Conformal Geometry of Arc length Functional on Space Curves and Space time

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Abstract: *The present manuscript is intend to the Quantization of the conformal Arclength functional, Conformal Change of Metric, Conformal geometry of space curve, Conformal structure of two-dimensional spacetimes.*

Key words: *Conformal geometry of curves, conformal strings, Griffiths' formalism, linking numbers.*

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1. Introduction

By a conformal string in Euclidean space is meant a closed critical curve with non constant conformal curvature of the conformal arclength functional. The conformal geometry of space curves was mainly developed in the first half of the past century and later taken up starting from the early 1980's. This subject has got much attention for its many fields of application, including the theory of integrable systems [3], [12], [8], topology and Möbius energy of knots [2], [6], [9], and the geometric approach to shape analysis and medical imaging [13].

Suppose that $\gamma \subset R^n, n \geq 3$, be a smooth curve parameterized by arclength s. The conformal arclength parameter ζ of γ is defined by

$$d\zeta = (\langle \dddot{\gamma}, \dddot{\gamma} \rangle - \langle \ddot{\gamma}, \ddot{\gamma} \rangle^2)^{\frac{1}{4}}\, ds =: \eta_\gamma,$$

where $\langle\, , \rangle$ is the standard scalar product on R^n and $\ddot{\gamma}, \dddot{\gamma}$ stands for double and triple derivative of γ. The 1-form η_γ, the infinitesimal conformal arclength of γ, is conformally invariant. If $\eta_\gamma\big|_s \neq 0$, for each s, the curve is called generic. The conformal arclength ζ gives a conformally invariant parameterization of a generic curve. We consider the conformally invariant variational problem on generic curves defined by the conformal arclength functional $\mathcal{L}[\gamma] = \int_\gamma \eta_\gamma$.

For $n = 3$ and higher dimensions this variational problem was studied in [7], [11]. A generic space curve is determined, up to conformal transformations, by the conformal arclength and two conformal curvatures. As for a closed critical curve with constant conformal curvatures, we can see that it is conformally equivalent to a closed rhumb line (loxodrome) of a torus of revolution.

1.1. The conformal group

Let $\mathbb{R}^{4,1}$ denote $\mathbb{R}^5$ with the Lorentz scalar product

$$(v, w) = -(v^0 w^4 + v^4 w^0) + \sum_{j=1}^{3} v^j w^j$$
$$= \sum_{a,b=0}^{4} g_{ab} v^a w^b, \quad g_{ab} = g_{ba}$$

where $v = (v^0, \dots\dots, v^4)$, and with the space and time orientations defined, respectively, by the volume form $dv^0 \wedge \dots \wedge dv^4$ and the positive light cone

$$L_+ = \{v \in \mathbb{R}^{4,1} : (v, v) = 0, v^0 + v^4 > 0\}$$

The conformal space M_3 is the projectivization of L_+, endowed with the oriented conformal structure induced by the scalar product and the space and time orientations.

If $(e_0, \dots, e_4)$ is the standard basis of $\mathbb{R}^{4,1}$, the map

$$J : x = (x^1, x^2, x^3) \in \mathbb{R}^3 \mapsto \left[\frac{t_{x.x}}{2} e_0 + \sum_{j=1}^{3} x^i e_j + e_4\right] \in M_3$$

is an orientation-preserving conformal diffeomorphism of $\mathbb{R}^3$ onto the conformal space minus the point $P_\infty = [e_0]$. The inverse of J is the conformal projection

$$p : \left[\sum_{a=0}^{4} v^a e_a\right] \in M_3 \setminus \{P_\infty\} \mapsto \frac{1}{v^4}(v^1, v^2, v^3) \in \mathbb{R}^3$$

The conformal group G consists of all pseudo-orthogonal transformations preserving the volume form. It is a 10-dimensional Lie group with two connected components. The first component is the subgroup G_+ consisting of all $V \in G$ preserving the positive light cone and the second one consists of all $V \in G$ switching the positive light cone with the negative one. The group G acts effectively and transitively on the left of M_3 preserving the conformal structure. The classical Liouville theorem [5] asserts that every conformal automorphism of M_3 is induced by a unique element of G. Consequently, the conformal group can be viewed as the pseudo-group of all conformal transformations of Euclidean 3-space. The orientation-preserving conformal transformations are induced by the elements of G_+, while the conformal transformations induced the elements of G are orientation-reversing. $\forall V \in G_+$, we denote by $V_0, \dots, V_4$ its column vectors. Then, $(V_0, \dots, V_4)$ is a positive light cone basis of $\mathbb{R}^{4,1}$, that is a positive-oriented basis such that

$$(V_a, V_b) = g_{ab}, \quad V_0, V_4 \in L_+, \quad a, b = 0, \dots, 4$$

Conversely, if $(V_0, \dots, V_4)$ is a positive light-cone basis, then the matrix F with column vectors $V_0, \dots, V_4$ is an element of G_+. The Lie algebra of G consists of all skew-adjoint matrices of the scalar product, that is $g = \{X \in gl(5, \mathbb{R}): {}^t X.g + g.X = 0\}$.

The maximal compact abelian subgroups of G are conjugate to the 2-dimensional torus

$$K = \{R(\theta_1, \theta_2): \theta_1, \theta_2 \in [0, 2\pi)\} \cong SO(2) \times SO(2)$$

Where

$$R(\phi_1, \phi_2) = \begin{pmatrix} \frac{1+\cos\theta_2}{2} & 0 & 0 & -\frac{\sin\theta_2}{\sqrt{2}} & \frac{1-\cos\theta_2}{2} \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ \frac{\sin\theta_2}{\sqrt{2}} & 0 & 0 & \cos\theta_2 & -\frac{\sin\theta_2}{\sqrt{2}} \\ \frac{1-\cos\theta_2}{2} & 0 & \frac{\sin\theta_2}{\sqrt{2}} & \frac{\sin\theta_2}{\sqrt{2}} & \frac{1+\cos\theta_2}{2} \end{pmatrix}$$

Note that $R(\theta_1, \theta_2)$ is the composition of the Euclidean rotation of angle θ_1 around the z-axis with the toroidal rotation of angle θ_2 around the Clifford circle $C = \{(x, y, 0): x^2 + y^2 = 2\}$. The z-axis and the Clifford circle are the rotational axes of K. The rotational axes of any other maximal torus $V \cdot K \cdot V^{-1}$ are the images under V of the axes of K.

1.2. **Conformal geometry of space curve:** Let $\gamma: I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a smooth curve parameterized by arclength s, I an open interval. Points where the infinitesimal conformal arclength η_γ vanishes are called vertices of γ. Generic curves can be parameterized by the conformal arclength parameter ζ, defined by $d\zeta = \eta_\gamma$. If such a conformal parametrization is defined for every $\zeta \in \mathbb{R}$, the curve is said complete. A frame field along γ is a smooth map $V: I \longrightarrow G_+$, such that $\mathcal{P} \circ V_4 = \gamma$.

Proposition 1.1.: For any oriented generic curve $\gamma: I \longrightarrow \mathbb{R}^3$, there is a unique frame field $V: I \longrightarrow G_+$ along , the Vessiot frame, such that

$$V^{-1} dV = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ f_2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & f_1 & 0 \\ 0 & 0 & -f_1 & 0 & 0 \\ 0 & f_2 & 1 & 0 & 0 \end{bmatrix} \eta_\gamma,$$

where f_1, f_2 are smooth functions, called the conformal curvatures. We call $\Gamma = V_4: I \longrightarrow \mathcal{L}_+$ the canonical null lift of γ.

1.3. **Conformal structure of spacetimes:** Liouville's Theorem states that there are some kind of rigidity on conformal structures of semi-Euclidean space $\mathbb{R}^n_v$ when $n \geq 3$. In two-dimensional Euclidean space, it is known that any conformal diffeomorphisms defined on an open subset of $\mathbb{R}^2$ are homography or anti-homography and these can be seen as a conformal map defined on Riemann sphere. Though some authors introduce conformal compactification of two-dimensional Minkowski spacetime, if we confine the subject to spacetimes with Cauchy surfaces, we can explicitly obtain their groups of conformal diffeomorphisms without compactifications. It is known that the group of conformal diffeomorphisms can be obtained by the group of causal automorphisms if the dimension of the Lorentzian manifold is bigger than two and so, in high dimensional Lorentzian manifolds to study conformal structures is equivalent to study causal structures.

2. Quantization of the conformal Arclength functional

Here we will obtain the proof of our three main results.

Theorem 2.1.: The conformal classes of conformal strings are in 1-1 correspondence with the rational points of the complex domain

$$\Omega = \left\{ q \in \mathbb{C}: \frac{1}{2} < Re\, q < \frac{1}{\sqrt{2}}, Im\, q > 0, |q| < \frac{1}{\sqrt{2}} \right\}$$

The rational points of Ω are called the moduli of conformal strings.

Proof: Since the periodic map $\theta = (\theta_1, \theta_2): \Sigma \longrightarrow \mathbb{R}^2$ is a real-analytic diffeomorphism onto the domain

$$\bar{\Omega} = \left\{ (x, y) \in \mathbb{R}^2: -\frac{1}{\sqrt{2}} < x < -\frac{1}{2}, x^2 + y^2 < \frac{1}{2}, y > 0 \right\}$$

Where

$$\theta_1(a, b) := \frac{1}{2\pi} \int_0^{\omega(a,b)} \frac{\mu(a, b)}{\kappa_{(a,b)}(t)^2 - \mu(a, b)^2} dt$$

$$\theta_2(a, b) := \frac{1}{2\pi} \int_0^{\omega(a,b)} \frac{\nu(a, b)}{\kappa_{(a,b)}(t)^2 - \nu(a, b)^2} dt$$

Where $\mu(a, b) = \frac{1}{\sqrt{2}} \sqrt{a + b + \sqrt{4 + (a - b)^2}}$,

$$\nu(a, b) = \frac{1}{\sqrt{2}} \sqrt{a + b - \sqrt{4 + (a - b)^2}}$$

Now by partially differentiating θ_1, θ_2

$$\partial_a \theta_1|_{(a,b)} = \frac{X_{11}(a, b) E\left(\frac{a - b}{a}\right) + Y_{11}(a, b) K\left(\frac{a - b}{b}\right)}{Z_{11}(a, b)}$$

$$\partial_b \theta_1|_{(a,b)} = \frac{X_{21}(a, b) E\left(\frac{a - b}{a}\right) + Y_{21}(a, b) K\left(\frac{a - b}{b}\right)}{Z_{21}(a, b)}$$

$$\partial_a \theta_2|_{(a,b)} = \frac{X_{12}(a,b)E\left(\frac{a-b}{a}\right) + Y_{12}(a,b)K\left(\frac{a-b}{b}\right)}{Z_{12}(a,b)}$$

$$\partial_b \theta_2|_{(a,b)} = \frac{X_{22}(a,b)E\left(\frac{a-b}{a}\right) + Y_{22}(a,b)K\left(\frac{a-b}{b}\right)}{Z_{22}(a,b)}$$

Where the coefficients $X_{ij}(a,b), Y_{ij}(a,b)$ and $Z_{ij}(a,b)$ rae

$$X_{11}(a,b) = \sqrt{2}\big(2\zeta(a,b) - a^2 b + a(4+\zeta(a,b)b) + b(4+\zeta(a,b)b)\big)$$

$$Y_{11}(a,b) = -2\sqrt{2}\Big(a+\zeta(a,b) - ab^2 + b\big(3 + b(\zeta(a,b)+b)\big)\Big)$$

$$Z_{11}(a,b) = \pi\sqrt{a}\zeta(a,b)(a-b)\big(a-b-\zeta(a,b)\big)\big(a+b+\zeta(a,b)\big)^{3/2}$$

$$X_{21}(a,b) = a\big(2b+\zeta(a,b)\big),$$
$$Y_{21}(a,b) = -b\big(a+b+\zeta(a,b)\big)$$

And by

$$Z_{21}(a,b) = \sqrt{2}\pi b(a-b)\zeta(a,b)\sqrt{a\big(a+b+\zeta(a,b)\big)}$$

$$X_{12}(a,b) = \sqrt{2}\Big(2\zeta(a,b) + a^2 b + a\big(-4+b\zeta(a,b)\big) - b\big(4 - b\zeta(a,b) + b^2\big)\Big)$$

$$Y_{12}(a,b) = 2\sqrt{2}\big(a - \zeta(a,b) - ab^2 + b(3 - b\zeta(a,b) + b^2)\big)$$

$$Z_{12}(a,b) = \pi\sqrt{a}\zeta(a,b)(a-b)\big(a-b+\zeta(a,b)\big)\big(a+b-\zeta(a,b)\big)^{3/2}$$

$$X_{22}(a,b) = a(\zeta(a,b) - 2b),$$
$$Y_{22}(a,b) = b\big(a+b-\zeta(a,b)\big),$$
$$Z_{22}(a,b) = \sqrt{2}\pi(a-b)b\zeta(a,b)\sqrt{a\big(a+b-\zeta(a,b)\big)}$$

Where $\zeta(a,b)$ stands for $\sqrt{4+(a-b)^2}$. These formulae have been derived with the help of the software *Mathematica*.

Since the partial derivatives of θ_1 and θ_2 are strictly positive on $\Sigma' = \{(a,b) : a > 1, ab > 1, b \le a\}$ [14]. The Jacobian of θ is strictly positive on Σ. In particular, the image $\theta(\Sigma)$ is a connected open set and $\theta : \Sigma \longrightarrow \theta(\Sigma)$ is a local diffeomorphism. The mapping θ is a real-analytic local diffeomorphism onto $\widetilde{\Omega}$ [14].

θ is one-one: First order partial derivatives of θ_1 and θ_2 are strictly positive on Σ'. Then there is an open neighborhood W of Σ' such that the first order partial derivatives of θ_1 and θ_2 are positive on $W' = W \cap Int(\overline{\Sigma})$. on this set we consider the nowhere vanishing vector fields

$$C_1 = \left(1, -\frac{\partial_a\theta_1}{\partial_b\theta_1}\right), C_2 = \left(1, -\frac{\partial_a\theta_2}{\partial_b\theta_2}\right).$$

The trajectories of the integral curves of C_1 and C_2 are graphs of strictly decreasing functions and hence they intersect $\partial_+\Sigma$ in at most one point.

In addition $\forall\, Q = (x,y) \in \widetilde{\Omega}$, we have, the connected components of the lavel curve $\mathcal{V}_1(x) = \Phi_1^{-1}(x) \cap \Sigma$ are contained in the intersection of a trajectory of C_1 with Σ. The connected components of the lavel curve $\mathcal{V}_2(y) = \Phi_2^{-1}(y) \cap \Sigma$ are contained in the intersection of a trajectory of C_2 with Σ. The level curve $\mathcal{V}_1(x')$ is connected, $\forall\, Q' = (x',y') \in \widetilde{\Omega}$. Also the level curves $\mathcal{V}_2(y')$ are connected, $\forall\, Q' = (x',y') \in \widetilde{\Omega}$.

It sufficient to show that $\big(\mathcal{V}_1(x) \cap \mathcal{V}_2(y)\big) = 1$ $\forall\,(x,y) \in \widetilde{\Omega}$. By contradiction suppose the existence of $(x,y) \in \widetilde{\Omega}$, such that $\big(\mathcal{V}_1(x) \cap \mathcal{V}_2(y)\big) > 1$. Let (a,b) and (a_1,b_1) be two distinct elements of Σ, such that $\Phi(a_1,b_1) = \Phi(a,b) = (x,y)$. From the above discussions, we know that the level curves $\mathcal{V}_1(x)$ and $\mathcal{V}_2(y)$ are connected and graphs of two strictly decreasing functions, denoted by u and v respectively. The domain of definition is an open interval $I \subset (1,+\infty)$, containing a and a_1. By construction, $u(a) = v(a) = b$, $u(a_1) == v(a_1) = b_1$, with $a \ne a_1$.

On the other hand, $\mathcal{V}_1(x)$ and $\mathcal{V}_2(y)$ are contained in the trajectories of the vector fields C_1 and C_2, respectively. From this, we have

$$u'(t) = -\frac{\partial_a\theta_1}{\partial_b\theta_1}\bigg|_{(t,u(t))} \quad v'(t) = -\frac{\partial_a\theta_2}{\partial_b\theta_2}\bigg|_{(t,v(t))} \quad \forall\, t \in I$$

With Jacobian

$$\frac{\partial_a\theta_1}{\partial_b\theta_1}\bigg|_{(\alpha,\beta)} - \frac{\partial_a\theta_2}{\partial_b\theta_2}\bigg|_{(\alpha,\beta)} > 0\ \forall(\alpha,\beta) \in \Sigma.$$

Then, the function $h = v - u$ satisfies $h(a) = h(a_1) = 0, h'(a) > 0$ and $h'(a_1) > 0$. This implies the existence of $a_2 \in I$, different from a and a_1, such that $h(a_2) = 0$ and $h'(a_2) \le 0$. Consequently, the Jacobian of θ is non positive at $(a_2,b_2) = \big(a_2,u(a_2)\big) = \big(a_2,v(a_2)\big) \in \Sigma$. Hence the required result.

Theorem 2.2: The conformal strings corresponding to a modulus $q \in \Omega$ are conformal equivalent to a model string $\gamma_q = \big(x(t),y(t),z(t)\big): \mathbb{R} \longrightarrow \mathbb{R}^3$.

Proof: Since conformal classes of conformal strings are in 1-1 correspondence with the elements of the countable set

$$\Omega_* = \Big\{(q_1, q_2) \in \mathbb{Q}^2 : \tfrac{1}{2} < q_1 < \tfrac{1}{\sqrt{2}}, q_2 > 0, q_1^2 +$$
q22<12.

Then, for every $q = (q_1, q_2) \in \Omega_*$, there is a unique $(a, b) \in \Sigma$ such that $\theta_1(a, b) = -q_1$ and $\theta_2(a, b) = q_2$. For every $(q_1, q_2) \in \Omega_*$, let

$$\Theta_1(t) = \int_0^t \frac{\mu}{\mu^2 - k(u)^2} du,$$

$$\Theta_2(t) = \int_0^t \frac{v}{v^2 - k(u)^2} du \text{ and}$$

$$r(t) = \sqrt{\mu^2 - v^2 k(t)} + v\sqrt{\mu^2 - k(t)^2} \cos \Theta_1(\mathrm{t}),$$

Where a, b are the parameters of q and k, μ, v stand for $k_{a,b,}$ $\mu(a, b)$ and $v(a, b)$, respectively. Note that (q_1, q_2) and (a, b) are related by

$$q_1 = \frac{1}{2\pi} \int_0^w \Theta_1^{'}(t)dt\,,\ q_2 = -\frac{1}{2\pi} \int_0^w \Theta_2^{'}(t)dt.$$

where $w = w(a, b)$ is the minimal period of k.

The symmetrical configuration of the conformal strings with modulus $q = (q_1, q_2)$ is the parametrized curve $\gamma_q = (x, y, z) : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by

$$x(t) = \frac{\sqrt{2}}{r(t)} \mu \sqrt{k(t)^2 - v^2} cos\Theta_2(t),$$

$$y(t) = \frac{\sqrt{2}}{r(t)} \mu \sqrt{k(t)^2 - v^2} sin\Theta_2(t),$$

$$z(t) = \frac{\sqrt{2}}{r(t)} v \sqrt{\mu^2 - k(t)^2} sin\Theta_1(t)$$

We have $\gamma_q = \big(x(t), y(t), z(t)\big) : \mathbb{R} \longrightarrow \mathbb{R}^3$. and modulus $q \in \Omega$ Where called the symmetrical configuration of q. Here

$$k(t) = \begin{cases} \sqrt{a} cn\left(\sqrt{a - b} t, \dfrac{a}{a - b}\right), & b < 0 \\ \sqrt{a} dn\left(\sqrt{a} t, \dfrac{a - b}{a}\right), & b > 0 \end{cases}$$

$$r(t) = \sqrt{\mu^2 - v^2} k(t) + v\sqrt{\mu^2 - k(t)^2} cos\Theta_1(t)$$

Where

$$\mu = \frac{1}{\sqrt{2}} \sqrt{a + b + \sqrt{4 + (a + b)^2}},$$

$$v = \frac{1}{\sqrt{2}} \sqrt{a + b + \sqrt{4 + (a + b)^2}}$$

$$\Theta_1(t) = \int_0^t \frac{\mu}{\mu^2 - k(u)^2} du,$$

$$\Theta_2(t) = \int_0^t \frac{v}{v^2 - k(u)^2} du,$$

And where a and b are real parameters, uniquely defined by q, such that $a > 0, a > b, b \neq 0, and\ ab > 1$.

Consider the unique conformal parametrization $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$of a conformal string with parameters a, b whose Vessiot frame F satisfies the initial condition $F(0) = Id$. It suffices to prove that γ is conformally equivalent to γ_q. Since the canonical lift $\Gamma : \mathbb{R} \longrightarrow \mathcal{L}_+$ of γ takes the form $\tilde{Y} \cdot W$, where $\tilde{Y} \in GL(5, \mathbb{C})$ and $W = (w_0, \ldots, w_4)$ is the $\mathbb{C}^5$-valued map defined by

$$w_0 = k,$$

$$w_1 = \sqrt{\mu^2 - k^2 e^{i\Theta_1(t)}},$$

$$w_2 = \sqrt{\mu^2 - k^2 e^{-i\Theta_1(t)}}$$

$$w_3 = \sqrt{v^2 - k^2 e^{i\Theta_2(t)}}$$

$$w_4 = \sqrt{v^2 - k e^{-i\Theta_2(t)}}$$

On the other hand, the curve $\widetilde{\Gamma} = (\tilde{\gamma}_0, \ldots, \tilde{\gamma}_4) : \mathbb{R} \longrightarrow \mathcal{L}_+$, defined by

$$\tilde{\gamma}_0(t) = \frac{1}{\sqrt{2}} \Big(\sqrt{\mu^2 - v^2} k(t) - v\sqrt{\mu^2 - k(t)^2} cos\Theta_1(t) \Big)$$

$$\tilde{\gamma}_1(t) = \mu \sqrt{k(t)^2 - v^2} cos\Theta_2(t)$$

$$\tilde{\gamma}_2(t) = \mu \sqrt{k(t)^2 - v^2} sin\Theta_2(t)$$

$$\tilde{\gamma}_3(t) = v \sqrt{\mu^2 - k(t)^2} sin\Theta_1(t)$$

$$\tilde{\gamma}_4(t) = \frac{1}{\sqrt{2}} \Big(\sqrt{\mu^2 - v^2} k(t) + v\sqrt{\mu^2 - k(t)^2} cos\Theta_1(t) \Big)$$

is a null lift of γ_q. From above conditions, it follows that $W = Z \cdot \widetilde{\Gamma}$, for some $Z \in GL(5, \mathbb{C})$. Consequently, $\Gamma = L \cdot \widetilde{\Gamma}$, for a suitable $L \in GL(5, \mathbb{C})$. This yields $V = L \cdot V_q$, where V_q is the Vessiot frame along γ_q. Thus $L \in G_+$, which implies that γ and γ_q are equivalent to each other. Hence the required result.

Theorem 2.3: Let $\gamma_q : \mathbb{R} \longrightarrow \mathbb{R}^3$ be the symmetrical configuration corresponding to the modulus $q = q_1 + iq_2$, where $q_1 = \frac{m_1}{n_1}, q_2 = \frac{m_2}{n_2}$ the pairs $(m_1, n_1), (m_2, n_2)$ are coprime integers. Let n be the least common multiple of n_1 and n_2, and consider the coprime integers $h_1 = \frac{n}{n_1}$ and $h_2 = \frac{n}{n_2}$.

Then,

(1) n is the order of the symmetry group of γ_q;

(2) $m_1 h_1$ and $m_2 h_2$ are the linking numbers of γ_q with the Clifford circle and the z-axis, respectively.

Proof: The curve γ is a real-analytic closed curve with positive chirality. Therefore, its symmetry group is generated by the monodromy. By construction, the canonical null lift $\Gamma : \mathbb{R} \longrightarrow \mathcal{L}_+$ and the first conformal curvature is a strictly positive periodic function, with minimal period ω. Using Theorem 2.2 We find $\Gamma_q(t + \omega) = R(2\pi q_2, 2\pi q_1) \cdot \Gamma_q(t)$, where $R(2\pi q_2, 2\pi q_1) \in K$. Then, $R(2\pi q_2, 2\pi q_1)$ is the monodromy of γ. This implies that the symmetry group of γ is generated by $R(2\pi q_2, 2\pi q_1)$. Let $[z]$ denote the z-axis with the downward orientation induced by the parametrization $\alpha(s) = (0, 0, -s)$, and let $[C]$ be the Clifford circle equipped with the orientation induced by the rational parametrization

$$\beta : t \in \mathbb{R} \mapsto \left(\frac{-\sqrt{2}(t^2 - 2)}{(t^2 + 2)}, \frac{4t}{(2 + t^2)}, 0 \right) \in \mathbb{R}^3$$

Now we compute the Gauss linking integrals $lk(\gamma,[z])$ and $lk(\gamma,[C])$. If $\gamma(t)=\big(x(t),y(t),z(t)\big)$, the Gauss linking integral $lk(\gamma,[z])$ is given by

$$lk(\gamma,[z])=\frac{1}{4\pi}\int_{[\gamma]}\int_{z}\frac{\gamma-\alpha}{\|\gamma-\alpha\|^3}\cdot d\gamma\times d\alpha=-\frac{1}{4\pi}\int_0^{n\omega}\left(\int_{-\infty}^{+\infty}\left(\frac{x(t)y'(t)-x'(t)y(t)}{(x(t)^2+y(t)^2+(z(t)-s)^2)^{3/2}}\right)ds\right)dt$$
$$=-\frac{1}{2\pi}\int_0^{n\omega}\frac{x(t)y'(t)-x'(t)y(t)}{x(t)^2+y(t)^2}dt=-\frac{n}{2\pi}\int_0^{\omega}\Theta_2'(t)dt=n\theta_2=nq_2=h_2m_2$$

To compute the linking integral of γ with the Clifford circle we consider the orientation preserving conformal involution

$$\psi:(x,y,z)\in\mathbb{R}^3\mapsto\frac{1}{x^2+y^2+z^2+2\sqrt{2}x+2}\left(x^2+y^2+z^2+2\sqrt{2},4z,4y\right)\in\mathbb{R}^3.$$

This map takes $\left(-\sqrt{2},0,0\right)$ to the point at infinity and exchanges the roles of the two axes of symmetry. A direct computation shows that the parametric equations of $\gamma_*=\theta\circ\gamma$ are

$$x_*(t)=\frac{\sqrt{2}}{r^*(t)}v\sqrt{\mu^2-k(t)^2}cos\Theta_1(t)$$
$$y_*(t)=\frac{\sqrt{2}}{r^*(t)}v\sqrt{\mu^2-k(t)^2}sin\,\Theta_1(t)$$
$$z_*(t)=\frac{\sqrt{2}}{r^*(t)}\mu\sqrt{k(t)^2-v^2}sin\,\Theta_2(t)$$

Where $r^*(t)=\sqrt{\mu^2-v^2}k(t)+\mu\sqrt{k(t)^2-v}cos\Theta_2(t)$. We then have

$$lk(\gamma,[C])=-lk(\gamma_*,[z])$$
$$=-\frac{1}{4\pi}\int_{[\gamma_*]}\int_{z}\frac{\gamma_*-\alpha}{\|\gamma_*-\alpha\|^3}\cdot d\gamma_*\times d\alpha$$
$$=\frac{1}{4\pi}\int_0^{n\omega}\left(\int_{-\infty}^{+\infty}\left(\frac{x_*(t)y_*'(t)-x_*'(t)y_*(t)}{(x_*(t)^2+y_*(t)^2+(z_*(t)-s)^2)^{3/2}}\right)ds\right)dt$$
$$=\frac{1}{2\pi}\int_0^{n\omega}\frac{x_*(t)y_*'(t)-x_*'(t)y_*(t)}{x_*(t)^2+y_*(t)^2}dt=\frac{n}{2\pi}\int_0^{\omega}\Theta_1'(t)dt=-n\theta_1=nq_1=h_1m_1$$

Which the required result.

3. Conformal Change of Metric

An interesting kind of metric changes is the conformal change, the simplest way to vary a metric. Let us first glance at a basic case to understand our gluing method for conformal change.

Theorem 3.1.: Suppose (X,g) is a compact Riemannian manifold and M is an oriented compact connected m-dimensional submanifold representing a nonzero class in $H_m(X;R)$. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a conformal change of g supported in U such that M is strongly calibrated by some calibration $\hat{\varphi}$ in $(X,\hat{g})$.

Proof: Since φ is pointwise a multiple of $\pi_g^*\omega$ in $U_{\frac{3}{5}}\in(M)$, $\|\Phi\|_g^*$ is smooth on $U_{\frac{3}{5}}\in(M)$. Let $g'=\left(\|\Phi\|_g^*\right)^{\frac{2}{m}}g$. Then $\|\Phi\|_{g'}^*=1$ on $U_{\frac{3}{5}}\in(M)$. Take $\tilde{g}=\sigma^{\frac{1}{m}}(1+d^2)g'+\alpha(1-\sigma)^{\frac{1}{m}}g$. Set $\hat{g}\triangleq\alpha^{-1}\tilde{g}$ and $\hat{\Phi}\triangleq\alpha^{-\frac{m}{2}}\Phi$. Then M is strongly calibrated by $\hat{\Phi}$ under $\hat{g}$. Furthermore, $\hat{g}$ is conformal to g and $\hat{g}=g$ on $X-U\in(M)$. The same local gluing ideas and elimination tricks on calibrations lead to the following results.

Theorem 3.2.: Suppose M is a finite mutually disjoint collection in a compact Riemannian manifold $(X;g)$ and every nonempty level M_k satisfy the convex hull condition. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a conformal change of g supported in U such that there exist a family of calibrations $\{\hat{\Phi}_k\}$ in $(X;\hat{g})$ and every nonzero current $T=\sum_{i=1}^{r_k}t_i[[M_i]]$ with $M_i\in M_k$ and $t_i\geq 0$ is calibrated by $\hat{\Phi}_k$. Consequently T is mass-minimizing in $[T]$ with $M(T)=\sum_{i=1}^{s}t_iVol\hat{g}(M_i)$

Corollary 3.3.: Suppose M is a finite mutually disjoint collection in a compact Riemannian manifold $(X;g)$ and each component represents a nonzero class in the R-homology of X. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a conformal change of g supported in U such that each M_k can be tamed in $(X;\hat{g})$.

Theorem 3.4.: Suppose M is a finite mutually disjoint collection in $(X;g)$ and each component represents a nonzero class in the R-homology of X. Then a new metric $\hat{g}$ can be constructed by a conformal change of g such that every M_k can be tamed in $(X;\hat{g})$.

Theorem 3.5.: Suppose M is a neat mutually disjoint collection in $(X;g)$ and each component represents a nonzero class in the R-homology of X. In addition, assume every level of M consist of finite components except the lowest level. Then a new metric $\hat{g}$ can be constructed by a conformal change of g such that each M_k can be tamed in $(X;\hat{g})$.

Remark 3.6.: If $(X;g)$ is hermitian with an (almost) complex J, so are the resulted metrics.

As an application, we strengthen Tasaki's "equivariant" theorem in [4].

Theorem 3.7.: (Tasaki) Let K be a compact connected Lie transformation group of a manifold X and M be a (connected) compact oriented submanifold in X. Assume M is invariant under the action of K and it represents a nonzero R-homology class of X. Then there exists a K-invariant Riemannian metric g on X such that M is mass-minimizing in homology class with respect to g.

Theorem 3.8.: Let K be a compact Lie transformation group of a manifold X and M be a compact connected oriented submanifold in X. Assume M is invariant under the action of K and the action is orientation preserving. Then for any K-invariant Riemannian metric g^K, there exists a K-invariant metric $\hat{g}^K$ conformal to g^K such that M can be calibrated in $(X;\hat{g}^K)$.

Theorem 3.9.: Suppose M is a neat mutually disjoint collection with only the lowest level possibly consisting of infinite components, and that each component represents a nonzero class in the R-homology of X. Let K be a compact connected Lie transformation group of X. Assume M is invariant under the action of K. Then for any K-invariant Riemannian metric g^K, there exists a K-invariant metric $\hat{g}^K$ conformal to g^K such that every M_k can be tamed in $(X;\hat{g}^K)$.
Proof: Without loss of generality, one only needs to consider the case of a single level. Since K is compact, there is a Haar-measure $d\mu$ with $\int_K d\mu = 1$. One can use $d\mu$ to average for a K-invariant Φ. (Note that ω^* and d are K-invariant.) Then average the corresponding α. By (2.1) one can get a K-invariant calibration pair $(\Phi,\hat{g}^K)$.

3.2.: On Mean Curvature Vector Fields: Let us take a short digression about mean curvature vector fields. By local calibrations, we have the following.
Corollary 3.10.: Suppose M is an oriented compact submanifold in $(X;g)$. Then there exists $\hat{g}$ conformal to g such that M is minimal in $(X;\hat{g})$.

Remark 3.11.: Since either local orientation of a submanifold leads to the same metric by our method and being minimal is really a local property, the orient ability and compactness requirements can be removed.
What is more, by a direct computation, a concrete relation between mean curvature vector fields through a conformal change can be given explicitly.

Proposition 3.12.: Let M be an m-dimensional submanifold in $(X;g)$ and $\tilde{g} = f.g$ where f is a positive function. Then at a point $p \in M$,

$$f(p).\widetilde{H}_p = Hp - \frac{m}{2f(p)}.grad^{\perp}_{g,p}(f)$$

Here H and $\widetilde{H}$ are mean vector fields of M under g and $\tilde{g}$ respectively and $grad^{\perp}_{g}(\cdot)$ stands for the normal part of $grad^{\perp}_{g}(\cdot)$ along M.

Remark 3.13.: M can be realized totally geodesic via a conformal change if and only if M is pointwise totally umbilical.

Theorem 3.14.: For a submanifold M (not necessarily oriented or compact) in $(X;g)$ and any (smooth) section ξ of the normal bundle over M, there exists some metric $\tilde{g}$ conformal to g such that $\widetilde{H} = \xi$
Proof: Suppose the ε-neighborhood U_ε of M for some suitable positive function ε on M can be identified with the normal ε-disc bundle B of M via the exponential map restricted to normal directions. Consider the smooth function f on B by $f_x(y) = 1 - \frac{2}{m} < \xi_x - H_g, y > g^{\frac{1}{x}}$ where x is a point of M and y lies in the ϵ-disc fiber through x. Let F be the induced positive (shrink ϵ if needed) function on U_ϵ. Take $\breve{g} = F.\hat{g}$. Since the differential of the identification map along M is identity, by (2.2) $H_{\hat{g}} = H_g - \frac{m}{2}\cdot grad^{\perp}_{g}F = H_g + \frac{m}{2}\cdot\frac{2}{m}\cdot(\xi_x - H_g)$.

4. Conformal structure of two-dimensional spacetimes

In general, it is well-known that any causal isomorphism between two Lorentzian manifolds is a smooth conformal diffeomorphism if the dimension of manifolds is bigger than two. However, this is not the case when the dimension is two. Even if a causal automorphism is $\mathbb{C}^\infty$, it is not necessarily a conformal diffeomorphism. For example, if we take $\phi = \psi = x^3$, then the function V defined as $\mathbb{C}^\infty$[15] and causal automorphism on $\mathbb{R}^2_1$. However, it is not a conformal diffeomorphism since its inverse is not differentiable at $(0,0)$. Therefore, if we want to get a conformal diffeomorphism on $\mathbb{R}^2_1$ from causal automorphism we need one more condition and we state the corresponding result in two-dimensional spacetimes with non-compact Cauchy surfaces.

Lemma 4.1: Let M and N be two-dimensional spacetimes with non-compact Cauchy surfaces and $V: M \to N$ be a causal isomorphism. If both V and V^{-1} are $\mathbb{C}^\infty$, then, V is a $\mathbb{C}^\infty$ conformal diffeomorphism.
Proof: It suffices to show that V_* sends null vectors to null vectors, by Lemma 2.1 in [1]. Let $v \in T_pM$ be a null vector and let γ be a future-directed null geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Then, since M has non-compact Cauchy surfaces, γ has no null cut points, and so, for any $t > 0$, we have $\gamma(0) \leq \gamma(t)$ but not $\gamma(0) << \gamma(t)$. Since V is a causal isomorphism, we have $V(\gamma(0)) \leq V(\gamma(t))$ but not $V(\gamma(0)) << V(\gamma(t))$. Therefore, any future-

directed causal curve from $V(\gamma(0))$ to $V(\gamma(t))$ is a null pregeodesic. Since V is a $\mathbb{C}^\infty$ causal isomorphism $V \circ \gamma$ is a future-directed causal curve and thus $V \circ \gamma$ is a null pregeodesic. Therefore, $V_*(v)$ is a null vector. Likewise, we can apply the same argument to V^{-1} to obtain the desired result.
Let $V: M \longrightarrow N$ be an anti-causal isomorphism. When the time-orientation of N is given by a vector field X, if we replace the time-orientation of N by $-X$, the map V becomes a causal isomorphism. Since conformal map is irrelevant to time-orientations, we have the following.

Corollary: Let M and N be two-dimensional spacetimes with non-compact Cauchy surfaces and $V: M \to N$ be an anti-causal isomorphism. If both V and V^{-1} are $\mathbb{C}^\infty$, then, V is a $\mathbb{C}^\infty$ conformal diffeomorphism.
If M is a two-dimensional spacetime with a non-compact Cauchy surface Σ, then Σ is homeomorphic to $\mathbb{R}$. If we identify $\mathbb{R}$ and $\mathbb{R}_0 = \{(x,0)|x \in \mathbb{R}\}$ which is a Cauchy surface of $\mathbb{R}_1^2$, we can choose a homeomorphism $f: \Sigma \to \mathbb{R}_0$. For given $p \in J^+(\Sigma)$, let $S_p = J^-(p) \cap \Sigma$. Then, since M is globally hyperbolic, S_p is compact and connected and thus $f(S_p)$ is also compact and connected subset of $\mathbb{R}_0$ and we can choose unique $q \in J^+(\mathbb{R}_0)$ such that $J^-(q) \cap \mathbb{R}_0 = f(S_p)$. In this way, we can extend f to a map from $J^+(\Sigma)$ into $J^+(\mathbb{R}_0)$. Likewise, we can extend f from $J^-(\Sigma)$ into $J^-(\mathbb{R}_0)$ and thus we have a map from M into $\mathbb{R}_1^2$. It can be shown that this extended map is a causal isomorphism from M into its image in $\mathbb{R}_1^2$ and that $\mathbb{R}_0$ is a Cauchy surface of the image of the extended map.
In the above argument, if we take f to be a $\mathbb{C}^\infty$ diffeomorphism between $\boldsymbol{\Sigma}$ and $\mathbb{R}_{\mathbf{0}}$, then the extended map is a $\mathbb{C}^\infty$ conformal diffeomorphism.

Theorem 4.1: Let M be a two-dimensional spacetime with non-compact Cauchy surfaces. Then M can be imbedded into $\mathbb{R}_1^2$ in such a way that the imbedding is a conformal diffeomorphism onto a globally hyperbolic subset of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface.
Proof: Let Σ be a Cauchy surface of M and take a $\mathbb{C}^\infty$ diffeomorphism $f: \Sigma \longrightarrow \mathbb{R}_0$. Then, by the above argument, f can be extended to a causal isomorphism $V: M \longrightarrow \mathbb{R}_1^2$ onto an open subset that contains $\mathbb{R}_0$ as a Cauchy surface, which is a topological imbedding.
By the previous lemma, it is sufficient to show that V and V^{-1} are $\mathbb{C}^\infty$.
For given $p \in J^+(\Sigma)$, since Σ is a non-compact smooth one-dimensional manifold, S_p is uniquely determined by two boundary points, say x and y. Since there are two unique null geodesics γ_1 from x to p and γ_2 from y to p, the dependence of p on x and y is $\mathbb{C}^\infty$. Likewise, the dependence of $V(p)$ on $V(x)$ and $V(y)$ is also $\mathbb{C}^\infty$. Therefore, since f is $\mathbb{C}^\infty$, F is $\mathbb{C}^\infty$.
By exactly the same manner, we can show that V^{-1} is $\mathbb{C}^\infty$.
From the above theorem, we can see that, to analyze conformal structure of two-dimensional spacetimes with non-compact Cauchy surfaces, it is sufficient to study conformal structures of an open subset of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface.

Lemma 4.2: Let U be a globally hyperbolic open subset of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface and let $V: U \longrightarrow R_1^2$ be a $\mathbb{C}^\infty$ conformal diffeomorphism into an open subset of R_1^2 that contains x-axis. Then, there exists unique $\mathbb{C}^\infty$ diffeomorphisms ϕ and ψ of R such that $\phi'\psi' > 0$ and V is given by one of the following form.
(1) $V(x,t) = (\phi(x+t) + \psi(x-t), \phi(x+t) - \psi(x-t))$.
(2) $V(x,t) = (\phi(x-t) + \psi(x+t), \phi(x-t) - \psi(x+t))$.
Proof: We only sketch outlines of the proof since it can be obtained from calculations and simple arguments. If a map $V: (x,t) \mapsto (X,T)$ is a conformal map, from the definitions of conformal map, we obtain two cases.
i) $X_x^2 < T_x^2$ and $(X_x = T_t \; or \; X_t = T_x)$
ii) $X_x^2 < T_x^2$ and $(X_x = -Y_t \; or \; X_t = -T_x)$
We show the case (i) since the case (ii) can be solved by exactly the same manner.
From $X_x = T_t$ and $X_t = T_x$, we can see that both X and T satisfies wave equation and thus from the general solution of wave equations in one spatial coordinate, we have $X = \phi(x+t) + \psi(x-t)$ and $T = \alpha(x+t) + \beta(x-t)$. From the system of partial differential equations $X_x = T_t$ and $X_t = T_x$, we have $X = \phi(x+t) + \psi(x-t)$ and $T = \phi(x+t) - \psi(x-t) + c$ for some $c \in R$.
By replacing ϕ by $\phi + \frac{c}{2}$ and $\psi \; by \; \psi - \frac{c}{2}$, we have $X = \phi(x+t) + \psi(x-t)$ and $T = \phi(x+t) - \psi(x-t)$. From $X_x^2 < T_x^2$, we obtain $\phi'\psi' > 0$. Since the domains of definitions and ranges of V must contain x-axis, ϕ and ψ must be defined on the whole of R and their ranges are R. For V to be a diffeomorphism, ϕ and ψ must be diffeomorphisms.

Theorem 4.2: Let M be a two-dimensional spacetime with non-compact Cauchy surfaces. Then, the group of all conformal diffeomorphisms of M is isomorphic to a subgroup of $Con(\mathbb{R}_1^2)$, the group of all conformal diffeomorphisms of $\mathbb{R}_1^2$.
Proof: By Theorem 4.1, we only need to study the group of all conformal diffeomorphisms of a globally hyperbolic open subset U of $\mathbb{R}_1^2$ that contains x-axis as a Cauchy surface. Then, for any conformal diffeomorphism on U, by the previous lemma, we have two unique diffeomorphisms ϕ

and ψ defined on $\mathbb{R}$ in such a way that V, defined as in the previous lemma, is the conformal diffeomorphism of U. Since ϕ and ψ are defined on the whole of R, we can uniquely extend V to a map V defined on $\mathbb{R}_1^2$ and the extension is a conformal diffeomorphism of $\mathbb{R}_1^2$ by the previous lemma. Then, the map $V \mapsto \bar{V}$ is a group isomorphism from $Con(M)$ into a subgroup of $Con(\mathbb{R}_1^2)$.

Theorem 4.3.: Let M be a two-dimensional spacetime with compact Cauchy surfaces and $\pi: M \longrightarrow M$ be a universal covering map. Then, we have the following.

(1) The group of covering transformation D consists of those functions ϕ given by $\phi(u,v) = (u+m, v+m)$ in null coordinates. The group A, the normalizer of D in $Con(M)$ consists of pairs of two diffeomorphisms $(\phi,\psi) \in Aut(M)$ on R that satisfy the condition: for any $n \in Z$, there exists $m \in Z$ such that $f(x+n) - f(x) = \frac{m}{2}$ for all x.

(2) The general form of conformal diffeomorphism on M is given by

$$g(e^{2\pi i x}, t) = \left(e^{\pi i\{\varphi(u)+\psi(v)\}}, \frac{1}{2}(\varphi(u) - \psi(v))\right)$$

Where ϕ and ψ are given from (1).

Theorem 4.4.: Let M be a two-dimensional spacetime with compact Cauchy surfaces. Then, there exists two-dimensional spacetime M with non-compact Cauchy surfaces of which the group of conformal diffeomorphisms contains D as a subgroup such that $Con(M)$ is isomorphic to A/D where A is the normalizer of D in $Aut(M)$. Conversely, given two-dimensional spacetime M with non-compact Cauchy surfaces, if the group of conformal diffeomorphisms of M contains D as a subgroup, then there exists two-dimensional spacetime M with compact Cauchy surfaces such that $Con(M)$ is isomorphic to A/D.

Theorem 4.5.: Let M be a two-dimensional spacetime with compact Cauchy surfaces. Then, $Con(M)$ is isomorphic to a subgroup of $Con(E)$.

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