On $\pi\beta$-Generalized Closed Sets in Topological Spaces

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Abstract : This paper is devoted to the study of $\pi\hat{\beta}$-generalized closed sets and $\pi\hat{\beta}$-generalized open sets in topological spaces and its properties.

Keywords: β – open set, $\pi\hat{\beta}$ g – closed sets, $\pi\hat{\beta}$ g – open sets

1. Introduction

The concept of generalized closed sets and generalization of closed sets in topological spaces was introduced by Levine [7] in 1970. Regular open sets have been introduced and investigated by Stone [17]. Benchalli and Wali[3] introduced the concept of rw-closed sets in topological spaces. Andrijevic[1] introduced semi preopen sets in general topology.

In this paper we study the properties of generalized $\pi\hat{\beta}$ -closed sets (briefly $\pi\hat{\beta}$ g- closed sets). Moreover in this paper, we defined $\pi\hat{\beta}$ g – open sets and obtained some of its properties.

2. Preliminaries

Definition 2.1:
A subset A of a topological space (X, τ) is said to be

(a) a pre open set if A⊆ int(cl(A)) and a preclosed set if cl(int(A))⊆ A. [16]

(b) a semiopen set if A⊆ cl(int(A)) and a semi closed set if int(cl(A))⊆ A. [8]

(c) a α-open set if A⊆ int(cl(int(A))) and a α -closed set if cl(int(cl(A)))⊆ A. [14]

(d) a semi-preopen set if A ⊆ cl(int(cl(A))) and a semi-preclosed set if int(cl(int(A))) ⊆ A. [1]

(e) a regular open set if A = int(cl(A)) and a regular closed set if A= cl(int(A)). [17]

(f) a generalized closed set (briefly, g-closed) if cl(A) ⊆ U whenever A⊆ U and U is open in X. [7]

(g) a semi-generalized closed set (briefly, sg-closed) if scl(A)⊆ U whenever A⊆ U and U is semiopen in X. [4]

(h) a generalized semi closed set (briefly, gs-closed) if scl(A)⊆ U whenever A⊆ U and U is open in X. [2]

(i) a generalized α-closed set (briefly, gα-closed) if αcl(A)⊆ U whenever A⊆ U and U is α open in X. [10]

(j) a α- generalized closed set (briefly, αg-closed) if αcl(A)⊆ U whenever A⊆ U and U is open in X. [9]

(k) a generalized semi -preclosed set (briefly, gsp closed) if spcl(A)⊆ U whenever A⊆ U and U is open in X. [5]

(i) a regular generalized closed set (briefly, rg-closed) if cl(A)⊆ U whenever A⊆ U and U is regular open in X. [16]

(m) a generalized preclosed set (briefly, gp-closed) if pcl(A)⊆ U whenever A⊆ U and U is open in X. [11]

(n) a generalized preregular closed set (briefly, gpr-closed) if pcl(A)⊆ U whenever A⊆ U and U is regular open in X. [6]

(o) a weakly closed set (briefly, w-closed) if cl(A)⊆ U whenever A⊆ U and U is semiopen in X. [17]

(p) a weakly generalized closed set (briefly, wg-closed) if cl(int(A))⊆ U whenever A⊆ U and U is open in X. [13]

(q) a semi weakly generalized closed set (briefly, swg-closed) if cl(int(A))⊆ U whenever A⊆ U and U is semiopen in X.

(r) a regular weakly generalized closed set (briefly, rwg-closed) if cl(int(A))⊆ U whenever A⊆ U and U is regular open in X.

Remark 2.2: The complements of the closed sets are known as the corresponding open sets and vice versa.

Definition 2.3[18]:
A subset A of a space (X, τ) is called:
(i) regular open if A=int(cl(A)).
(ii) π open if A is the union of regular open sets.

3. On $\pi\hat{\beta}$ -generalized closed sets

In this section we introduced the concept of $\pi\hat{\beta}$ -generalized closed set in topological spaces

Definition 3.1 A subset A of a topological space (X ,τ)is called $\pi\hat{\beta}$ g-closed set ($\pi\hat{\beta}$ -generalized closed set) if cl(int(cl(A))) ⊆ U whenever A ⊆ U and U is π open in X.

Theorem 3.2 The union of two $\pi\hat{\beta}$ g-closed subsets of X is also $\pi\hat{\beta}$ g-closed subset of X.

Proof: Assume that A and B are $\pi\hat{\beta}$ g-closed set in (X ,τ). Let U is π open in X such that A∪B⊂ U. Then A⊂ U and B⊂ U. Since A and B are $\pi\hat{\beta}$ g-closed, clintcl(A) ⊂ U and clintcl(B) ⊂ U. Hence clintcl(A ∪ B) = clintcl(A) ∪ clintcl(B) ⊂ U . Therefore clintcl(A∪B) ⊂ U. Hence A∪B is $\pi\hat{\beta}$ g-closed set in X.

Remark 3.3 The intersection of two $\pi\hat{\beta}$ g-closed sets in (X ,τ) is generally not $\pi\hat{\beta}$ g-closed sets in X.

Example 3.4 Let X ={a , b, c}with the topology τ ={X ,φ,{a}}. If A ={a , b}and B ={a , c}. Then A and B are $\pi\hat{\beta}$ g-closed sets in X, but A ∩ B ={a}is not a $\pi\hat{\beta}$ g-closed set in X.

Theorem 3.5 If a subset A of X is $\pi\hat{\beta}$ g-closed set in X Then clintcl(A)-A does not contain any non empty open set in X.

Proof: Suppose that A is $\pi\hat{\beta}$ g-closed set in X. We prove the result by contradiction. Let U be open set such that clintcl(A)-A⊃ U and U≠ ϕ. Now U⊂ clintcl(A)-A. Therefore,U⊂ X-U. Since U π is open set, X-U is also π open in X. Since A is $\pi\hat{\beta}$ g-closed sets in X, by definition we have clintcl(A)⊂ X-U. So U⊂ X-clintcl(A). Also U⊂ clintcl(A). Therefore U ⊂ clintcl(A)(X-clintcl(A))= ϕ. This shows that U= ϕ which is contradiction. Hence clintcl(A)-A does not contains any non empty open set in X.

Remark 3.6 The converse of the above theorem need not be true as seen from the following example.

Example 3.7 If clintcl(A)-A contains no non-empty open set in X, then A need not be $\pi\hat{\beta}$ g-closed. Consider X ={a, b, c} with the topology τ ={X ,φ,{a}{,a , b}} and A ={a , b}. Then clintcl(A)-A=X-{a, b}={c} does not contain any non-empty open set, but A is not an $\pi\hat{\beta}$ g-closed set in X.

Theorem 3.8 If A is regular closed in (X ,τ) then A is $\pi\hat{\beta}$ g-closed subset of (X ,τ).

Proof: Suppose that A⊆ U and U is π open in X. Now U⊆ X is open if and only if U is the union of a semi open set and pre open set. Let A be a regular closed subset of (X , τ). So A= clintcl(A). Every regular closed set is semi open set and every semi open set is open set. Hence clintcl(A)⊆ U where U is π open in X. Therefore A is $\pi\hat{\beta}$ g-closed set in X.

Remark 3.9 The converse of the above theorem need not be true as seen from the following example.

Example 3.10 Consider X ={a, b, c} with the topology τ ={X ,φ,{a}{,b}{,a , b}{,b.c}}. Let A ={a, c}. Clearly A is $\pi\hat{\beta}$ g-closed set but not regular closed. Since, A≠rclA. This implies that A is not regular closed.

Theorem 3.11 For an element x∈X, the set X-{x} is $\pi\hat{\beta}$ g-closed or open.

Proof: Suppose X-{x} is not open. Then X is the only open set containing X-{x}. This implies clintcl(X-{x})⊂ X. Hence X-{x} is an $\pi\hat{\beta}$ g-closed set in X.

Theorem 3.12 If A is regular open and $\pi\hat{\beta}$ g-closed, then A is regular closed and hence clopen.

Proof: Suppose A is regular open and $\pi\hat{\beta}$ g-closed. As every regular open set is open and A⊂ A, we have clintcl(A)⊂ A. Since cl(A)⊂ clintcl(A). We have cl(A)⊆ A. Also A⊆ cl(A) . Therefore cl(A)=A that means A is closed. Since A is regular open, A is open. Now cl(int(A)) = cl(A) = A. Therefore, A is regular closed and clopen.

Theorem 3.13 If A is regular open and rg-closed, then A is $\pi\hat{\beta}$ g-closed set in X.

Proof: Let A be regular open and rg-closed in X. We prove that A is an $\pi\hat{\beta}$ g-closed set in X. Let U be π open set in X such that A⊂ U. Since A is regular open and rg-closed, we have cl(A)⊂ A. Then cl(A)⊂ A⊂ U. Hence A is $\pi\hat{\beta}$ g-closed set in X.

Theorem 3.14 If Ais an $\pi\hat{\beta}$ g-closed subset in X such that A⊂ B⊂ cl(A), then B is an $\pi\hat{\beta}$ g-closed set in X.

Proof: Let A be an $\pi\hat{\beta}$ g-closed set in X such that A ⊂ B⊂ cl(A). Let U be a π open set of X such that B ⊂ U. Then A⊂ U. Since A is $\pi\hat{\beta}$ g-closed. We have cl(A) ⊂ U. Now cl(B) ⊂ cl(cl(A)) = cl(A) ⊂ U. Therefore B is an $\pi\hat{\beta}$ g-closed set in X.

Remark 3.15 The converse of the above theorem need not be true as seen from the following example.

Example 3.16 Consider the topological space (X ,τ), where X ={a , b, c} be with the topologyτ ={X ,φ,{b},{b.c}}. Let A ={a}and B ={a, c}. Then A and B are π β˄ g-closed set in (X ,τ), but A⊂ B is not subset in cl(A).

Theorem 3.17 Let A be $\pi\hat{\beta}$ g-closed in (X ,τ). Then A is closed if and only if cl(A)-A is open.

Proof: Suppose A is closed in X. Then cl(A)=A and so cl(A)-A= ϕ , which is open in X. Conversely, suppose cl(A)-A is open in X. Since A is $\pi\hat{\beta}$ g-closed, by Theorem 3.5, cl(A)-A does not contain any non-empty open set in X. Then cl(A)-A= ϕ , hence A is closed in X.

Theorem 3.18 If A is both open and g-closed in X then it is $\pi\hat{\beta}$ g-closed set in X.

Proof: Let A be an open and g-closed in X. Let A⊂ U and let U be π open in X. Now A ⊂ A, By hypothesis cl(A)⊂ A. That is cl(A)⊂ U. Thus A is $\pi\hat{\beta}$ g-closed set in X.

Theorem 3.19 Every gα -closed set in a topological space X is $\pi\hat{\beta}$ g-closed set.

Proof: Let A be a gα -closed in (X ,τ) and A⊂ U where α is open. Now α is open implies that U is π open. Also clintcl(A) ⊆ cl(A) ⊆ α cl(A) ⊆ U. Hence A is $\pi\hat{\beta}$ g-closed set in X.

Remark 3.20 The converse of the above theorem need not be true as seen from the following example.

Example 3.21 Consider the topological space (X ,τ), where X ={a , b, c} Be with the topology τ ={X ,φ,{a},{b},{a, b}}. Then let A ={a}is $\pi\hat{\beta}$ g-closed Set in (X ,τ), but not gα -closed set in X .

4. On $\pi\hat{\beta}$ -generalized open sets and $\pi\hat{\beta}$ -generalized neighbourhoods

In this section, we introduce an study $\pi\hat{\beta}$ g – open sets in topological spaces and obtain some of their properties. Also, we introduce $\pi\hat{\beta}$ g– neighborhood (briefly $\pi\hat{\beta}$ g – nbhd) in topological spaces by using the notion of $\pi\hat{\beta}$ g – open sets.

Also,we prove that every nbhd of x in X is $\pi\hat{\beta}$ g – nbhd of x but not conversely.

Definition 4.1 A subset A in X is called $\pi\hat{\beta}$ generalized open (briefly $\pi\hat{\beta}$ g – open) in X if A^c is $\pi\hat{\beta}$ g – closed in X. We denote the family of all $\pi\hat{\beta}$ g – open sets in X by $\pi\hat{\beta}$ gO(X).

Theorem 4.2 If A and B are $\pi\hat{\beta}$ g – open sets in a topological space X. Then A∩B is also $\pi\hat{\beta}$ g – open set in X.

Proof: Let A and B are $\pi\hat{\beta}$ g – open sets in a space X. Then A^c and B^c are $\pi\hat{\beta}$ g – closed set in X. By Theorem 3.2, $A^c \cup B^c$ is also $\pi\hat{\beta}$ g – closed set in X. That is $A^c \cup B^c = (A\cap B)^c$ is a $\pi\hat{\beta}$ g – closed set in X. Therefore A∩B is also $\pi\hat{\beta}$ g – open set in X.

Definition 4.3 Let X be a topological space and let x∈X. A subset N of X is said to be a $\pi\hat{\beta}$ g – nbhd of x iff there exists a π β˄ g-open set G such that x∈G ⊂ N.

Definition 4.4 A subset N of space X, is called a $\pi\hat{\beta}$ g – nbhd of A⊂ X iff there exists a $\pi\hat{\beta}$ g-open set G such that A⊂ G⊂ N.

Remark 4.5 The $\pi\hat{\beta}$ g – nbhd N of x∈X need not be a $\pi\hat{\beta}$ g-open set in X.

Example 4.6 Consider the topological space (X ,τ), whereX={a,b,c}be with the topology τ ={X ,φ,{c}}. The $\pi\hat{\beta}$ gO(X) ={X ,φ,{a},{b},{c},{b, c}{,a , c}}. Note that {a , b}is not a $\pi\hat{\beta}$ g – open set in (X ,τ), but it is a $\pi\hat{\beta}$ g – nbhd of{a}. Since{a}is a $\pi\hat{\beta}$ g-open set such that a∈{a} ⊂ {a, b}.

Theorem 4.7 Every nbhd N of x∈X is a $\pi\hat{\beta}$ g –

nbhd of X.

Proof: Let N be a nbhd of point x∈X. To prove that N is a $\pi\hat{\beta}$ g – nbhd of x. By definition of nbhd, there exists an open set G such that x∈G⊂ N. As every open set is $\pi\hat{\beta}$ g – open set G such that x∈G ⊂ N. Hence N is $\pi\hat{\beta}$ g – nbhd of X.

Remark 4.8 In general, a $\pi\hat{\beta}$ g – nbhd N of x∈X need not be a nbhd of x in X, as seen from the following example.

Example 4.9 Consider the topological space (X ,τ), where X ={a, b, c} be with the topology τ ={X ,φ,{c}}. The $\pi\hat{\beta}$ gO(X) ={X ,φ,{a},{b},{c},{b, c}{,a , c}}. The set{a , b}is $\pi\hat{\beta}$ g – nbhd of the point b, since the $\pi\hat{\beta}$ g – open set{b}is such that b ∈{b} ⊂ {a , b}. However the set {a, b} is not a nbhd of the point b, since no open set G exists such that b ∈G ⊂ {a,b}.

Theorem 4.10 If a subset N of a space X is $\pi\hat{\beta}$ g – open, then N is a $\pi\hat{\beta}$ g – nbhd of each of its points.

Proof: Suppose N is $\pi\hat{\beta}$ g – open. Let x∈N. We claim that N is $\pi\hat{\beta}$ g – nbhd of x. For N is a $\pi\hat{\beta}$ g – open set such that x∈N⊂ N. Since x is an arbitrary point of N, it follows that N is a $\pi\hat{\beta}$ g – nbhd of each of its points.

Remark 4.11 The converse of the above theorem need not be true as seen from the following example.

Example 4.12 Consider the topological space (X ,τ), where X = {a , b, c} be with the topology τ ={X ,φ,{c}}.

Then $\pi\hat{\beta}$ gO(X) ={X ,φ,{a},{b},{c}{,b, c}{,a , c}}. The set {a , b} is $\pi\hat{\beta}$ g – nbhd of the point a, since the $\pi\hat{\beta}$ g – open set{a}is such that a ∈{a} ⊂ {a , b}. Also the set {a, b} is $\pi\hat{\beta}$ g – nbhd of the point b, since the $\pi\hat{\beta}$ g – open set{b}is such that b ∈ {b}⊂ {a, b}. That is, {a, b} is $\pi\hat{\beta}$ g – nbhd of each of its points. However the set {a , b}is not a $\pi\hat{\beta}$ g – open set in X.

Theorem 4.13 Let X be a topological space. If F is a $\pi\hat{\beta}$ g – closed subset of X and x∈F^c. Prove that there exists a $\pi\hat{\beta}$ g – nbhd N of x such that N∩F= φ.

Proof: Let F is a $\pi\hat{\beta}$ g – closed subset of X and x∈F^c. Then F^c is $\pi\hat{\beta}$ g – open set of X. Therefore, F^c contains a $\pi\hat{\beta}$ g – nbhd of each of its points. Hence there exists a $\pi\hat{\beta}$ g – nbhd N of x such that N ⊂ F^c. That is N∩F=φ.

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