Initial value Linear Evolutions equations involving parabolic operator on the plane

Gadagkar P.V. *

*Modern College of Arts, Science and Commerce, Pune 5, India.

A unique solution of an initial value problem involving a parabolic operator on the plane of linear evolution equations is obtained by constructing its Lax pair and performing spectral analysis.

Lax pair, evolution equation.

Introduction

A unique solution of the initial value problem on the plane of linear evolution equations is obtained by constructing and analyzing its Lax pair [1]. This problem can be analysed by performing spectral analysis of the t-independent Lax pair which reproduces the solution obtained by two dimensional Fourier transform. An equation will be called integrable if it admits a Lax pair formulation. The equation can be written as the compatibility condition of two linear eigenvalue equations. In particular, for evolution equations, these two equations are referred to as the x-part and as the t-part of the Lax pair. The spectral analysis of the x-part yeilds the direct and inverse Fourier transform in two dimensions, while the t-part is used to determine the time evolution of the Fourier data, which is called spectral data.

Results

The following results have been shown in [2]

Proposition 1. *Let $L(\partial_{x_1}, \partial_{x_2})$ be a linear differential operation in ∂_{x_1} and ∂_{x_2} with constant coefficients. A Lax pair of the equation $Lq(x_1, x_2) = 0$ is given by*

$$\mu_{x_1} - ik\mu = q, \qquad [1]$$

$$L\mu = 0. \qquad [2]$$

Proposition 2. *Let $q(z, \overline{z}, t)$ satisfy the linear evolution equation with constant coefficients*

$$L(\partial_z, \partial_{\overline{z}}, \partial_t)q = 0. \qquad [3]$$

Then the equations

$$\mu_{\overline{z}} - k\mu = q, k \in \mathbb{C} \qquad [4]$$

$$L\mu = 0 \qquad [5]$$

form a Lax pair for this equation.

Notations.

1. $S(\mathbb{R}^2)$ will denote the space of complex valued Schwartz functions on $\mathbb{R}^2$.
   i.e. $S(\mathbb{R}^2) = \left\{q(x) \in C^\infty : sup_{x\in\mathbb{R}^2} \left|x^\alpha D^\beta q(x)\right| < \infty, \forall \alpha, \beta\right\}$
2. $^-$ denotes complex conjugation.
3. $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

Aim of the paper

The aim of this paper is to derive a unique solution of the following initial value problem :

$$iq_t + \frac{1}{2}q_{xy} = 0, \qquad [6]$$

$$q(x, y, 0) = q_0(x, y) \qquad [7]$$

where $x, y \in \mathbb{R}$, $t > 0$, $q_0(x, y, 0) \in S(\mathbb{R}^2)$, α is a real constant.

Initial value problem

Theorem 3. *Let $q(x, y, t)$ satisfy (6) and (7). Then the unique solution of this initial value problem is given by*

$$q(z, \overline{z}, t) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} e^{k\overline{z} - \overline{k}z + i(\overline{k}^2 - k^2)t}\rho_0(k, \overline{k})dk \wedge d\overline{k} \qquad [8]$$

where $\rho(k, \overline{k}, 0) = \rho(k, \overline{k}, t)$ at $t = 0$ and

$$\rho(k, \overline{k}, t) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} e^{-k\overline{z} + \overline{k}z} q(z, \overline{z}, t)dz \wedge d\overline{z}. \qquad [9]$$

Proof. Let $q(x, y, t)$ satisfy (7). Equation (6) can be rewritten in the form

$$iq_t + +q_{zz} - q_{\overline{z}\overline{z}} = 0 \qquad [10]$$

Equations (4) and

$$\mu_t - i\mu_{zz} + ik^2\mu = -ikq - iq_{\overline{z}} \qquad [11]$$

form its Lax pair.

Indeed, if q satisfies (10), differentiating (4) w.r.t. $\overline{z}$ and eliminating $\mu_{\overline{z}\overline{z}}$ from (5), we get (11)

If we assume that $q(z, \overline{z}, t)$ exists, has sufficient smoothness and decay, and perform the spectral analysis of equation (4). Now this means that we construct a solution μ, which for every fixed z and $\overline{z}$ is bounded in k, $k \in \mathbb{C}$, and which is of $O(1/k)$ as $k \to \infty$ [3]. Equation (4) can be written as

$$\partial_{\overline{z}}(\mu e^{-k\overline{z} + \overline{k}z}) = qe^{-k\overline{z} + \overline{k}z}, \qquad [12]$$

where the term $exp(\overline{k}z)$ is used to ensure the boundedness of $exp(-k\overline{z} + \overline{k}z)$. This equation together with the assumption that $q \to 0$ as $z \to \infty$ [3], imply

$$\mu(z, \overline{z}, t, k) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} \frac{e^{k(\overline{z} - \overline{\xi}) - (z - \xi)}}{\xi - z} q(\xi, \overline{\xi}, t)d\xi \wedge d\overline{\xi} \qquad [13]$$

But

$$\frac{\partial\mu}{\partial\overline{k}} = e^{k\overline{z} - \overline{k}z}\rho(k, \overline{k}, t), \qquad [14]$$

where $\rho(k, \overline{k}, t)$ is defined by (9)

Equation (13) implies that $\mu = O(1/k)$, $k \to \infty$. This estimate and (13) imply

$$\mu(z, \overline{z}, t, k) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} \frac{e^{l\overline{z} - \overline{l}z}\rho(l, \overline{l}, t)}{l - k} dl \wedge d\overline{l}. \qquad [15]$$

Substituting, this equation into (4), we find

$$q(z,\bar{z},t)=\frac{1}{2\pi i}\int_{\mathbb{R}^2} e^{l\bar{z}-\bar{l}z}\rho(l,\bar{l},t)dl\wedge d\bar{l}. \qquad [16]$$

Thus, the spectral analysis of (4) yields Eqns(9) and (16) which are direct and the inverse Fourier transform in two dimensions.

In order to find the time evolution of the Fourier data, we use the t part of the Lax pair : Equation (13) implies that

$$\rho(k,\bar{k},t)=-\lim_{z\to\infty} ze^{-k\bar{z}+\bar{k}z}\mu(z,\bar{z},t,k). \qquad [17]$$

Differentiating wrt z we get,

$$0=\lim_{z\to\infty} ze^{-k\bar{z}+\bar{k}z}\mu_z+\bar{k}ze^{-k\bar{z}+\bar{k}z}\mu+e^{-k\bar{z}+\bar{k}z}\mu$$

The last term in the absolute value tends to zero as $z\to\infty$ since exponential is bounded under absolute value for all z and k and $\mu\to\infty$ by the definition of μ. Thus we get,

$$0=\lim_{z\to\infty} ze^{-k\bar{z}+\bar{k}z}\mu_z(z,\bar{z},t,k)+\bar{k}(-\rho)+0.$$

Now equation (11), equation (17) and the assumption $q\to 0$ as $z\to\infty$ yields

$$\rho_t-i(\bar{k}^2-k^2)\rho=0 \qquad [18]$$

Solving this equation in terms of $\rho(k,\bar{k},0)$ and using Eq.(9) at $t=0$, (8) follows where

$$\rho_0(k,\bar{k})=\frac{1}{2\pi i}\int_{\mathbb{R}^2} e^{-k\bar{z}+\bar{k}z}q_0(z,\bar{z})dz\wedge d\bar{z} \qquad [19]$$

Equations (8) and (19) hence lead the solution of the initial value problem (6) and (7) without a priori assumption of existence : Given $q_0\in S(\mathbb{R}^2)$ define $\rho_0(k,\bar{k})$ by (19). Given ρ_0 define $q(z,\bar{z},t)$ by (19). Since the dependence of q on $z,\bar{z},t$ is of the form $e^{k\bar{z}-\bar{k}z+i(\bar{k}^2-k^2)}$, it immediately follows that q satisfies the (10). All that remains is to show that $q(z,\bar{z},0)=q_0(z,\bar{z})$ i.e.

$$q_0(z,\bar{z})=\frac{1}{2\pi i}\int_{\mathbb{R}^2} e^{k\bar{z}-\bar{k}z}\rho_0(k,\bar{k})dk\wedge d\bar{k}. \qquad [20]$$

If one assumes the validity of the inversion formula for the two dimensional Fourier transform, then (20) is a consequence of the definition (19).

1. P.D. Lax , Commun. Pure Appl., (1968), 21, 467.
2. A.S. Fokas, On the integrability of linear and nonlinear partial differential equations, Journal of Mathematical Physics, (2000), 41, 4188-4237.
3. M.J. Ablowitz and A.S. Fokas, Introduction and Applications of Complex Variables (Cambridge University Press, Cambridge, UK, 1997).