, \lambda, p) = \left\{ x = (x_{ij}) : \sum\sum_{i+j \ge N} \| \lambda_{ij} x_{ij} \|^{n^{-1/p_{ij}}} < \infty \ \forall \ n \in N \right\};$$

$$\ell_0^2(X, \lambda, p) = \left\{ x = (x_{ij}) : \sum\sum_{i+j\geq N} \|\lambda_{ij} x_{ij}\|^{p_{ij}} M^{-p_{ij}} < \infty \text{ for some } M \in N \right\};\ p_{ij} > 1\ \forall\ i,j \in N$$

$$cs[X'] = \left\{ (f_{ij}) \subset X' : \sum\sum_{i+j\geq N} f_{ij}(x) \text{ converges } \forall\ x \in X \right\} \qquad (2.4)$$

When X = K, the scalar field of X, the corresponding first two sequence spaces are written as $M_0(p)$ and M = (p), respectively. These two spaces were first introduced by Grosse-Erdmann [1].

3. Main Results

We begin by giving some general properties of β-dual of vector-valued sequence spaces.

Proposition 3.1 : Let X be a Banach space and let E^2, E_1^2 and E_2^2 be X-valued sequence spaces. Then

(i) ${}^2E^\alpha \subseteq {}^2E^\beta$

(ii) If $E_1^2 \subseteq E_2^2$ then ${}^2E_2^\beta \subseteq {}^2E_1^\beta$

(iii) If $E^2 = E_1^2 + E_2^2$, then ${}^2E^\beta = {}^2E_1^\beta\ {}^2E_2^\beta$

(iv) If E is normal then ${}^2E^\alpha = {}^2E^\beta$

Proof : Assertions (i), (ii) and (iii) are immediately obtained by the definitions. To prove (iv), by (i), it suffices to show only that ${}^2E^\beta = {}^2E^\alpha$. Let $f_{ij} \in {}^2E^\beta$ and $x = (x_{ij}) \in E^2$. Then $\sum\sum_{i+j\geq N} f_{ij}(x_{ij})$ converges. Choose a scalar sequence (t_{ij}) with $|t_{ij}| = 1$ and $f_{ij}(t_{ij}X_{ij}) = |f_{ij}(x_{ij})|$ for all $i,j \in N$. Since E is normal, $(t_{ij}x_{ij}) \in E$. It follows that $\sum\sum_{i+j\geq N} f_{ij}(x_{ij})$ converges, hence $(f_{ij}) \in {}^2E^\alpha$.

It E is a BK-space, we define a norm on E^β by the formula

$$\|(f_{ij})\|_{{}^2E^\beta} = \sup_{\|x_{ij}\|\leq 1} \left| \sum\sum_{i+j\geq N} f_{ij}(x_{ij}) \right| \qquad (3.1)$$

It is easy to show that $\|.\|_{{}^2E^\beta}$ is a norm on ${}^2E^\beta$.

Next, we give a relationship between β-dual of a sequence space and its continuous dual. Indeed, we need a lemma.

Lemma 3.2 : Let E be an X-valued sequence space which is an FK-space containing φ(X). Then for each $i,j \in N$, the mapping $T_{i,j} : X \to E$, defined by $T_{ij}x = e^{ij}(x)$, is continuous.

Proof : Let $V = e^{ij}(x) : x \to X$. Then V is a closed subspace of E, so it is an FK-space because E is an FK-space. Since E is a K-space, the coordinate mapping $p_{ij} : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_{ij} is open, which implies that $p_{ij}^1 : X \to V$ is continuous. But since $T_{ij} = p_{ij}^1$, we thus obtain that T_{ij} is continuous.

Theorem 3.3 : If E is a BK-space having property AK, then ${}^2E^\beta$ and ${}^2E'$ are isometrically isomorphic.

Proof : We first show that for $x = (x_{ij}) \in E^2$ and $f \in {}^2E'$.

$$f(x) = \sum\sum_{i+j\geq N} f(e^{ij}(x_{ij})) \qquad (3.2)$$

To show this, let $x = (x_{ij}) \in E$ and $f \in E'$. Since E has property AK,

$$X = \lim_{m+n\to\infty} \sum_{2\le i+j\le N}\sum (e_{ij}(x_{ij})) \qquad (3.3)$$

By the continuity of f, it follows that

$$f(x) = \sum_{2\le i+j\le m+n}\sum f(e^{ij}(x_{ij})) = \sum_{i+j\ge N}\sum f(e^{ij}(x_{ij})) \qquad (3.4)$$

so (3.2) is obtained. For each $i,j \in N$, let $T_{i,j} : X \to E$ be defined as in Lemma 3.2. Since E is a BK-space, be Lemma 3.2, T_{ij} is continuous. Hence $f \circ T_{i,j} \in X'$ for all $i,j \in N$. It follows from (3.2) that

$$f(x) = (f \text{ o } T_{i,j})(x_{ij}) \quad \forall\, x = (x_{ij}) \in E \qquad (3.5)$$

It implies, by (3.5), thet $(f \circ T_{ij})_{i,j=1}^{\infty} \in {}^2E^{\beta}$. Define $\varphi : {}^2E' \to {}^2E^{\beta}$ by

$$\varphi(f) = (f \circ T_{ij})_{i,j=1}^{\infty} \quad \forall f \in {}^2E' \qquad (3.6)$$

It is easy to see that φ is linear. Now, we show that φ is onto. Let $(f_{ij}) \in {}^2E^{\beta}$. Define $f : E \to K$, where K is the scalar field of X, by

$$f(x) = \sum_{i+j\ge N}\sum f_{ij}(x_{ij}) \quad \forall\, x = (x_{ij}) \in E^2 \qquad (3.7)$$

For each $i,j \in N$, let p_{ij} be the (i,j)th coordinate mapping on E. Then we have

$$f(x) = \sum_{i+j\ge N}\sum (f_{ij} \text{ o } p_{ij}) = \lim_{m+n\to\infty} \sum_{i+j\ge N}\sum (f \text{ o } p_{ij})(x) \qquad (3.8)$$

Since f_{ij} and p_{ij} are continuous linear, so is also continuous f o p_{ij}. It follows by Banach Steinhaus theorem that $f \in {}^2E'$ and we have by (3.7) that; for each $i,j \in N$ and each $z \in X$, $(f \text{ o } T_{ij})(z) = f(e^{(ij)}(z)) = f_{ij}(z)$. Thus $f \text{ o } T_{ij} = f_{ij}$ for all $i,j \in N$, which implies that $\varphi(f) = (f_{ij})$, hence φ is onto.

Finally, we show that φ is linear isometry. For $f \in E$, we have

$$\|f\| = \sup_{\|(x_{ij})\|=1} |f(x_{ij})|$$

$$= \sup_{\|(x_{ij})\|=1} \Big|\sum_{i+j\ge N}\sum (f(e^{ij})(x_{ij}))\Big| \qquad \text{by (3.2)}$$

$$= \sup_{\|(x_{ij})\|=1} \Big|\sum_{i+j\ge N}\sum (f \circ T_{ij})(x_{ij})\Big| \qquad (3.9)$$

$$= \|(f \circ T_{ij})_{i,j=1}^{\infty}\|_{{}^2E^{\beta}}$$

$$= \|\varphi(f)\|_{{}^2E^{\beta}}.$$

Hence φ is isometry. Therefore, $\varphi : {}^2E' \to {}^2E^{\beta}$ is an isometrically isomorphism from E' onto E^{β}. This completes the proof.

We next give characterizations of β-dual of the sequence space $\ell(X, p)$ when $p_{ij} > 1$ for all $i,j \in N$.

Theorem 3.4 : Let $p = (p_{ij})$ be a bounded sequence of positive real numbers with $p_{ij} > 1$ for all $i,j \in N$. Then $\ell(X, \lambda, p)^{\beta} = \ell_0^2(X', \lambda, q)$ where $q = (q_{ij})$ is a sequence of positive real numbers such that $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$ for all $i,j \in N$.

Proof : Suppose that $(f_{ij}) \in \ell_0^2(X', \lambda, q)$. Then $\sum_{i+j\ge N}\sum \|f_{ij}\|^{q_{ij}} M^{-q_{ij}} < \infty$ for some $M \in N$. Then for each $x = (x_{ij}) \in \ell(X, \lambda, p)$, we have

$$\sum_{i+j\geq N}\sum |f_{ij}(x_{ij})| \leq \sum_{2\leq i+j\geq N}\sum \|f_{ij}\| \, M^{-1/p_{ij}} \, M^{1/p_{ij}} \, \|\lambda_{ij} x_{ij}\|$$

$$\leq \sum_{i+j\geq N}\sum \left(\|f_{ij}\|^{q_{ij}} M^{-q_{ij}/p_{ij}} + M\|\lambda_{ij}x_{ij}\|^{p_{ij}}\right)$$

$$\leq \sum_{i+j\geq N}\sum \|f_{ij}\| \, M^{-(q_{ij}-1)} + M \sum_{2\leq i+j\geq N}\sum \|\lambda_{ij}x_{ij}\|^{p_{ij}}$$

which implies that $\sum_{i+j\geq N}\sum f_{ij}(\lambda_{ij}x_{ij})$ converges, so $(f_{ij}) < \ell_0^2(X, \lambda, p)^{\beta}$.

On the other hand, assume that $(f_{ij}) \in \ell^2(X, \lambda, p)^{\beta}$, then $\sum_{i+j\geq N}\sum f_{ij}(\lambda_{ij}x_{ij})$ converges for all $x = (x_{ij}) \in \ell^2(X, \lambda, p)$.

For each $x = (x_{ij}) \in \ell^2(X, \lambda, p)$ choose scalar sequence (t_{ij}) with $|t_{ij}| = 1$ such that $f_{ij}(t_{ij}x_{ij}) = |f_{ij}(x_{ij})|$ for all $i,j \in N$. Since $(t_{ij}x_{ij}) \in \ell^2(X, \lambda, p)$, by our assumption, we have $\sum_{i+j\geq N}\sum f_{ij}(\lambda_{ij}x_{ij})$ converges, so that

$$\sum_{i+j\geq N}\sum |f_{ij}(x_{ij})| < \infty \quad \forall\, x \in \ell^2(X, \lambda, p) \tag{3.11}$$

We want to show that $(f_{ij}) \in \ell_0^2(X', \lambda, q)$, that is $\sum_{i+j\geq N}\sum \|f_{ij}\|^{q_{ij}} M^{-q_{ij}} < \infty$ for some $M \in N$.

If it is not true, then

$$\sum_{i+j\geq N}\sum \|f_{ij}\|^{q_{ij}} m^{-q_{ij}} < \infty \quad \forall\, m \in N \tag{3.12}$$

It implies by (3.12) that for each $i,j \in N$,

$$\|f_{i'i''}\|^{q_{i'i''}} \, M^{-q_{i'i''}} = \infty \quad \forall\, r,s \in N \tag{3.13}$$

By (3.12), let $r,s = 1$, then there is a $i_1,j_1 \in N$

$$\sum_{2\leq i+j\leq i_1+j_1}\sum \|f_{ij}\|^{q_{ij}} \, (r+s)^{-q_{ij}} > 1 \tag{3.14}$$

By (3.13), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that

$$\sum_{k_1\leq i+j\leq k_2}\sum \| f_{ij} \|^{q_{ij}} \, m_2^{-q_{ij}} > 1 \tag{3.15}$$

Proceeding in this way, we can choose sequences of positive integers (k_i) and (m_i) with $i = k_0 < k_1 < k_2 < \ldots\ldots$ and $m_1 < m_2 < \ldots..$, such that $m_1 > 2^i$ and

$$\sum_{k_{i-1}\leq i+j\leq k_i}\sum \| f_{ij} \|^{q_{ij}} \, m_i^{-q_{ij}} > 1 \tag{3.16}$$

For each $i \in N$, choose $x_{i,j}$ in X with $\|x_{ij}\|=1$ for all $i,j \in N$, $K_{i-1} < k \leq k$, such that

$$\sum_{k_{i-1}\leq i+j\leq k_i}\sum | f_{ij}(x_{ij}) |^{q_{ij}} \, m^{-q_{ij}} > 1 \tag{3.17}$$

Let $a_i = \sum_{k_{i-1}\leq i+j\leq k_i}\sum | f_{ij}(x_{ij}) |^{q_{ij}} \, m_i^{-q_{ij}}$. Put $y = (y_{ij})$, $(y_{ij}) = a_i^{-1} m^{-q_{ij}} \, | f_{ij}(x_{ij}) |^{q_{i-1,j-1}} \, x_{ij}$ for all $k \in N$ with $k_{i-1} < k \leq k_i$. By using the fact that $p_k q_k = p_k + q_k$ and $p_k(q_k - 1) = q_k$ for all $i,j \in N$, we have that for each $i \in N$,

$$\sum_{k_{i-1}\le i+j\le k_i}\sum \| f_{ij} \|^{p_{ij}} = \sum_{k_{i-1}\le i+j\le k_i}\sum \| a_i^{-1} m_i^{-q_{ij}} | f_{ij}(x_{ij}) |^{q_{i-1,j-1}} x_{ij} \|^{q_{ij}}$$

$$= \sum_{k_{i-1}\le i+j\le k_i}\sum a_i^{-p_{ij}} m_i^{-p_{ij} q_{ij}} | f_{ij}(x_{ij}) |^{q_{ij}}$$

$$= \sum_{k_{i-1}\le i+j\le k_i}\sum a_i^{-p_{ij}} m_i^{-p_{ij}} m_i^{p_{ij}} m_i^{q_{i-1,j-1}} | f_{ij}(x_{ij}) |^{q_{ij}}$$

$$\le a_i^{-1} m_i^{-1} \sum_{k_{i-1}\le i+j\le k_i}\sum m_i^{-q_{ij}} | f_{ij}(x_{ij}) |^{q_{ij}} \qquad (3.18)$$

$$\le a_i^{-1} m_i^{-1} a_i$$

$$\le m_i^{-1}$$

So we have that $\sum_{i,j\ge N}\sum \| y_{ij} \|^{p_{ij}} \le \sum_{2\le i+j\le N}\sum \frac{1}{2^i} < \infty$. Hence, y = ($y_{ij}$). For each i ∈ N, we have mm

$$\sum_{k_{i-1}\le i+j\le k_i}\sum | f_{ij}(y_{ij}) | = \sum_{k_{i-1}\le i+j\le k_i}\sum | f_{ij} | (a_i^{-1} m_i^{-q_{ij}}) | | f_{ij}(x_{ij}) |^{q_{i-1,j-1}} x_{ij} |$$

$$= \sum_{k_{i-1}\le i+j\le k_i}\sum a_i^{-1} m_i^{-q_{ij}} | f_{ij}(y_{ij}) |^{q_{ij}} \qquad (3.19)$$

$$= a_i^{-1} \sum_{k_{i-1}\le i+j\le k_i}\sum m_i^{-q_{ij}} | f_{ij}(y_{ij}) |^{q_{ij}}$$

$$= 1$$

So that $\sum_{i,j\ge N}\sum | f_{ij}(y_{ij}) | = \infty$, which contradicts (3.11). Hence $(f_{ij}) \in \ell_0^2(X', \lambda, q)$. The proof is now complete.

The following theorem gives a characterization of β-dual of ℓ^2 (X, λ, p), when $p_{ij} \le 1$ for all i,j ∈ N. To do this, the following lemma is needed.

Lemma 3.5 : Let p = (p_{ij}) be a bounded sequence of positive real numbers. Then ℓ_∞^2(X, λ, p) = $\bigcup_{n=1}^{\infty} \ell_\infty^2(X, \lambda, p)_{n^{-1/p_{ij}}}$.

Proof : Let x ∈ ℓ_∞^2(X, λ, p), then there is some n ∈ N with $\| x_{ij} \|^{p_{ij}} \le n$ for all i,j ∈ N. Hence $\| x_{ij} \| n^{-1/p_{ij}} \le 1$ for all x ∈ $\ell_\infty^2(X, \lambda, p)_{n^{-1/p_{ij}}}$. On the other hand, if x ∈ $\bigcup_{n=1}^{\infty} \ell_\infty^2(X, \lambda, p)_{n^{-1/p_{ij}}}$, then there are some n ∈ N and M > 1 such that $\| x_{ij} \|_{n^{-1/p_{ij}}} \le M$ for every i,j ∈ N. The we have $\| x_{ij} \|^{p_{ij}} \le n\, M^{p_{ij}} \le n\, M^{\alpha}$ for all i,j ∈ N, where α = $\sup_{ij} p^{ij}$. Hence x ∈ ℓ_∞^2(X, λ, p).

Theorem 3.6 : Let p = (p_{ij}) be a bounded sequence of positive real numbers with $p_{ij} \le 1$ for all i,j ∈ N. Then ℓ^2 (X, λ, p)$^\beta$ = $\ell_\infty^2(X', \lambda, p)$.

Proof : If $(f_{ij}) = \ell^2(X, \lambda, p)^\beta$, then $\sum\sum_{i+j\geq N} f_{ij}(x_{ij})$ converges for every $x = (x_{ij}) \in \ell^2(X, \lambda, p)$ using the same proof as in Theorem 3.4, we have

$$\sum\sum_{i,j\geq N} |f_{ij}(x_{ij})| < \infty \quad \forall\, x = (x_{ij}) \in \ell^2(X, \lambda, p) \tag{3.20}$$

If $(f_{ij}) \notin \ell^2_\infty(X', \lambda, p)$, it follows by Lemma 3.5 that $\sup_{ij}\|f_{ij}\|\, m^{-1/p_{ij}} = \infty$ for all $m \in N$. For each $i \in N$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \ldots..$ and $k_1 < k_2 < \ldots..$ such that $m_i > 2^i$ and $\|f_{ij}\|\, m_i^{-1/p_{ij}} > 1$. Choose $x_{ij} \in X$ with $\|x_{ij}\| = 1$ such that

$$|f_{ij}(x_{ij})|\, m^{-1/p_{ij}} > 1 \tag{3.21}$$

Let $y = (y_{ij})$, $y_{ij} = m_i^{-1/p_{ij}} x_{ij}$ if some i, and 0 otherwise. Then $\sum\sum_{i+j\geq N} \| y_{ij} \|^{p_{ij}} = \sum_{i=1}^{\infty} \frac{1}{m_i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$, so that $(y_{ij}) \in \ell^2(X, \lambda, p)$ and

$$\begin{aligned}\sum\sum_{i,j\geq N} f_{ij}(y_{ij}) &= \sum\sum_{i,j\geq N} \left| f_{k_i}\left(m_i^{-1/p_{k_i}} x_{k_i} \right) \right| \\ &= \sum\sum_{i,j\geq N} m_i^{-1/p_{k_i}} \left| f_{k_i}(x_{k_i}) \right| \\ &= \infty \quad \text{(by 3.21)},\end{aligned} \tag{3.22}$$

and this is contradictory to (3.20), hence $(f_{ij}) \in \ell^2_\infty(X, \lambda, p)$.

Conversely assume that $(f_{ij}) \in \ell^2_\infty(X', \lambda, p)$. By Lemma 3.3 there exists $M \in N$, such that $\sup_{ij}\|f_{ij}\|\, m^{-1/p_{ij}} < \infty$, so there is a $K > 0$ such that

$$\|f_{ij}\| \leq K\, M^{1/p_{ij}} \quad \forall\; i,j \in N. \tag{3.23}$$

Let $x = (x_{ij}) \in \ell^2(X, \lambda, p)$. Then there is a $k_0 \in N$ such that $M^{1/p_{ij}} \|x_{ij}\| \leq 1$ for all $k \geq k_0$. By $p_{ij} \leq 1$ for all $i,j \in N$, we have that for all $i,j \geq k_0$.

$$M^{1/p_{ij}} \|x_{ij}\| \leq \left(M^{1/p_{ij}} \| x_{ij} \| \right)^{p_{ij}} = M \| x_{ij} \|^{p_{ij}} \tag{3.24}$$

Then

$$\begin{aligned}\sum\sum_{i+j\geq N} |f_{ij}(x_{ij})| &\leq \sum\sum_{2\leq i+j\leq k_0} \|f_{ij}\|\, \|x_{ij}\| + \sum\sum_{k_{0+1}} \|f_{ij}\|\, \|x_{ij}\| \\ &\leq \sum\sum_{2\leq i+j\leq k_0} \|f_{ij}\|\, \|x_{ij}\| + K \sum\sum_{k_{0+1}\leq i+j\leq\infty} M^{1/p_{ij}} \|x_{ij}\| \quad \text{(by (3.23))} \quad (3.25) \\ &\leq \sum\sum_{2\leq i+j\leq k_0} \|f_{ij}\|\, \|x_{ij}\| + KM \sum\sum_{k_{0+1}\leq i+j\leq\infty} \| x_{ij} \|^{p_{ij}} < \infty \quad \text{(by (3.24))} \\ &\leq \infty.\end{aligned}$$

This implies that $\sum\sum_{2\leq i+j\leq\infty} f_{ij}(x_{ij})$ converges, hence $(f_{ij}) \in \ell^2(X, \lambda, p)^\beta$.

Theorem 3.7 : Let p = (p_{ij}) be a bounded sequence of positive real numbers. Then $\ell_\infty^2(X, \lambda, p)^\beta = M_\infty^2(X', \lambda, p)$.

Proof : If (f_{ij}) = $M_\infty^2(X', \lambda, p)$, then $\sum\sum_{i+j\geq N} \|f_{ij}\| m^{1/p_{ij}} < \infty$ for all m ∈ N, we have that for each x = (x_{ij}) ∈ $\ell_\infty^2(X, \lambda, p)$, there is m_0 ∈ N such that $\|x_{ij}\| \leq m_0^{1/p_{ij}}$ for all i,j ∈ N, hence $\sum\sum_{i+j\geq N} f_{ij}(x_{ij}) \leq \sum\sum_{i+j\geq N} \|f_{ij}\| \|x_{ij}\| \leq \sum\sum_{i+j\geq N} \|f_{ij}\| m_0^{1/p_{ij}} < \infty$, which imples that $\sum\sum_{i+j\geq N} f_{ij}(x_{ij})$ converges, so that (f_{ij}) ∈ $\ell_\infty^2(X, \lambda, p)^\beta$.

Conversely, assume that (f_{ij}) ∈ $\ell_\infty^2(X, \lambda, p)^\beta$, then $\sum\sum_{i+j\geq N} f_{ij}(x_{ij})$ converges for all x = (x_{ij}) ∈ $\ell_\infty^2(X, \lambda, p)$ by using the same proof as in Theorem 3.4, we have

$$\sum\sum_{i+j\geq N} |f_{ij}(x_{ij})| < \infty \qquad \forall\ x = (x_{ij}) \in \ell_\infty^2(X, \lambda, p) \qquad (3.26)$$

If (f_{ij}) ∉ $M_\infty^2(X', \lambda, p)$ then $\sum\sum_{i+j\geq N} \|f_{ij}\| M^{1/p_{ij}} = \infty$ for some M ∈ N. Then we can choose a sequence (k_i) of positive integers with θ = $k_0 < k_1 < k_2 < \ldots$ such that

$$\sum\sum_{k_{i-1}\leq i+j\leq k_i} \|f_{ij}\| M^{1/p_{ij}} > i \qquad \forall\ i \in N \qquad (3.27)$$

And we choose x_{ij} in X with $\|x_{ij}\| = 1$ such that for all i ∈ N,

$$\sum\sum_{k_{i-1}\leq i+j\leq k_i} |f_{ij}(x_{ij})|\ M^{1/p_{ij}} > i \qquad (3.28)$$

Put y = (y_{ij}), (y_{ij}) = $M^{1/p_{ij}} x_{ij}$ clearly, y ∈ $\ell_\infty^2(X, \lambda, p)$ and

$$\sum\sum_{i,j\geq N} |f_{ij}(y_{ij})| > \sum\sum_{k_{i-1}\leq i+j\leq k_i} |f_{ij}(x_{ij})|\ M^{1/p_{ij}} > i\ \forall\ i \in N \qquad (3.29)$$

Hence $\sum\sum_{i,j\geq N} |f_{ij}(y_{ij})| = \infty$, which contradicts (3.26). Hence (f_{ij}) ∈ $M_\infty^2(X', \lambda, p)$. The proof is now complete.

Theorem 3.8 : Let p = (p_{ij}) be a bounded sequence of positive real numbers. Then $c_0^2(X, \lambda, p)^\beta = M_0^2(X', \lambda, p)$.

Proof : Suppose (f_{ij}) = $M_0^2(X', \lambda, p)$, then $\sum\sum_{i,j\geq N} \|f_{ij}\| M^{-1/p_{ij}} < \infty$ for some M ∈ N. Let x = (x_{ij}) ∈ $c_0^2(X, \lambda, p)$. Then there is a positive integer K_0 such that $\| x_{ij} \|^{p_{ij}} < \frac{1}{m}$ for all k ≥ K_0, hence $\|x_{ij}\| < M^{-1/p_{ij}}$ for all i+j ≥ K_0. Then we have

$$\sum\sum_{i+j>k_0} |f_{ij}(x_{ij})| \leq \sum\sum_{i+j>k_0} \|f_{ij}\| \|x_{ij}\| \leq \sum\sum_{i+j>k_0} \|f_{ij}\| M^{-1/p_{ij}} < \infty \qquad (3.30)$$

It follows that $\sum\sum_{i+j>N} |f_{ij}(x_{ij})|$ converges, so that (f_{ij}) ∈ $c_0^2(X, \lambda, p)^\beta$.

On the other hand, assume that $(f_{ij}) \in c_0^2(X, \lambda, p)^\beta$, then $\sum\sum_{i+j>N} f_{ij}(x_{ij})$ converges for all $x=(x_{ij}) \in c_0^2(X, \lambda, p)$. For each $x = (x_{ij}) \in c_0^2(X', \lambda, p)$, choose scalar sequence (t_{ij}) with $|t_{ij}| = i$ such that $f_{ij}(t_{ij} x_{ij}) = |f_{ij}(x_{ij})|$ for all $i,j \in N$. Since $(t_{ij}x_{ij}) \in c_0^2(X, \lambda, p)$, by our assumption, we have $\sum\sum_{i+j>N} f_{ij}(t_{ij}x_{ij})$ converges, so that

$$\sum\sum_{i+j>N} |f_{ij}(x_{ij})| < \infty \qquad \forall\, x \in c_0^2(X, \lambda, p) \tag{3.31}$$

Now, suppose that $(f_{ij}) \notin M_0^2(X, \lambda, p)$.Then $\sum\sum_{i+j\geq N} \|f_{ij}\|\, m^{-1/p_{ij}} = \infty$ for all $m \in N$. Choose $m_1 k_1 \in N$ such that

$$\sum\sum_{2\leq i+j\leq k_1} \|f_{ij}\|\, m_1^{-1/p_{ij}} > 1 \tag{3.32}$$

And choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum\sum_{k_1\leq i+j\leq k_2} \|f_{ij}\|\, m_2^{-1/p_{ij}} > 2 \tag{3.33}$$

Proceeding in this way, we can choose $m_1 < m_2 < \ldots\ldots$, and $0 < k_1 < k_2 < \ldots$ such that

$$\sum\sum_{k_{i-1}\leq i+j\leq k_i} \|f_{ij}\|\, m_i^{-1/p_{ij}} > i, \tag{3.34}$$

Take x_{ij} in X with $\|x_{ij}\| = 1$ for all k, $k_{i-1} < k \leq k$, such that

$$\sum\sum_{k_{i-1}\leq i+j\leq k_i} |f_{ij}(x_{ij})|\, m_i^{-1/p_{ij}} > i \qquad \forall\, i \in N \tag{3.35}$$

Put $y = (y_{ij}) = m_i^{-1/p_{ij}} x_{ij}$ for $k_{i-1} < i+j \leq k_i$, then $y \in c_0^2(X, \lambda, p)$ and

$$\sum\sum_{i+j>N} |f_{ij}(x_{ij})| \geq \sum\sum_{k_{i-1}\leq i+j\leq k_i} |f_{ij}(x_{ij})|\, m_i^{-1/p_{ij}} > i \qquad \forall\, i \in N \tag{3.36}$$

Hence, we have $\sum\sum_{i,j>N} |f_{ij}(y_{ij})| = \infty$, which contradicts (3.31), therefore $(f_{ij}) \in M_\infty^2(X', \lambda, p)$.

This completes the proof.

Theorem 3.9 : Let $p = (p_{ij})$ be a bounded sequence of positive real numbers. Then $c^2(X, \lambda, p)^\beta = M_0^2(X', \lambda, p) \cap cs[X']$.

Proof : Since $c_0(X, \lambda, p) = c_0^2(X, \lambda, p)+E$, where $E=\{e(x) : x \in X\}$ it follows by proposition 3.1(iii) and Theorem 3.8 that $c_0(X,\lambda,p)^\beta = M_0^2(X',\lambda, p) \cap E^\beta$. It is obvious by definition that $E^\beta=\{(f_{ij}) \subset X' : \sum\sum_{i+j>N} f_{ij}(x_{ij}))$ converges for all $x \in X\}=cs[X']$. Hence we have the theorem.

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