Some fixed point theorems for contraction of rational expression on cone metric space

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***Abstract*—In this paper, we discuss generalised result on altering distance functions and fixed point theorems of integral type contraction through rational expression on cone metric space.**

***Index Terms*—Cone metric space, integral type contraction, Altering distance functions.**

I. INTRODUCTION

The notion of cone metric space is initiated by Huang and Zhang [7] and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mappings cone metric spaces.

In 1984, M.S. Khan, M. Swalech and S. Sessa [5] expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function. Introduced the concept of the property P in metric spaces by G.S.Jeong and B.E. Rhoades in [4]. In 2002, Branciari in [8] introduced a general contractive condition of integral type.Discuss Farshid Khojasteh et.al,[6] Some Fixed Point Theorems of Integral Type Contraction in Cone Metric Spaces.

In this paper, we discuss generalised result on altering distance functions and fixed point theorems of integral type contraction through rational expression cone metric space.

Definition 1. *Let E be a Banach space. A subset P of E is called a cone if and only if:*

i. *P is closed, nonempty and $P \neq 0$*
ii. *$ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b*
iii. *$P \cap (-P) = \{0\}$.*

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

Definition 2. *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

i. *$d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y \; \forall x, y \in X$,*
ii. *$d(x, y) = d(y, x), \; \forall x, y \in X$,*
iii. *$d(x, y) \leq d(x, z) + d(z, y), \; \forall x, y, z \in X$,*

Then (X, d) is called a cone metric space (CMS).

Example 1. *Let $E = R^2$*

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a CMS.

Definition 3. *Let (X, d) be a CMS and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x in X whenever for every $c \in E$ with $0 \ll c$, there is a natural number $n_0 \in N$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.*

Definition 4. *Let (X, d) be a CMS and $\{x_n\}_{n\geq 0}$ be a sequence in X. $\{x_n\}_{n\geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $n_0 \in N$, such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$.*

Definition 5. *Let (X, d) be a cone metric space and P is a normal cone, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

Definition 6. *A function $\varphi : P \to P$ is called an altering distance function if the following properties are satisfied: (1) $\varphi(t) = 0$ if and only if $t = 0$, (2) φ is monotonically non decreasing, (3) φ is continuous*

By Φ is denoted by set of all altering distance function

Definition 7. *Let T be a self mapping of a cone metric space (X, d) and P is a normal cone with a non empty fixed point set $F(T)$.Then T is said to satisfy the property P if $F(T) = F(T^n)$ for every n in N*

Definition 8. *[6] Suppose that P is a normal cone in E. $a, b \in E$ and $a < b$. we define*

$$\begin{aligned} [a, b] &= \{x \in E : x = tb + (1 - t)a, \text{for some} t \in [0, 1]\} \\ [a, b) &= \{x \in E : x = tb + (1 - t)a, \text{for some} t \in [0, 1)\} \end{aligned} \tag{1}$$

Definition 9. *The set $\{a = x_0, x_1.x_2 \cdots, x_n = b\}$ is called a partition for $[a, b]$ if and only if the sets $\{x_{t-1}, x_t\}_{t=1}^{n}$ are pairwise disjoint and $[a, b] = \{\bigcup_{t=1}^{n}[x_{i-1}, x_t) \cup \{b\}\}$*

Definition 10. *[6] For each partition Q of $[a, b]$ and each increasing function $\psi : [a, b] \to P$, we define cone lower* *summation and cone upper summation as*

$$L_n^{con}(\psi, Q) = \sum_{t=0}^{n-1} \psi(x_t)\|x_t - x_{t+1}\| \\ U_n^{con}(\psi, Q) = \sum_{t=0}^{n-1} \psi(x_{t+1})\|x_t - x_{t+1}\| \tag{2}$$

Respectively.

Definition 11. *[6] Suppose that P is a normal cone in E. $\psi : [a,b] \to P$ is called an integrable function on $[a,b]$ with respect to cone P or to simplicity, Cone integrable function, if and only if for all partition Q of $[a,b]$, $\lim_{n\to\infty} L_n^{con}(\psi, Q) = S^{con} = \lim_{n\to\infty} U_n^{con}(\psi, Q)$, where S^{con} must be unique. We show the common value S^{con} by $\int_a^b \psi(x) d_p(x)$ to simplicity $\int_a^b \psi d_p$*

Definition 12. *The function $\psi : P \to E$ is called subadditive cone integrable function if and only if for all $a, b \in P$, $\int_0^{a+b} \psi d_p \leq \int_0^a \psi d_p + \int_0^b \psi d_p$*

Example 2. *[6] Let $E = X = R, d(x,y) = |x-y|, P = (0,\infty)$, and $\psi(t) = \frac{1}{(t+1)}$ for all $t > 0$. Then for all $a, b \in P$, $\int_0^{a+b} \frac{dt}{(t+1)} = \ln(a+b+1)$, $\int_0^a \frac{dt}{(t+1)} = \ln(a+1)$, $\int_0^b \frac{dt}{(t+1)} = \ln(b+1)$ Since $ab \geq 0$, then $a+b+1 \leq a+b+1+ab = (a+1)(b+1)$. Therefore*

$$\ln(a+b+1) \leq \ln(a+1) \leq \ln(b+1)$$

This shows that ψ is an example of subadditive cone integrable function.

II. Main Result

Theorem 13. *Let (X,d) be a complete cone metric space and P is a normal cone, let $\varphi \in \Phi$ and let $T : X \to X$, be a given mapping which satisfies the following condition:*

$$\varphi(d(Tx,Ty)) \leq a\varphi(d(x,y)) + b(\varphi L(x,y)) \tag{3}$$

where $L(x,y) = d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}$, for all $x, y \in X$, $a > 0, b > 0, a+b < 1$. Then T has a unique fixed point $z \in X$, and for each $x \in X$ $\lim_{n\to\infty} T^n x = z$

Proof: Let $x \in X$ ne an arbitrary point and let $\{x_n\}$ be a sequence defined as follows $x_{n+1} = Tx_n = T^{n+1}x$ for every $n \geq 1$. Now,

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &= \varphi(d(Tx_{n-1}, Tx_n)) \\ &\leq a\varphi(d(x_{n-1}, x_n)) + b\varphi(L(x_{n-1}, x_n)) \\ &\leq a\varphi(d(x_{n-1}, x_n)) \\ &+ b\varphi(d(x_n, Tx_n)\frac{1+d(x_{n-1}, Tx_{n-1})}{1+d(x_{n-1}, x_n)}) \\ &\leq a\varphi(d(x_{n-1}, x_n)) \\ &+ b\varphi(d(x_n, x_{n+1})\frac{1+d(x_{n-1}, x_n)}{1+d(x_{n-1}, x_n)}) \\ &\leq a\varphi(d(x_{n-1}, x_n)) + b\varphi(d(x_n, x_{n+1}))\end{aligned} \tag{4}$$

implies that,

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &\leq (\frac{a}{1-b})\varphi(d(x_{n-1}, x_n)) \\ &\leq (\frac{a}{1-b})^2\varphi(d(x_{n-1}, x_n)) \\ &\leq \cdots \\ &\leq (\frac{a}{1-b})^n\varphi(d(x_{n-1}, x_n))\end{aligned} \tag{5}$$

Since $\alpha = \frac{a}{1-b} \in (0,1)$, from above equation we have,

$$\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0$$

From the fact that $\varphi \in \Phi$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

Claim: Prove that $\{x_n\}$ is a Cauchy sequence.
Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$d(x_{m_i}, x_{n_i}) \geq \epsilon \quad \text{and} \quad d(x_{m_i}, x_{n_{i-1}}) \leq \epsilon \tag{6}$$

$$\lim_{i\to\infty} d(x_{m_i}, x_{n_i}) = \epsilon \tag{7}$$

$$\lim_{i\to\infty} d(x_{m_{i+1}}, x_{n_{i+1}}) = \epsilon \tag{8}$$

For $x = x_{m_i}$ and $y = y_{n_i}$

$$\begin{aligned}\varphi(d(x_{m_{i+1}}, x_{n_{i+1}})) &= \varphi(d(Tx_{m_i}, x_{n_i})) \\ &\leq \varphi(d(x_{m_i}, x_{n_i})) \\ &+ b\varphi(d(x_{n_i}, x_{n_{i+1}})\frac{1+d(x_{m_i}, x_{n_{i+1}})}{1+d(x_{m_i}, x_{n_i})})\end{aligned}$$

Taking limit on both side we have,

$$\begin{aligned}\varphi(\epsilon) &= \lim_{i\to\infty} \varphi(d(x_{m_{i+1}}, x_{n_{i+1}})) \\ &\leq a \lim_{i\to\infty} \varphi(d(x_{m_i}, x_{n_i})) \\ &\leq a\varphi(\epsilon)\end{aligned}$$

Since $a \in (0,1)$, which is the contradiction. Thus $\{x_n\}$ is a Cauchy sequence in complete cone metric space X. Thus there exists $z \in X$ such that $\lim_{n\to\infty} = z$ Taking $x = x_n$ and $y = z$ in equation (3) we have

$$\begin{aligned}\varphi(d(x_{n+1}, Tz)) &= \varphi(d(Tx_n, z)) \\ &\leq a(\varphi(d(x_{n+1}, z))) \\ &+ b\varphi(d(z, Tz)\frac{1+d(x_n, Tx_n)}{1+d(x_n, z)})\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|\varphi(d(x_{n+1}, Tz))\| \leq bK\|\varphi(d(z, Tz))\|$
i,e

$$\lim_{n\to\infty} \|\varphi(d(z, Tz))\| \leq bK\|\varphi(d(z, Tz))\|$$

Since $b \in (0,1)$, then $\varphi(d(z,Tz)) = 0$ which is implies that $d(z,Tz) \ll 0$ thus $z = Tz$ Prove that T has unique common fixed point.

Let z, w be two fixed points of T such that $z \neq w$. Taking $x = z$ and $y = w$ in equation (3) we have

$$\begin{aligned}\varphi(d(Tz,Tw)) &\leq a(\varphi(d(z,w))) + b\varphi(d(z,Tz)\frac{1+d(w,Tw)}{1+d(w,z)}) \\ &= a\varphi(d(z,w))\end{aligned}$$

Which implies that $\varphi(d(z,w)) = 0$ Therefore $d(z,w) = 0$. Thus $z = w$ ■

Corollary 14. *Let (X,d) be a complete cone metric space and P is a normal cone, let $\varphi \in \varphi$ and let $T : X \to X$, be a given mapping which satisfies the following condition:*

$$\varphi(d(Tx,Ty)) \leq a\varphi(d(x,y)) \tag{9}$$

for all $x, y \in X$, $0 < a < 1$. Then T has a unique fixed point $z \in X$, and for each $x \in X$ $\lim_{n\to\infty} T^n x = z$

Theorem 15. *Let (X,d) be a complete cone metric space and P is a normal cone. Let $T : X \to X$ be a given mapping which satisfies the following condition:*

$$d(Tx,Ty) \leq ad(x,y) + bL(x,y) \tag{10}$$

where $L(x,y) = d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}$, for all $x, y \in X$, $a > 0, b > 0, a + b < 1$. Then $F(T) \neq \emptyset$ and T has the property P

Proof: By theorem (13) $F(T) \neq \emptyset$ for every $n \in N$. Fix $n > 1$, assume that $w \in F(T^n)$.
Show that $w \in F(T)$. Suppose that $w \neq Tw$. By equation (10)

$$\begin{aligned}d(w,Tw) &= d(T^n w, T^{n+1} w) \\ &\leq ad(T^{n-1}w, T^n w) \\ &+ bd(T^n w, T^{n+1} w)\frac{1+d(T^{n-1}w,T^n w)}{1+d(T^{n-1}w,T^n w)} \\ &= ad(T^{n-1}w,T^n w) + bd(T^n w, T^{n+1} w)\end{aligned}$$

Therefore

$$\begin{aligned}d(w,Tw) &= d(T^n w, T^{n+1} w) \\ &\leq \frac{a}{1-b} d(T^{n-1}w, T^n w) \\ &\cdots \\ &\leq (\frac{a}{1-b})^n d(w,Tw) \\ \|d(w,Tw)\| &\leq K(\frac{a}{1-b})^n \|d(w,Tw)\|\end{aligned}$$

Which is a contradiction. Consequently, $w \in F(T)$ and T has the property P ■

Corollary 16. *Let (X,d) be a complete cone metric space and P is a normal cone. Let $T : X \to X$ be a given mapping which satisfies the following condition:*

$$d(Tx,Ty) \leq ad(x,y) \tag{11}$$

for all $x, y \in X$, $0 < a < 1$. Then $F(T) \neq \emptyset$ and T has the property P

Theorem 17. *Let (X,d) be a complete cone metric space and P is a normal cone. Let $T : X \to X$ be a given mapping and let $\varphi \in \Phi$ which satisfies the following condition:*

$$\varphi(d(Tx,Ty)) \leq a\varphi(d(x,y)) + b(\varphi L(x,y)) \tag{12}$$

for all $x, y \in X$, $a > 0, b > 0, a + b < 1$ and $L(x,y) = d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}$. Then $F(T) \neq \emptyset$ and T has the property P

Proof: By theorem (13) $F(T) \neq \emptyset$ for every $n \in N$. Fix $n > 1$, assume that $w \in F(T^n)$.
Show that $w \in F(T)$. Suppose that $w \neq Tw$. By equation (12)

$$\begin{aligned}\varphi(d(w,Tw)) &= \varphi(d(T^n w, T^{n+1} w)) \\ &\leq a\varphi(d(T^{n-1}w, T^n w)) \\ &+ b\varphi(d(T^n w, T^{n+1} w)\frac{1+d(T^{n-1}w,T^n w)}{1+d(T^{n-1}w,T^n w)}) \\ &= a\varphi(d(T^{n-1}w,T^n w)) + bd(T^n w, T^{n+1} w)\end{aligned}$$

Therefore

$$\begin{aligned}\varphi(d(w,Tw)) &= \varphi(d(T^n w, T^{n+1} w)) \\ &\leq \varphi(\frac{a}{1-b} d(T^{n-1}w, T^n w)) \\ &\cdots \\ &\leq (\frac{a}{1-b})^n \varphi(d(w,Tw)) \\ \|\varphi(d(w,Tw))\| &\leq K(\frac{a}{1-b})^n \|\varphi(d(w,Tw))\|\end{aligned}$$

Which is a contradiction.therefore $\varphi(d(w,Tw)) = 0$, Since $\varphi \in \Phi$, We conclude that $d(z,Tz) = 0$. Hence $w \in F(T)$ and T has the property P ■

Corollary 18. *Let (X,d) be a complete cone metric space and P is a normal cone. Let $T : X \to X$ be a given mapping and let $\varphi \in \Phi$ which satisfies the following condition:*

$$\varphi(d(Tx,Ty)) \leq a\varphi(d(x,y)) \tag{13}$$

for all $x, y \in X$, $0 < a < 1$. Then $F(T) \neq \emptyset$ and T has the property P

III. Certain integral type contraction through rational expression in cone metric space

Theorem 19. *Let (X,d) be a compete cone metric space and P is a normal cone. Let $T : X \to X$ be a map for each $x, y \in X$*

$$\int_0^{d(Tx,Ty)} \psi d_p \leq a \int_0^{d(x,y)} \psi d_p + b \int_0^{d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}} \psi d_p \tag{14}$$

where $a > 0, b > 0, a + b < 1$ and $\psi : P \to P$ is a non-vanishing map and a subadditive cone integrable on each $[a,b] \subset P$ such that for each $\epsilon \gg 0$, $\int_0^{\epsilon} \psi d_p \gg 0$. Then T has unique fixed point in $z \in X$.

Proof: Let $x \in X$ ne an arbitrary point and let $\{x_n\}$ be a sequence defined as follows $x_{n+1} = Tx_n = T^{n+1}x$ for every $n \geq 1$. Now,

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \psi d_p &= \int_0^{d(Tx_{n-1},Tx_n)} \psi d_p \\
&\leq a \int_0^{d(x_{n-1},x_n)} \psi d_p + b \int_0^{L(x_{n-1},x_n)} \psi d_p \\
&\leq a \int_0^{d(x_{n-1},x_n)} \psi d_p + b \int_0^{d(x_n,Tx_n)\frac{1+d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},x_n)}} \psi d_p \\
&\leq a \int_0^{d(x_{n-1},x_n} \psi d_p) + b \int_0^{d(x_n,x_{n+1})\frac{1+d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)}} \psi d_p \\
&\leq a \int_0^{d(x_{n-1},x_n)} \psi d_p + b \int_0^{d(x_n,x_{n+1})} \psi d_p
\end{aligned} \tag{15}$$

implies that,

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \psi d_p &\leq (\frac{a}{1-b}) \int_0^{d(x_{n-1},x_n)} \psi d_p \\
&\leq (\frac{a}{1-b})^2 \int_0^{d(x_{n-1},x_n)} \psi d_p \\
&\leq \cdots \\
&\leq (\frac{a}{1-b})^n \int_0^{d(x_{n-1},x_n)} \psi d_p
\end{aligned} \tag{16}$$

Since $\alpha = \frac{a}{1-b} \in (0,1)$, from above equation we have,

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \psi d_p = 0$$

Claim: Prove that $\{x_n\}$ is a Cauchy sequence.
Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$d(x_{m_i}, x_{n_i}) \geq \epsilon \quad \text{and} \quad d(x_{m_i}, x_{n_{i-1}}) \leq \epsilon \tag{17}$$

$$\lim_{i\to\infty} d(x_{m_i}, x_{n_i}) = \epsilon \tag{18}$$

$$\lim_{i\to\infty} d(x_{m_{i+1}}, x_{n_{i+1}}) = \epsilon \tag{19}$$

For $x = x_{m_i}$ and $y = y_{n_i}$ in equation (14)

$$\begin{aligned}
\int_0^{d(x_{m_{i+1}},x_{n_{i+1}})} \psi d_p &= \int_0^{d(Tx_{m_i},x_{n_i})} \psi d_p \\
&\leq a \int_0^{d(x_{m_i},x_{n_i})} \psi d_p \\
&+ b \int_0^{d(x_{n_i},x_{n_{i+1}})\frac{1+d(x_{m_i},x_{n_{i+1}})}{1+d(x_{m_i},x_{n_i})}} \psi d_p
\end{aligned}$$

Taking limit on both side we have,

$$\begin{aligned}
\int_0^{\epsilon} \psi d_p = \lim_{i\to\infty} \int_0^{d(x_{m_{i+1}},x_{n_{i+1}})} \psi d_p \\
\leq a \lim_{i\to\infty} \int_0^{d(x_{m_i},x_{n_i})} \psi d_p \\
\leq a \int_0^{\epsilon} \psi d_p
\end{aligned}$$

Since $a \in (0,1)$, which is the contradiction. Thus $\{x_n\}$ is a Cauchy sequence in complete cone metric space X. Thus there exists $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$ Taking $x = x_n$ and $y = z$ in by equation (14) we have

$$\begin{aligned}
\int_0^{d(x_{n+1},Tz)} \psi d_p &= \int_0^{d(Tx_n,z)} \psi d_p \\
&\leq a \int_0^{d(x_{n+1},z)} \psi d_p + b \int_0^{d(z,Tz)\frac{1+d(x_n,Tx_n)}{1+d(x_n,z)}} \psi d_p
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \| \int_0^{d(x_{n+1},Tz)} \psi d_p \| \leq bK \| \int_0^{d(z,Tz)} \psi d_p \|$

Since $b \in (0,1)$, then $\int_0^{d(z,Tz)} \psi d_p = 0$ which is implies that $d(z,Tz) \ll 0$ thus $z = Tz$ Prove that T has an unique fixed point.
Let z, w be two fixed points of T such that $z \neq w$. Taking $x = z$ and $y = w$ in equation (14) we have

$$\begin{aligned}
\int_0^{d(z,w)} \psi d_p &= \int_0^{d(Tz,Tw)} \psi d_p \\
&\leq a(\int_0^{d(z,w)} \psi d_p) + b \int_0^{d(z,Tz)\frac{1+d(w,Tw)}{1+d(w,z)}} \psi d_p \\
&= a \int_0^{d(z,w)} \psi d_p
\end{aligned}$$

Which is a contradiction. Thus T has a unique fixed point $z \in X$

■

Corollary 20. *Let (X,d) be a compete cone metric space and P is a normal cone. Let $T : X \to X$ be a map for each $x, y \in X$*

$$\int_0^{\varphi(d(Tx,Ty))} \psi(t)d_p \leq a \int_0^{\varphi(d(x,y))} \psi(t)d_p + b \int_0^{\varphi(d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)})} \psi(t)d_p \tag{20}$$

where $a > 0, b > 0, a + b < 1$ and $\psi : P \to P$ is a non-vanishing map and a subadditive cone integrable on each $[a, b] \subset P$ such that for each $\epsilon \gg 0, \int_0^{\epsilon} \psi d_p \gg 0$. Then T has unique fixed point in $z \in X$ $\lim_{n\to\infty} T^n x = z$.

Proof: Assuming hypothesis $\varphi : P \to P$, we define $\varphi(t) = \int_0^{t} \psi d_p,\ t \ll P$, By Definition (6) it is clear that $\varphi(0) = 0$ and φ is monotonically non decreasing and by hypothesis φ is absolutely continuous, hence φ is continuous. Since $\varphi \in \Phi$. By equation (14)

$$\varphi(d(Tx,Ty)) \leq a\varphi(d(x,y)) + b\varphi(d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)})$$

Hence from theorem (19) there exits a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n = z$ ■

Corollary 21. *Let (X,d) be a compete cone metric space and P is a normal cone. Let $T : X \to X$ be a map for each $x, y \in X$*

$$\int_0^{d(Tx,Ty)} \psi d_p \leq a \int_0^{d(x,y)} \psi d_p \tag{21}$$

where $0 < a < 1$ and $\psi : P \to P$ is a non-vanishing map and a subadditive cone integrable on each $[0, a] \subset P$ such that for each $\epsilon \gg 0, \int_0^{\epsilon} \psi d_p \gg 0$. Then T has unique fixed point in $z \in X$.

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