# Discussion on Some Properties of an Infinite Non-abelian Group

Ankur Bala[#1], Madhuri Kohli[#2]

[#1] M.Sc. , Department of Mathematics, University of Delhi, Delhi

[#2] M.Sc., Department of Mathematics, University of Delhi, Delhi

India

**Abstract**: In this Article, we have discussed some of the properties of the infinite non-abelian group of matrices whose entries from integers with non-zero determinant. Such as the number of elements of order 2, number of subgroups of order 2 in this group. Moreover for every finite group $G$, there exists $k \in \mathbb{N}$ such that $GL(k,\mathbb{Z})$ has a subgroup isomorphic to the group $G$.

**Keywords:** Infinite non-abelian group, $GL(n,\mathbb{Z})$

**Notations:** $GL(n,\mathbf{Z}) = \{ \ [a_{ij}]_{n\times n} \ : \ a_{ij} \in \mathbf{Z}.$ & $\det([a_{ij}]_{n\times n}) = \pm 1 \ \}$

**Theorem 1:** $GL(n,\mathbb{Z})$ can be embedded in $GL(m,\mathbb{Z}) \ \forall \ m \geq n.$

**Proof:** If we prove this theorem for $m = n+1$, then we are done.

Let us define a mapping $\varphi : GL(n,\mathbb{Z}) \longrightarrow GL(n+1,\mathbb{Z})$

Such that

$$\varphi\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & & a_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Let A = $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ and B= $\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

Such that determinant of A and B is $\pm 1$, where $a_{ij}, b_{ij} \in \mathbb{Z}$.

Then A.B =

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} . \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

Where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj}$

$$\Rightarrow \varphi(A.B) = \begin{bmatrix} c_{11} & \cdots & c_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n1} & & c_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad \cdots (1)$$

$$\varphi(A).\varphi(B) = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & & a_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} . \begin{bmatrix} b_{11} & \cdots & b_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ b_{n1} & & b_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n1} & & c_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad \cdots (2)$$

Now from (1) and (2) one can easily say that $\varphi$ is homomorphism.

Now consider $ker\varphi$,

$Ker\varphi = \{ A \in GL(n,\mathbb{Z}) : \ \varphi(A) = I_{(n+1)\times(n+1)} \}$ *where I is identity matrix*

Clearly $Ker\varphi = \{ I_{n\times n} \}$

Hence $\varphi$ is injective homomorphism.

So, by fundamental theorem of isomorphism one can easily conclude that $GL(n,\mathbb{Z})$ can be embedded in $GL(m,\mathbb{Z})$ for all m $\geq n$.

**Theorem 2:** $GL(n,\mathbb{Z})$ is non-abelian infinite group $n \geq 2$.

**Proof:** Consider

$$GL(2,\mathbb{Z}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \ \in \mathbb{Z} \ and \det(A) = \pm 1 \right\}$$

Consider $A_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ *such that* $c \in \mathbb{Z}$

And let H= $\{A_c \ : c \in \ \mathbb{Z}\}$

Clearly H is subset of $GL(2,\mathbb{Z})$.

And H has infinite elements.

Hence $GL(2,\mathbb{Z})$ is infinite group.

Now consider two matrices $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \in GL(2,\mathbb{Z})$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

And

$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 1 & 3 \end{bmatrix}$

Hence, $GL(2,\mathbb{Z})$ is non-abelian.

And by a direct application theorem 1, one can conclude that $GL(n,\mathbb{Z})$ is infinite and non-abelian $\forall\, n \geq 2$

**Theorem 3:** Number of elements of order 2 in $GL(n,\mathbb{Z})$ is infinite and hence number of subgroups of order 2 is infinite.

**Proof:** Consider $GL(2,\mathbb{Z}) = \left\{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}\ and\ \det(A) = \pm 1\right\}$

Let $\in GL(2,\mathbb{Z})$, and order of A is 2.

Then

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $for\ some\ a,b,c,d \in \mathbb{Z}\ and\ A^2 = I$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ca+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow a^2 + bc = 1, ab + bd = 0, ca + cd = 0, bc + d^2 = 1$

Since $a^2 + bc = 1$, then:

Case (i):$a^2 = 1, bc = 0$

$\Rightarrow a = \pm 1\ and\ either\ b\ or\ c = 0$

Subcase (i): If we take $a = 1\ and\ b = 0$

Then $\begin{bmatrix} 1 & 0 \\ c+cd & d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow c + cd = 0$

$\Rightarrow c(1 + d) = 0$

$\Rightarrow either\ c = 0\ or\ d = -1$

$\Rightarrow if\ c \neq 0\ then\ d = -1$

Let $c \neq 0$

Then our case is $a = 1, b = 0, c \neq 0, d = -1$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}$ has order 2 if $c \neq 0\ and\ c \in \mathbb{Z}$

Clearly c has infinite choices.

$\Rightarrow$ Number of elements of order 2 in $GL(2,\mathbb{Z})$ is infinite.

And since number of elements of order 2 = Number of subgroups of order 2 in any group.

$\Rightarrow$ Number of subgroups of order 2 in $GL(2,\mathbb{Z})$ is infinite.

Also by a direct application of theorem 1, one can easily conclude that this theorem is valid for every value of $n$.

**Theorem 4**: Every finite group can be embedded in $S_n$ for some $n \in \mathbb{N}$

**Proof**: Let G be any group and A(G) be the group of all permutations of set G.

For any $a \in G$, define a map $f_a: G \longrightarrow G$ such that $f_a(x) = ax$

Then as $x = y \Rightarrow ax = ay$

$\Rightarrow f_a(x) = f_a(y)$

Hence, $f_a$ is well defined

Clearly $f_a$ is one-one.

Also for any $y \in G$, Since $f_a(a^{-1}y) = y$.

$\Rightarrow f_a$ is onto.

And hence $f_a$ is permutation.

Let K be set of all such permutations

Clearly K is subgroup of A(G).

Now define a mapping $\varphi : G \longrightarrow K$ such that $\varphi(a) = f_a$

Clearly $\varphi$ is well-defined and one-one map.

And consider the following equation

$\varphi(a.b) = f_{ab} = f_a o f_b = \varphi(a).\varphi(b)$

Which shows that $\varphi$ is homomorphism

Obviously, $\varphi$ is onto homomorphism

$\Rightarrow \varphi$ is isomorphism.

And hence the theorem.

**Theorem 5**: $S_n$ is isomorphic to some subgroup of $GL(n,\mathbf{Z})$ for all $n \in \mathbb{N}$.

**Proof**: Let $S_n$ be the permutation group on n symbols.

Define $\varphi : S_n \longrightarrow GL(n,\mathbf{Z})$ such that: $\varphi(\sigma) = [\sigma]_{n\times n}\ \forall \sigma \in S_n$

Where $[\sigma]_{n\times n}$ is permutation matrix obtained by $\sigma$

i.e. if $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix}$ then

$$[\sigma]_{n\times n} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Where $R_i$ is $R^{th}_{\beta_i}$ row of identity matrix.

Clearly $\varphi$ is a homomorphism.

Now consider the kernel of this homomorphism.

$\ker\varphi = \{\sigma : \varphi(\sigma) = I_{n\times n}\}$ $\Rightarrow i = \beta_i\ \ \forall i$

$\Rightarrow \ker\varphi$ is trivial.

Hence the homomorphism is injective.

$\Rightarrow S_n$ is isomorphic to some subgroup of $GL(n,\mathbf{Z})$ for all $n \in \mathbb{N}$.

**Theorem 6:** For every finite group $G$, there exists $k \in \mathbb{N}$ such that $GL(k,\mathbb{Z})$ has a subgroup isomorphic to the group $G$.

**Proof:** It is an obvious observation of theorem 4 and theorem 5.

## CONCLUSION

$GL(n,\mathbb{Z})$ is non-abelian infinite group having infinite number of elements of order 2 as well as subgroups of order 2. Also $GL(n,\mathbb{Z})$ can be embedded in $GL(m,\mathbb{Z})$ $\forall\ m \geq n$ , Moreover for every finite group $G$, there exists $k \in \mathbb{N}$ such that $GL(k,\mathbb{Z})$ has a subgroup isomorphic to the group $G$.

## REFERENCES


[1] Contemporary Abstract Algebra, J.A. Gallian, Cengage Learning, 2016 .ISBN: 1305887859, 9781305887855

[2] Abstract Algebra (3rd Edition), David S. Dummit , Richard M. Foote, ISBN: 978-0-471-43334-7

[3] On Embedding of Every Finite Group into a Group of Automorphism IJMTT, Vol.37, Number 2, ISSN: 2231-5373

[4] A Course in Abstract Algebra, Vijay K. Khanna, S.K. Bhambri, Second Edition, Vikas Publishing House Pvt Limited, 1999, 070698675X, 9780706986754