# On Multi Normed Linear Spaces

Ranajoy Roy [1], Sujoy Das [2], S. K. Samanta [3]

[1] Department of Mathematics, Visva Bharati, Santiniketan, West Bengal, India.

[2] Department of Mathematics, Suri Vidyasagar College, Suri, Birbhum -731101, West Bengal, India

[3] Department of Mathematics, Visva Bharati, Santiniketan, West Bengal, India.

***Abstract* —** *In the present paper notions of multi linear space and multi normed linear space are presented and some basic properties of such spaces have been investigated.*

**Keywords —** *Multi vector space, multi vector, multi norm, multi open set, multi limit point, multi derived set.*

## I. Introduction

Multiset (bag) is a well established notion both in mathematics and in computer science ([9], [10], [22]). In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained ([21], [23], [24]). In various counting arguments it is convenient to distinguish between a set like $\{a, b, c\}$ and a collection like $\{a, a, a, b, c, c\}$.The latter, if viewed as a set, will be identical to the former. However, it has some of its elements purposely listed several times. We formalize it by defining a multiset as a collection of elements, each considered with certain multiplicity. For the sake of convenience a multiset is written as $\{k_1/x_1,\ k_2/x_2, \ldots\ldots, k_n/x_n\}$ in which the element $x_i$ occurs $k_i$ times. We observe that each multiplicity $k_i$ is a positive integer.

From 1989 to 1991, Wayne D. Blizard made a thorough study of multiset theory, real valued multisets and negative membership of the elements of multisets ([1], [2],[3],[4]). K. P. Girish and S. J. John introduced and studied the concepts of multiset topologies, multiset relations, multiset functions, chains and antichains of partially ordered multisets ([11], [12],[13],[14],[15]). D. Tokat studied the concept of soft multi continuous function [25].Concepts of multigroups and soft multigroups are found in the studies of Sk. Nazmul and S. K. Samanta ([17], [18]). Many other authors like Chakrabarty et al. ([5], [6], [7], [8]), S. P. Jena et al. ([16]), J. L. Peterson ([19]) also studied various properties and applications of multisets.

In our previous paper we have introduced the notion of Multi metric Space. In this paper we are going to introduce a concept of Multi linear space and the idea of norm in such spaces. Furthermore we are going to investigate some properties of such Multi normed linear spaces.

## II. Priliminaries

***Definition 2.1. [11]*** A multi set $M$ drawn from the set $X$ is represented by a function $CountM$ or $C_M$ defined as $C_M : X \to N$where $N$ represents the set of non negative integers.

***Definition 2.2. [11]*** Let $M$ and $N$ be two msets drawn from a set $X$. Then, the following are defined:

$(i)\ M = N \ if\ C_M(x) = C_N(x) for\ all\ x \in X.$

$(ii)\ M \subset N\ if\ C_M(x) \leq C_N(x) for\ all\ x \in X.$

$(iii)\ P = M \cup N\ if\ C_P(x) = Max\{C_M(x), C_N(x)\}$

$for\ all\ x \in X.$ $(iv)\ P = M \cap N\ if\ C_P\ (x) = Min\{C_M(x), C_N(x)\}$

$for\ all\ x \in X.$

$(v)\ P = M \oplus N\ if\ C_P\ (x) = C_M(x) + C_N(x)\ for\ all\ x \in X:$

$(vi) P = M \ominus N\ if\ C_P\ (x) = Max\{C_M(x) - C_N(x), 0\}$

$for\ all\ x \in X$, where $\oplus$ and $\ominus$represents mset addition and mset subtraction respectively.

Let $M$ be an mset drawn from a set $X$. The **support set** of $M$, denoted by $M^*$, is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., $M^*$ is an ordinary set. $M^*$ is also called root set.

An mset M is said to be an empty mset if for all $x \in X, C_M(x) = 0$ . The **cardinality** of an mset $M$ drawn from a set $X$ is denoted by Card (M) or $|M|$ and is given by $Card(M) = \sum_{x \in X} C_M$ .

***Definition 2.3. [20]*** **Multi point:** Let M be a multi set over a universal set $X$, $x \in X$ and $k \in \boldsymbol{N}$ such that $k \leq C_M(x)$. Then a multi point of M is defined by a mapping $P_x^k : X \to \boldsymbol{N}$ such that $P_x^k\ (y) = \begin{cases} k \text{ if } y = x \\ 0 \text{ , otherwise} \end{cases}$

x and k will be referred to as the **base** and the **multiplicity** of the multi point $P_x^k$ , respectively. Collection of all multi points of an mset M is denoted by $M_{pt}$.

***Definition 2.4. [20]*** The **mset generated by a collection** $B$ of multi points is denoted by $MS(B)$ and is defined by $C_{MS(B)}(x) = Sup\{k : P_k^x \in B\}$.

***Definition 2.5. [20]*** Let $m\boldsymbol{R}^+$ denote the multi set over $\boldsymbol{R}^+$ (set of non-negative real numbers) having multiplicity of each element equal to $w$, $w \in N$. The members of $(m\boldsymbol{R}^+)_{pt}$ will be called non-negative multi real points.

***Definition 2.6. [20]*** Let $P_i^a$ and $P_j^b$ be two multi real points of $m\boldsymbol{R}^+$. We define $P_i^a > P_j^b$ if $a > b$ or $P_i^a > P_j^b$ if $i > j$ when $a = b$.

***Definition 2.7. [20]*** **(Addition of multi real points)** We define $P_i^a + P_j^b = P_{i+j}^k$ where $k = Max\{i,j\}$, $P_i^a; P_j^b \in (m\boldsymbol{R}^+)_{pt}$.

***Definition 2.8. [20]*** **(Multiplication of multi real points)** We define multiplication of two multi real points in $m\boldsymbol{R}^+$ as follows:

$$P_i^a \times P_j^b = \begin{cases} P_0^1 & \text{, if either } P_i^a \text{ or } P_j^b = P_0^1 \\ P_{ab}^k & \text{, otherwise where } k = Max\{i,j\} \end{cases}$$

.

***Definition 2.9. [20]*** **Multi Metric**:

Let $d : M_{pt} \times M_p t \rightarrow (m\boldsymbol{R}^+)_{pt}$ ($M$ being a multi set over a Universal set $X$ having multiplicity of any element at most equal to $w$) be a mapping which satisfy the following:

$(M1)\ d(P_x^l, P_y^m) \geq P_0^1, \forall P_x^l, P_y^m \in M_{pt}$

$(M2)\ d(P_x^l, P_y^m) = P_0^1\ iff\ P_x^l, = P_y^m \quad \forall P_x^l, P_y^m \in M_{pt}$.

$(M3)\ d(P_x^l, P_y^m) = d(P_y^m, P_x^l)\ \forall P_x^l, P_y^m \in M_{pt}$.

$(M4)\ d(P_x^l, P_y^m) + d(P_y^m, P_z^n) \geq d(P_x^l, P_z^n)$, $\forall P_x^l, P_y^m, P_z^n \in M_{pt}$.

$(M5)\ For\ l \neq m,\ d(P_x^l, P_y^m) = P_0^k \Leftrightarrow x = y$ $and\ k = Max\{l,m\}$.

Then d is said to be a multi metric on M and $(M,d)$ is called a Multi metric (or an M - metric)space.

***Definition 2.10. [20]*** Let $(M,d)$ be an M-metric space, $r > 0$ and $P_a^k \in Mpt$. Then the open ball with centre $P_a^k$ and radius $P_r^1$ $[r > 0]$ is denoted by $B(P_a^k, P_r^1)$ and is defined by $B(P_a^k, P_r^1) = \{P_x^l : d(P_x^l, P_a^k) < P_r^1\}$.

$MS[B(P_a^k, P_r^1)]$ will be called a multi open ball with centre $P_a^k$ and radius $P_r^1 > P_0^1$.

***Definition 2.11. [20]*** $B[P_a^k, P_r^1] = \{P_x^l : d(P_x^l, P_a^k) \leq P_r^1\}$ is called the **closed ball** with centre $P_a^k$ and radius $P_r^1$ $[r > 0]$. $MS[B[P_a^k, P_r^1]]$ will be called a **multi closed ball** with centre $P_a^k$ and radius $P_r^1$ $[r > 0]$.

***Definition 2.12. [20]*** Let $(M,d)$ be an M-metric space. Then a collection B of multi points of M is said to be open if every multi point of B is an interior point of B i.e., for each $P_a^k \in B$, there esists an open ball $B(P_a^k, P_r^1)$ with centre at $P_a^k$ and r > 0 such that $B(P_a^k, P_r^1) \subset B$.

$\phi$ is separately considered as an open set.

***Definition 2.13. [20]*** Let $(M,d)$ be an M-metric space. Then $N \subset M$ is said to be multi open in $(M,d)$ iff there esists a collection B of multi points of N such that B is open and MS(B) = N.

***Definition 2.14. [20]*** A multi set N in an M-metric space $(M,d)$ is said to be multi closed if its complement $N^c$ is multi open in $(M,d)$.

***Definition 2.15. [20]*** Let $(M,d)$ be an M-metric space and B be a collection of multi points of M. Then a multi point $P_x^l$ of M is said to be a limit point of B if every open ball $B(P_x^l, P_r^1)$ $(r > 0)$ containing $P_x^l$ in $(M,d)$ contains at least one point of B other than $P_x^l$. The set of all limit points of B is said to be the derived set of B and is denoted by $B^d$.

***Definition 2.16. [20]*** Let $(M,d)$ be an M-metric space and $N \subset M$. Then $P_x^l \in M_{pt}$ is said to be a multi limit point of N if it is a limit point of $M_{pt}$ ie. if every open ball $B(P_x^l, P_r^1)$ $(r > 0)$ containing $P_x^l$ in $(M,d)$ contains at least one point of $N_{pt}$ other than $P_x^l$.

***Definition 2.17. [20]*** Let $(M,d)$ be an M-metric space and $B \subset M_{pt}$. Then the collection of all points of B together with all limit points of B is said to be the closure of B in $(M,d)$ and is denoted by $\bar{B}$. Thus $\bar{B} = B \cup B^d$.

***Definition 2.18. [20]*** Let $(M,d)$ be an M-metric space and $N \subset M$. Then the multi set generated by all multi points and all multi limit points of N is said to be the multi closure of N and is denoted by $\bar{N}$.

***Definition 2.19. [20]*** A sequence $\{P_{x_n}^{l_n}\}$ of multi points in $m\boldsymbol{R}^+$ is said to converge to $P_0^1$ if for any $\epsilon > 0$, $\exists\ n_0 \in N$ such that $P_{x_n}^{l_n} < P_\epsilon^1 \ \forall n \geq_n 0$.

Since $l_n \geq 1\, \forall n \in N, P_{x_n}^{l_n} < P_\epsilon^1 \Leftrightarrow x_n < \epsilon$.

$\therefore P_{x_n}^{l_n} \rightarrow P_0^1 \Leftrightarrow x_n \rightarrow 0\ as\ n \rightarrow \infty$ in $\boldsymbol{R}^+$.

***Definition 2.20. [20]*** Let $\{P_{x_n}^{l_n}\}$ be a sequence of multi points in an M-metric space $(M,d)$. The sequence $\{P_{x_n}^{l_n}\}$ is said to converge in $(M,d)$ if $\exists\ P_x^l \in M_{pt}$ such that $d(P_{x_n}^{l_n}, P_x^l) \rightarrow P_0^1\ as\ n \rightarrow \infty$.

## III. MULTI VECTOR SPACE

***Definition 3.1.*** **Multi vector space :** Let V be vector space over a field K (to be denoted by $V_K$). A multiset X over V is said to be a multi vector space or a multi linear space or Mvector space of V over K if every element of X has the same multiplicity and the support $X^*$ of X is a subspace of V .

The multiplicity of every element of X will be denoted by $w_X$.

***Example 3.2.*** Let $\boldsymbol{R}^3$ be the Euclidean 3-dimensional space over $\boldsymbol{R}$. Let $X = \{5/(a,b,0) : a,b \in \boldsymbol{R}\}$. Then X is a multi vector space of $\boldsymbol{R}^3$ over $\boldsymbol{R}$.

***Definition 3.3***. (i) **Addition of two multisets over V** : Let P and Q be two multisets over V . Then P + Q is a multiset over V such that $C_{P+Q}(x) = Sup\{C_P(u) \vee C_Q(v) : x = u + v, \forall u \in P^*, v \in Q^*\}$. Clearly $(P + Q)^* = P^* + Q^* = \{u + v : u \in P^*, v \in Q^*\}$

(ii) Let P be a multiset over V and $a \in K$, then for $a \neq 0$, we define $aP$ as $C_{aP}(x) = C_P(y)$ where $y \in P^*$ and $x = ay$ ie $C_{aP}(x) = C_P(a^{-1}x)$ and for $a = 0$, we define $aP$ as $C_{aP}(x) = \begin{cases} 0, \text{if } x \neq \theta \\ \bigvee_{y \in P^*} C_P(y), if\ x = \theta \end{cases}$

Clearly for $a \neq 0$, $(aP)^* = P^*$ and for a = 0, $(aP)^* = \{\theta\}$.

(iii) If F is a multiset over V and $x \in V$, we define $x + F$ to be a multiset over V defined as $(x + F)^* = x + F^* = \{x + f : f \in F^*\}$ and $C_{x+F}(u) = C_F(v)$ where $v \in F^*$ and $u = x + v$.

(iv) If $U \subset V$ and F is a multiset over V. Then $U + F$ is a multiset over $V$ defined as $U + F = \{m/u + f : u \in U, f \in F^*$ and $m = C_F(f)\}$.

Clearly $U + F = \bigcup_{u \in U}(u + F)$.

***Theorem 3.4.*** If F and G are two multisets over a vector space $V_K$, then for $a \in K$, $a(F + G) = aF + aG$.

***Theorem 3.5.*** If , $G_i$, $i = 1, 2, 3 \dots\dots\dots.n$, are multisets over a vector space $V$, $F = F_1 + F_2 + \cdots + F_n$, $G = G_1 + G_2 + \cdots + G_n$ and $H = F_1 + F_2 + \cdots + F_n + G_2 + \cdots + G_n$, then $H = F + G$.

**Proof:** The proof is straight forward.

***Definition 3.6***. Let X be an Mvector space over a vector space $V_K$. Then $F \subset X$ is said to be a multi subspace or Msubspace of X if F is an Mvector space over $V_K$ ie $F^*$ is a subspace of $V_K$ and every element of F has the same multiplicity.

***Example 3.7***. $Y = \{4/(a,0,0) : a \in \boldsymbol{R}\}$ is an Msubspace of the Mvector space defined in example 3.2.

***Theorem 3.8.*** Let X be an Mvector space over $V_K$. Then $F \subset X$ is an Msubspace of X iff every element of F has the same multiplicity and for any $a, b \in K$, $aF + bF \subset F$.

**Proof.** Let $F \subset X$ is an Msubspace of X. Then by definition, every element of F has the same multiplicity $w_F$ and $F^*$ is a subspace of $V_K$.

Consequently $\forall a, b \in K$, $(aF + bF) = F^*$. Also for $x \in (aF + bF)^* = F^*$,

$C_{aF+bF}(x) = Sup\{C_{aF}(u) \vee C_{bF}(v) : u \in (aF)^*, v \in (bF)^*\ and\ x = u + v\}$.

$= Sup\{C_{aF}(ap) \vee C_{bF}(bq) : p, q \in F\_\ and\ x = ap + bq\}$

$= Sup\{C_F(p) \vee C_F(q) : p, q \in F\_\ and\ x = ap + bq\}$

$= Sup\{w_F \vee w_F\} = w_F = C_F(x)$.

Conversely, let $F \subset X$ such that every element of F has the same multiplicity and for any $a, b \in K$, $aF + bF \subset F. \Rightarrow (aF + bF)^* = aF^* + bF^* \subset F^* \forall a, b \in K$

$\Rightarrow au + bv \in F^* \forall a, b \in K\ and\ \forall u, v \in V_K$

$\Rightarrow F^*$ is a subspace of $V_K$.

Hence F is an Msubspace of X.

***Proposition 3.9.*** (i) If F and G are two Msubspaces of an Mvector space X over $V_K$, then $F + G$ and ∀ a ∈ K, $aF$ are also Msubspaces of X over $V_K$:

(ii) If $\{F_i : i \in \Delta\}$ be a family of Msubspaces of an Mvector space X over $V_K$, then $\bigcap_{i \in \Delta} F_i$ is also an Msubspace of X.

Proof: The proof is straight forward.

## IV. MULTIVECTORS IN MVECTOR SPACE

***Definition 4.1***. **Multivectors:** Let X be an Mvector space over a vector space $V_K$. Then every multi point of X ie., every element of $X_{pt}$ will be called a multivector or Mvector of X.

***Definition 4.2.*** **Multi scalar field:** Let K be a field. Then a multi set L over K is called a multi scalar field or Mscalar field if every element of L has the same multiplicity (denoted by $w_L$ ) and the support $L^*$ of L is a subfield of K.

Multi points of L will be referred to as multi scalars or Mscalars of L.

Multiplicity of each element of L will be denoted by $w_L$.

***Example 4.3.*** In example 3.2, $P^1_{(1,1,0)}$, $P^2_{(1,1,0)}$, $P^4_{(1,5,0)}$ etc. are Mvectors of the given Mvector space.

***Definition 4.4.*** Let X be an Mvector space over $V_K$. Then an Mvector $P_x^k$ of X will be called a null Mvector if its base $x = \theta$ ($\theta$ being the null vector of $X^*$ ie $V_K$) and it will be denoted by $\Theta^k$, ie. $P_x^k = \Theta^k$ if $x = \theta$.
An Mvector$P_x^k$ will be called non null if $x \neq \theta$.

***Note 4.5***. Null Mvector of an Mvecotr space is not unique.

***Definition 4.6.*** Let X be an Mvector space over a vector space $V_K$, L be an Mscalar field over K such that $w_L \le w_X$, $P_x^l, P_y^m \in X_{pt}$ and $P_a^i \in L_{pt}$ .Then we define

$$P_x^l + P_y^m = \begin{cases} P_\theta^1 & iff\ x = -y\ and\ l = m \\ P_{x+y}^{l \vee m} & otherwise \end{cases}$$

and

$$P_a^i . P_x^l = \begin{cases} P_\theta^1 & iff P_a^i = P_0^1\ or\ P_x^l = P_\theta^1 \\ P_{ax}^{i \vee l} & otherwise \end{cases},$$

where 0 is the null element of K.

***Proposition 4.7.*** Let X be an Mvector space over a vector space $V_K$ and L be an Mscalar field over K such that$w_L \le w_X$. Then for $P_x^l \in X_{pt}$ and $P_a^i \in L_{pt}$ with $l, i > 1$,
(i) $P_a^i . P_x^l = \Theta^n$ where $n = i \vee l$.
(ii) $P_a^i . \Theta^l = \Theta^n$ where $n = i \vee l$.
(iii) $P_{-1}^i . P_x^l = . P_{-x}^n$ where $n = i \vee l$.
(iv) $\forall P_x^l \in X_{pt}$ and $P_a^i \in L_{pt}$ ; $Pia, P_a^i . P_x^l = \Theta^n$
$\Rightarrow either\ a = 0\ or\ x = \theta$.
**Proof:** Proofs are straight forward.

***Theorem 4.8.*** Let X be an Mvector space over a vector space $V_K$. Then $Y \subset X$ is an Msubspace of X iff every element of Y has the same multiplicity $w_Y$ and for any Mscalar field L over K with $w_L \le w_Y$, , $P_a^i . P_x^l + P_b^j . P_y^m \in Y_{pt}\ \forall\ P_a^i ; P_b^j \in L_{pt}$ and $P_x^l, P_y^m \in Y_{pt}$.
**Proof.** Let $Y \subset X$ is an Msubspace of X. Then every element of Y has the same multiplicity $w_Y$ and the support $Y^*$ is a subspace of $X^*$. Let L be an Mscalar field over K such that
$w_L \le w_Y$ . Then $\forall\ x, y \in Y^*$ and $a, b \in L^* \subset K$, $ax + by \in Y^* \Rightarrow P_a^i . P_x^l + P_b^j . P_y^m \in Y_{pt}\ \forall\ P_a^i ; P_b^j \in L_{pt}$ and $P_x^l, P_y^m \in Y_{pt}$ .

Conversely let the given conditions hold. Then $\forall\ x, y \in Y^*$ and $a, b \in L^* \subset K$, $ax + by \in Y^* \Rightarrow L^*$ is a subspace $X^*$_ and since every element of Y has the same multiplicity, Y is an Msubspace of X.

***Definition 4.9.*** **Multi linear combination:** Let X be an Mvector space over a vector space $V_K$ and L be an Mscalar field over K such that $w_L \le w_X$ . Then an Mvector $P_x^l \in X_{pt}$ is said to be a multi linear combination or Mlinear combination of the vectors $P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n} \in X_{pt}$ over L if $P_x^l$ can be expressed as
$P_x^l = P_{a_i}^{i_1} . P_{x_1}^{l_1} +$
$P_{a_2}^{i_2} . P_{x_2}^{l_2}, + \cdots \dots\dots\dots\dots . + P_{a_n}^{i_n} . P_{x_n}^{l_n}$ for some Mscalars $P_{a_i}^{i_1}, P_{a_2}^{i_2} \dots\dots\dots\dots\dots\dots\dots . . P_{a_n}^{i_n} \in L_{pt}$.

***Example 4.10.*** Let us consider the Mvector space given in Example 3.2. Let L be an Mscalar field over ***R*** given by $L = \{3/r : r \in \boldsymbol{R}\}$. $Let\ x_1 = (1, 2, 0)$, $x_2 = (-2, 5, 0), x_3 = (0, 1, 0)$. Then $P_{x_1}^1 + c$,
$P_{x_1}^1 + P_{x_2}^2 + P_{x_3}^3, P_2^3 . P_{x_1}^1 + P_5^2\ P_{x_2}^2$ ; $P_{\frac{5}{6}}^3 . P_{x_2}^2 + P_{7/8}^2 . P_{x_3}^3$
are Mlinear combinations of the Mvectors $P_{x_1}^1, P_{x_2}^2, P_{x_3}^3$ over L.

***Definition 4.11.*** **Multi linearly dependent and multi linearly independent:** Let X be an Mvector space over a vector space $V_K$. Then a finite collection of Mvectors $\{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ of X is said to be multi linearly dependent or Mlinearly dependent or ML.D if for some
Mscalar field L over K with $L^* \neq \{0\}$ and $w_L \le w_X$ , there exist Mscalars $P_{a_i}^{i_1}, P_{a_2}^{i_2} \dots\dots\dots\dots\dots\dots\dots . . P_{a_n}^{i_n} \in L_{pt}$ for some i = 1, 2, ............ , n such that
$P_{a_i}^{i_1} . P_{x_1}^{l_1} + P_{a_2}^{i_2} . P_{x_2}^{l_2} + \cdots \dots\dots\dots\dots\dots\dots . . P_{a_n}^{i_n} . P_{x_n}^{l_n} = \Theta^l$.

The collection of Mvectors $\{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ of X is said to be multi linearly independent or Mlinearly independent or ML.Id it is not ML.D ie. if for any Mscalar field L over K with $L^* \neq \{0\}$ and $w_L \le w_X$, a relation $P_{a_i}^{i_1} . P_{x_1}^{l_1} + P_{a_2}^{i_2} . P_{x_2}^{l_2} + \cdots \dots\dots\dots\dots\dots\dots . . P_{a_n}^{i_n} . P_{x_n}^{l_n} = \Theta^l$ where $P_{a_k}^{i_k} \in L_{pt}$, $k = 1, 2, \dots\dots\dots . n$, holds only when $a_k = 0$
$\forall\ k = 1, 2, \dots\dots . n$.

***Note 4.12.*** (i) Every superset of a finite collection of ML.D Mvectors in an Mvector space is ML.D.

(ii) Every nonempty subset of a finite collection of ML.Id Mvectors in an Mvector space is ML.Id.

***Definition 4.13.*** (i) An infinite collection of Mvectors in an Mvector space is ML.D if it has a finite ML.D subset.
(ii) An infinite collection of Mvectors in an Mvector space is ML.Id if every nonempty finite subset of it is ML.Id.

***Theorem 4.14.*** Let X be an Mvector space over a vector space $V_K$. If a collection of Mvectors $\{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ of X is ML.Id (or ML.D), then $\{P_{x_1}^{k_1}, P_{x_2}^{k_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{k_n}\}$ is ML.Id (or ML.D)
$\forall\, k_1, k_2, \dots\dots\dots\dots, k_n \in \{1, 2, \dots\dots\dots\dots, w_X\}$.
Proof. The proof is straight forward.

***Definition 4.15.*** An arbitrary multiset $G \subset X$ is said to be ML.D or ML.Id according as the collection
$\{P_x^l : x \in G^*\}$
is ML.D or ML.Id.

***Note 4.16.*** Any collection of Mvectors having at least two elements with same base is ML.D.

***Theorem 4.17.*** Let X be an Mvector space over a vector space $V_K$. Then a finite collection of Mvectors $\{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ of X is ML.Id iff $\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$ is L.Id.
**Proof:** Let $\{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ is ML.Id , L be an Mscalar field over $V_K$ with $L^* = K, w_L \leq w_X$ and $a_1, a_2, \dots\dots\dots\dots\dots\dots a_n \in K = L^*$ such that $a_1.x_1 + a_2.x_2 + \cdots\dots\dots + a_n x_n = \theta \Rightarrow P_{a_i}^{i_1}.P_{x_1}^{l_1} + P_{a_2}^{i_2}.P_{x_2}^{l_2} + \cdots\dots\dots\dots\dots\dots\dots P_{a_n}^{i_n}.P_{x_n}^{l_n} = \Theta^l$ where $P_{a_k}^{i_k} \in L_{pt}$ , $k = 1, 2, \dots\dots\dots. n \Rightarrow a_1 = a_2 = \cdots\dots\dots\dots\dots = a_n = 0 \Rightarrow$
$\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$ is ML.Id.

Conversely let $\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$ is L.Id and L be an Mscalar field over $V_K$ with with $L^* = K, w_L \leq w_X$. Now for Mscalars $P_{a_i}^{i_1}, P_{a_2}^{i_2} \dots\dots\dots\dots\dots\dots P_{a_n}^{i_n} \in L_{pt}$ with $P_{a_i}^{i_1}.P_{x_1}^{l_1} + P_{a_2}^{i_2}.P_{x_2}^{l_2} + \cdots\dots\dots\dots\dots\dots P_{a_n}^{i_n}.P_{x_n}^{l_n} = \Theta^l \Rightarrow$
$a_1.x_1 + a_2.x_2 + \cdots\dots\dots + a_n x_n = \theta \Rightarrow a_1 = a_2 = \cdots\dots\dots\dots = a_n = 0$
$\Rightarrow \{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ is ML.Id.

***Theorem 4.18.*** Let X be an Mvector space over a vector space $V_K$ . Then a finite collection of Mvectors $\{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ of X is ML.D iff $\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$ is L.D.
Proof. The proof follows similarly as the previous one.

***Definition 4.19.* Linear span :** Let $X$ be an Mvector space over a vector space $V_K$ , L be an Mscalar field over K such that $w_L \leq w_X$ and $S = \{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ be a collection of Mvectors of $X$. Then the linear span of S over L denoted by $LS(S, L)$ is defined as $LS(S, L) = \{P_{a_i}^{i_1}.P_{x_1}^{l_1} + P_{a_2}^{i_2}.P_{x_2}^{l_2} + \cdots\dots\dots\dots\dots\dots\dots P_{a_n}^{i_n}.P_{x_n}^{l_n} : P_{a_i}^{i_1}, P_{a_2}^{i_2}, \dots\dots\dots\dots\dots\dots P_{a_n}^{i_n} \in L_{pt}\}$.
$MS[LS(S, L)]$ will be referred to as the multi linear span or Mlinear span of S over L.

***Theorem 4.20.*** Let X be an Mvector space over a vector space $V_K$, L be an Mscalar field over K such that $w_L \leq w_X$ and $S = \{P_{x_1}^{l_1}, P_{x_2}^{l_2}, \dots\dots\dots\dots\dots\dots P_{x_n}^{l_n}\}$ be a collection of Mvectors of X. If $MS[LS(S, L)] = X$ then $X^* = LS\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$ and either $w_L = w_X$ *or* $l_k = w_X$ for some $k = 1, 2, \dots\dots\dots. n$ and conversely.
**Proof:** Let $MS[LS(S, L)] = X$. Then for any $x \in X^*, C_X(x) = w_X \Rightarrow C_{MS[LS(S,L)]}(x) = w_X \Rightarrow Sup\,\{l : P_x^l \in LS(S, L)\} = w_X \Rightarrow P_x^{w_X} \in LS(S, L) \Rightarrow \exists\, P_{a_i}^{i_1}, P_{a_2}^{i_2}, \dots\dots\dots\dots\dots\dots P_{a_n}^{i_n} \in L_{pt}$ *such that* $P_x^{w_X} = P_{a_i}^{i_1}.P_{x_1}^{l_1} + P_{a_2}^{i_2}.P_{x_2}^{l_2} + \cdots\dots\dots\dots\dots\dots P_{a_n}^{i_n}.P_{x_n}^{l_n}$
$\Rightarrow x =$
$a_1.x_1 + a_2.x_2 + \cdots\dots\dots + a_n x_n$ *and* $w_X = Sup\,\{i_k, l_k : k = 1, 2, \dots\dots\dots\dots, n\} \Rightarrow x \in LS\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$ and either $i_k = w_X$ or $l_k = w_X$ for some $k = 1, 2, \dots\dots\dots. n$. Since $i_k \leq w_L \leq w_X\ \forall\, k = 1, 2, \dots\dots\dots. n$, so for some $k = 1, 2, \dots\dots\dots. n,\ i_k = w_X \Rightarrow w_L = w_X$.
Also $\forall\, x \in X^*, x \in LS\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\} \Rightarrow X^* \subset LS\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\} \subset X^* \Rightarrow X^* = LS\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$.

Conversely let $X^* = LS\,\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$ *and either* $l_k = w_X$ for some $k = 1, 2, \dots\dots\dots. n$ or $w_L = w_X$. Let $x \in^{w_X} X \Rightarrow x \in X^* =$
$LS\{x_1, x_2, \dots\dots\dots\dots\dots\dots x_n\}$

$\Rightarrow \exists\, a_1, a_2, \dots \dots, a_n \in L^*$ such that $x = a_1.x_1 + a_2.x_2 + \cdots \dots \dots + a_n x_n$ . If $l_k = w_X$ for some $k = 1, 2, \dots, n$, then

$P^1_{a_i}.P^{l_1}_{x_1} + P^1_{a_2}.P^{l_2}_{x_2} + \cdots \dots \dots \dots \dots + P^1_{a_n}.P^{l_n}_{x_n}$
$= P^{\vee_{k=1}^{n} l_k}_{a_1.x_1 + a_2.x_2 + \cdots \dots + a_n x_n} = P^{w_X}_x \Rightarrow P^{w_X}_x \in LS(S, L)$

$\Rightarrow C_{MS[LS(S,L)]}(x) = w_X$ [since $MS[LS(S, L)] \subset X$]
$\Rightarrow X = MS[LS(S, L)]$.

***Definition 4.21.*** An Mvector space X over $V_K$ is said to be finite dimensional if there is a finite set of ML.Id Mvectors in X that also generates X ie. there exists a finite set $S = \{P^{l_1}_{x_1}, P^{l_2}_{x_2}, \dots \dots \dots \dots P^{l_n}_{x_n}\}$ of Mvectors of X which is ML.Id and $MS[LS(S, L)] = X$ for some Mscalar field L over K with $w_L \leq w_X$.

The number of elements of such a set S is called the dimension of X and is denoted by Dim(X).

***Theorem 4.22.*** Let X be an Mvector space over $V_K$. Then $dim(X^*) = n$ iff there exists a collection of n ML.Id Mvectors of X generating X.

**Proof.** Let there is a finite collection $S = \{P^{l_1}_{x_1}, P^{l_2}_{x_2}, \dots \dots \dots \dots P^{l_n}_{x_n}\}$ of n ML.Id Mvectors of X such that $MS[LS(S, L)] = X$ for some Mscalar field L over K with $w_L \leq w_X$.
Now as S is ML.Id, $\{x_1, x_2, \dots \dots \dots \dots . x_n\}$ is L.Id in $X^*$. Also $MS[LS(S, L)] = X \Rightarrow X^* = LS\{x_1, x_2, \dots \dots \dots \dots . x_n\} \Rightarrow \{x_1, x_2, \dots \dots \dots \dots . x_n\}$ is a basis of $X^* \Rightarrow Dim(X^*) = n$.
Conversely let $Dim(X^*) = n$ and $\{x_1, x_2, \dots \dots \dots \dots . x_n\}$ is a basis of $X^*$. Then clearly $S = \{P^{w_X}_x : k = 1, 2, \dots . n\} \subset X_{pt}$ and $MS[LS(S)] = X$ where LS(S) can be considered over any Mscalar field L over K such that $L^* = K$ and $w_L \leq w_X$.

**Note 4.23.** Since $dim(X^*)$ unique, it follows that $Dim(X)$ is also unique and $dim(X^*) = Dim(X)$.

## V. MULTI NORMED LINEAR SPACE

**Notation:** Throughout this section we shall consider V as a vector space over $K = \boldsymbol{R}/\boldsymbol{C}$, $X$ as an Mvector space over $V_K$ with $w_X \leq w$ (w being the multiplicity of every element of $m\boldsymbol{R}^+$) and L as an Mscalar field over K with support $L^* = K$ and $w_L \leq w_X$.

***Definition 5.1***. A mapping $\| \ \| : X_{pt} \to (m\boldsymbol{R}^+)_{pt}$ will be called a multi norm or Mnorm on $X$ if it satisfies the following:
$(N1) \|P^l_x\| \geq P^1_0 \ \forall P^l_x \in X_{pt}$.
$(N2) \|P^l_x\| = P^k_0$ iff $x = \theta$ and $l = k$.
$(N3) \|P^i_a.P^l_x\| = P^i_{|a|} \|P^l_x\| \ \forall P^i_a \in L_{pt}, P^l_x \in Xpt$.
$(N4) \|P^l_x + P^m_y\| \leq \|P^l_x\| + \|P^m_y\| \ \forall P^l_x, P^m_y \in X_{pt}$ .

An Mvector space $X$ with an Mnorm $\| \ \|$ on $X$ is called a multi normed linear space or Mnormed linear space and is denoted by $(X, \| \ \|)$. $(N1), (N2), (N3)$ and $(N4)$ are called norm axioms.

***Example 5.2.*** Let us consider the Mvector space $m\boldsymbol{R} = \{w/r : r \in R\}$ over $\boldsymbol{R}$ and L be an Mscalar field over $\boldsymbol{R}$. Also let $\| \ \| : (m\boldsymbol{R})_{pt} \to (m\boldsymbol{R}^+)_{pt}$ defined by $\|P^i_a\| = P^i_{|a|} \ \forall P^i_a \in (m\boldsymbol{R})_{pt}$ where $| \ |$ denotes the modulus of real numbers. Then :
$(N1)$ Clearly $\forall P^i_a \in (m\boldsymbol{R})_{pt}, a \in \boldsymbol{R}$ and $1 \leq i \leq w \Rightarrow \|P^i_a\| = P^i_{|a|} \geq P^1_0$ .

$(N2) \ \forall P^i_a \in (m\boldsymbol{R})_{pt}, \|P^i_a\| = P^k_0 \Leftrightarrow P^i_{|a|} = P^k_0 \Leftrightarrow |a| = 0$ and $i = k$.

$(N3)$ For $P^i_a \in (m\boldsymbol{R})_{pt}$, $P^m_\alpha \in L_{pt}$ , $\|P^m_\alpha P^i_a\| = \|P^{m \vee i}_{\alpha a}\| = P^{m \vee i}_{|\alpha a|} = P^{m \vee i}_{|\alpha||a|} = P^m_{|\alpha|} P^i_{|a|} = P^m_{|\alpha|} \|P^i_a\|$.

$(N4)$ For $P^i_a, P^j_b \in (m\boldsymbol{R})_{pt}$, $\|P^i_a + P^j_b\| = \|P^{i \vee j}_{a+b}\| = P^{i \vee j}_{|a+b|} \leq P^{i \vee j}_{|a|+|b|} = P^i_{|a|} + P^j_{|b|} = \|P^i_a\| + \|P^j_b\|$.

Thus $(m\boldsymbol{R}, \| \ \|)$ is an Mnormed linear space.

***Example 5.3***. Let $(V, \| \ \|)$ be a normed linear space over $K = \boldsymbol{R}/\boldsymbol{C}$ and X be an Mvector space over V with $w_X = w$. Let $\| \ \|_m : X_{pt} \to (m\boldsymbol{R}^+)_{pt}$ such that $\| P^l_x \|_m = P^l_{\|x\|} \ \forall P^l_x \in X_{pt}$. Then $\| \ \|_m$ is an Mnorm over X and $(X, \| \ \|_m)$ is an Mnormed linear space.

***Note 5.4.*** Corresponding to every normed linear space, there exists an Mnormed linear space.

***Theorem 5.5.*** Let $(X, \| \ \|)$ be an Mnormed linear space over a vector space $V\_K$. Then $d : X_{pt} \times$

$X_{pt} \rightarrow (m\boldsymbol{R}^+)_{pt}$ defined by $d(P_x^l, P_y^m) = \|P_x^l - P_y^m\| \ \forall\ P_x^l, P_y^m \in X_{pt}$ is a multimetric on $X$.

**Proof.** $(M1)$ Clearly from $(N1)$, $d(P_x^l, P_y^m) \geq P_0^1$ $\forall\ P_x^l, P_y^m \in X_{pt}$.

$(M2)$ Let $P_x^l, P_y^m \in X_{pt}$, then $d(P_x^l, P_y^m) = P_0^1 \Leftrightarrow \|P_x^l - P_y^m\| = P_0^1 \Leftrightarrow P_x^l - P_y^m = P_\theta^1$ $[From\ (N2)] \Leftrightarrow P_x^l = P_y^m$.

$(M3)$ Let $P_x^l, P_y^m \in X_{pt}$. If $P_x^l = P_y^m$, then there is nothing to prove. Let $P_x^l \neq P_y^m$ .Then $d(P_x^l, P_y^m) = \|P_x^l - P_y^m\| = \|P_{x-y}^{l \vee m}\| = \|P_{-(y-x)}^{l \vee m}\| = \|P_{-1}^1 . P_{y-x}^{l \vee m}\| = P_{|-1|}^1 \|P_{y-x}^{l \vee m}\| = P_{|1|}^1 \|P_{y-x}^{l \vee m}\| = \|P_1^1 . P_{y-x}^{l \vee m}\| = \|P_{y-x}^{l \vee m}\| = \|P_y^m - P_x^l\| = d(P_y^m, P_x^l)$ .

$(M4)$ Let $P_x^l, P_y^m, P_n^z \in X_{pt}$. Then $\|P_x^l - P_z^n\| = \|P_x^l - P_y^m + P_y^m - P_z^n\| \leq \|P_x^l - P_y^m\| + \|P_y^m - P_z^n\|$ $[From\ (N4)]) \Rightarrow d(P_x^l, P_z^n) \leq d(P_x^l, P_y^m) + d(P_y^m, P_z^n)$.

$(M5)$ Let $P_x^l, P_y^m \in X_{pt}$ with $l \neq m$. Now $d(P_x^l, P_y^m) = p_0^k \Leftrightarrow \|P_x^l - P_y^m\| = p_0^k \Leftrightarrow \|P_{x-y}^{l \vee m}\| = p_0^k$ [ since $l \neq m \Rightarrow P_x^l \neq P_y^m \Rightarrow P_x^l - P_y^m = P_x^l + P_{-y}^m = P_{x-y}^{l \vee m}$ ]
$\Leftrightarrow x - y = \theta$ and $l \vee m = k$ $[By\ (N2)]$
$\Leftrightarrow x = y$ and $l \vee m = k$.
Thus
if $P_x^l, P_y^m \in X_{pt}$ with $l \neq m$, then $d(P_x^l, P_y^m) = p_0^k \Leftrightarrow x = y$ and $l \vee m = k$.
$\therefore$ d is a multi metric on $X$.

***Definition 5.6*. Mnorm subspace:** Let $(X, \|\ \ \|_X)$ be an Mnormed linear space over $V_K$ and $Y \subset X$ is an Msubspace of X. Then $\|\ \ \|_Y : Y_{pt} \rightarrow (m\boldsymbol{R}^+)_{pt}$ defined by $\|P_x^l\|_Y = \|P_x^l\|_X \ \forall\ P_x^l \in Y_{pt}$ is an Mnorm on Y. This Mnorm is known as the relative Mnorm on Y induced by $\|\ \ \|_X$. The Mnormed linear space $(\|\ \ \|_Y, Y)$ is called a an Mnorm subspace or simply an Msubspace of the Mnormed linear space $(X, \|\ \ \|_X)$ .

## VI. Sequence and their convergence in an Mnormed linear space

***Definition 6.1*.** Let $(X, \|\ \ \|)$ be an Mnormed linear space over a vector space $V_K$ and $r > 0$. We define the following:

(i) $B(P_x^l, P_r^1) = \{P_y^m \in X_{pt} : \|P_x^l - P_y^m\| < P_r^1\}$ is called an open ball with center $P_x^l$ and radius $P_r^1$.
(ii) $\bar{B}(P_x^l, P_r^1) = \{P_y^m \in X_{pt} : \|P_x^l - P_y^m\| \leq P_r^1\}$ is called a closed ball with center $P_x^l$ and radius $P_r^1$.
(iii) $S(P_x^l, P_r^1) = \{P_y^m \in X_{pt} : \|P_x^l - P_y^m\| = P_r^1\}$ is called a sphere with center $P_x^l$ and radius $P_r^1$.

$MS[B(P_x^l, P_r^1)], MS[\bar{B}(P_x^l, P_r^1)]$ and $MS[S(P_x^l, P_r^1)]$ are respectively called an Mopen ball, an Mclosed ball and an Msphere with center $P_x^l$ and radius $P_r^1$ .

***Definition 6.2.* Convergence of a sequence:** A sequence $\{P_{x_n}^{l_n}\}$ of Mvectors in an Mnormed linear space $(X, \|\ \ \|)$ over $V_K$ is said to be convergent and converges to an Mvector $P_x^l$ if $\|P_{x_n}^{l_n} - P_x^l\| \rightarrow P_0^1$ as $n \rightarrow \infty$ which means, for any $\epsilon > 0$, $\exists\ n_0 \in \boldsymbol{N}$ such thatk $\|P_{x_n}^{l_n} - P_x^l\| < P_\epsilon^1$ $\forall\ n \geq n_0$ ie. $n \geq n_0 \Rightarrow P_{x_n}^{l_n} \in B(P_x^l, P_\epsilon^1)$. We denote this by $P_{x_n}^{l_n} \rightarrow P_x^l$ as $n \rightarrow \infty$ or by $\lim_{n\rightarrow\infty} P_{x_n}^{l_n} = P_x^l$. $P_x^l$ is said to be the limit of $\{P_{x_n}^{l_n}\}$ as $n \rightarrow \infty$ .

**Example 6.3.** In Example 5.2, let us consider a sequence $\{P_{x_n}^{l_n}\}$ of Mvectors in $(m\boldsymbol{R}, \|\ \ \|)$ where $x_n = 1/n$ and $l_n = w \ \forall\ n \in N$. Then for any $\in > 0\ \exists\ n_0 \in \boldsymbol{N}$ such that $\|P_{x_n}^{l_n} - P_0^w\| < P_\epsilon^1$ $\forall\ n \geq n_0 \Rightarrow P_{x_n}^{l_n} \rightarrow P_0^w\ as\ n \rightarrow \infty$.

***Note 6.4.*** All convergent sequences of Mvectors having the same base will converge to Mvectors having the same base ie. if for a sequence $\{P_{x_n}^{l_n}\}$ of Mvectors, $P_{x_n}^{l_n} \rightarrow P_x^l$, then $P_{x_n}^{k_n} \rightarrow P_x^k$ for any $1 \leq k \leq C_X(x)$ and for any sequence $\{k_n\}$ of natural numbers with $k_n \leq C_X(x)$. To prove this, let $\epsilon > 0$ be taken arbitrarily. Then as $P_{x_n}^{l_n} \rightarrow P_x^l$, $\exists\ n_0 \in \boldsymbol{N}$ such that $\|P_{x_n}^{l_n} - P_x^l\| < P_{\epsilon/3}^1$ $\forall\ n \geq n_0 \Rightarrow$ for any sequence $\{k_n\}$ of natural numbers and $k \in \boldsymbol{N}$ with $k_n \leq C_X(x)$ and $1 \leq k \leq C_X(x)$, $\|P_{x_n}^{k_n} - P_x^k\| = \|P_{x_n}^{k_n} - P_{x_n}^{l_n} + P_{x_n}^{l_n} - P_x^l + P_x^l - P_x^k\| \leq \|P_{x_n}^{k_n} - P_{x_n}^{l_n}\| + \|P_{x_n}^{l_n} - P_x^l\| + \|P_x^l - P_x^k\| < P_{\frac{\epsilon}{3}}^1 + P_{\frac{\epsilon}{3}}^1 + P_{\frac{\epsilon}{3}}^1 = P_\epsilon^1 \ \forall\ n \geq n_0 \Rightarrow P_{x_n}^{k_n} \rightarrow P_x^k$ $n \rightarrow \infty$.

***Definition 6.6.* Boundedness :** (i) In an Mnormed linear space $(X, \|\ \ \|)$, a multi subset $Y \subset X$ is said to be bounded if $\exists\ r > 0$ such that $\|P_x^l\| < P_r^1 \ \forall\ P_x^l \in Y_{pt}$.

(ii) A sequence $\{P_{x_n}^{l_n}\}$ of Mvectors in an Mnormed linear space $(X, \| \quad \|)$ is bounded if $\exists\, r > 0$ such that
$\|P_{x_n}^{l_n} - P_{x_m}^{l_m}\| < P_r^1 \ \forall\, m, n \in \boldsymbol{N}$.

***Definition 6.7.*** **Cauchy sequence :** A sequence $\{P_{x_n}^{l_n}\}$ of Mvectors in an Mnormed linear space $(X, \| \quad \|)$ is said to be Cauchy if for any $\in > 0, \exists\, n_0 \in \boldsymbol{N}$ such that
$\|P_{x_n}^{l_n} - P_{x_m}^{l_m}\| < P_{\in}^1 \ \forall\, m, n \geq n_0 \ ie. \|P_{x_n}^{l_n} - P_{x_m}^{l_m}\| \to P_0^1$
$as\ m, n \to \infty$.

***Theorem 6.8.*** Every convergent sequence in an Mnormed linear space is Cauchy and every Cauchy sequence is bounded.
Proof. Since Mnorm induces multi metric, the result follows obviously.

***Definition 6.9.*** **Completeness:** An Mnormed linear space $(X, \| \quad \|)$ is said to be complete if every Cauchy sequence of Mvectors in $(X, \| \quad \|)$ converges to an Mvector of X.

***Example 6.10.*** $(m\boldsymbol{R};\ \| \quad \|)$ is complete where $m\boldsymbol{R}$ is the multiset over $\boldsymbol{R}$ having multiplicity of every element equal to $w$ and $\|P_x^l\| = P_{|x|}^l \ \forall\, P_x^l \in (m\boldsymbol{R})_{pt}$.

***Theorem 6.11.*** In an Mnormed linear space $(X, \| \quad \|)$, if
$P_{x_n}^{l_n} \to P_x^l$ and $P_{y_n}^{k_n} \to P_y^k$ , then $P_{x_n}^{l_n} + P_{y_n}^{k_n} \to P_x^l + P_y^k$. .

***Theorem 6.12.*** In an Mnormed linear space $(X, \| \quad \|)$ over a vector space $V_K$, if $\{P_{x_n}^{l_n}\}$ be a sequence of Mvectors such that $P_{x_n}^{l_n} \to P_x^l$ and $\{P_{a_n}^{k_n}\}$ be a sequence of Mscalars such that $P_{a_n}^{k_n} \to P_a^k$, then $P_{a_n}^{k_n}.P_{x_n}^{l_n} \to P_a^k.P_x^l$.

***Theorem 6.13.*** In an Mnormed linear space $(X, \| \quad \|)$ over a vector space $V_K$, if $\{P_{x_n}^{l_n}\}, \{P_{y_n}^{m_n}\}$ are Cauchy sequences of Mvectors and $\{P_{a_n}^{k_n}\}$ is a Cauchy sequence of Mscalars, then $\{P_{x_n}^{l_n} + P_{y_n}^{m_n}\}$, $\{P_{a_n}^{k_n}.P_{x_n}^{l_n}\}$ are Cauchy sequences of Mvectors.
**Proof.** The proof is straight forward.

***Theorem 6.14.*** If M be an Msubspace of an Mnormed linear space$(X, \| \quad \|)$, then $\bar{M}$ is also an Msubspace of $(X, \| \quad \|)$.

**Proof.** Let $P_x^l, P_y^m \in (\bar{M})_{pt} = \overline{M_{pt}}$. Then $P_x^l, P_y^m \in (\bar{M})_{pt}$
$\forall\, l, m \in \{1, 2, \ldots\ldots\ldots\ldots, w_X\}$ ie. every element of M has the same multiplicity equal to $w_X$. Since $P_x^l, P_y^m \in (\bar{M})_{pt} = \overline{M_{pt}}$ , for any $\epsilon > 0, \exists\, P_{x_1}^{l_1}$, $P_{y_1}^{m_1} \in M_{pt}$ such that
$\|P_x^l - P_{x_1}^{l_1}\| < P_\epsilon^1$ , $\|P_y^m - P_{y_1}^{m_1}\| < P_\epsilon^1$.
Let $P_a^p, P_b^q \in L_{pt}$ [ $L$ being an Mscalar field over $K$ with $L^* = K$ ]. Since M is an Msubspace of $(X, \| \quad \|)$,
$P_a^p.P_{x_1}^{l_1} + P_b^q.P_{y_1}^{m_1} \in M_{pt}$. Now $\|P_a^p.P_{x_1}^{l_1} + P_b^q.P_{y_1}^{m_1} - (P_a^p.P_x^l + P_b^q.P_y^m)\| \leq P_{|a|}^p\|P_{x_1}^{l_1} - P_x^l\| + P_{|b|}^q\|P_{y_1}^{m_1} - P_y^m\| \leq P_{|a|}^p.P_\epsilon^1 + P_{|b|}^q.P_\epsilon^1 = P_{(|a|+|b|)\epsilon}^{p\vee q} < P_\eta^1$ where
$0 < (|a| + |b|)\epsilon < \eta$ . Since $\epsilon > 0$ is arbitrary so is $\eta$ and hence $P_a^p\ P_{x_1}^{l_1} + P_b^q.P_{y_1}^{m_1} \in B(P_a^p.P_x^l + P_b^q.P_y^m, P_\eta^1)$ for any arbitrary $\eta > 0 \Rightarrow$
$B(P_a^p.P_x^l + P_b^q.P_y^m, P_\eta^1) \cap M_{pt} \neq \phi$for any $\eta > 0 \Rightarrow$
$P_a^p.P_x^l + P_b^q.P_y^m \in \overline{M_{pt}} = (\bar{M})_{pt}$.

## VII. CONCLUSIONS

Functional analysis is an important branch of Mathematics and it has many applications in Mathematics and Sciences. Metric space is the beginning of functional analysis and it has several applications in many branch of functional analysis. In this paper an extension of the concept of metric is made by using multi set and multi number instead of crisp set and crisp real number. There is an ample scope for further research on multi metric space. Research on Multi norm and multi inner product can be of special interest.

Acknowledgements. The present work is supported by the Minor Research Project (MRP) of UGC, New Delhi, India [Grant No. PSW{203/13{14(ERO)] and Department of Mathematics, Visva-Bharati under the Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F 510/3/DRS-III/2015 (SAP-I)].

## VIII. REFERENCES

[1] Wayne D. Blizard, *Multiset theory*, Notre Dame Journal of Formal Logic 30(1), 36 - 66 (1989).

[2] Wayne D. Blizard, *Real-valued multisets and fuzzy sets, Fuzzy Sets and Systems* 33(1),77 - 97 (1989).

[3] Wayne D. Blizard, *Negative membership*, Notre Dame Journal of Formal Logic 31(3),346 - 368 (1990).

[4] Wayne D. Blizard, *The development of multiset theory*, Modern Logic 1(4), 319 - 352 (1991).

[5] K. Chakrabarty, *Bags with interval counts*, Foundations of Computing and Decision Sciences 25(1), 23 - 36 (2000).

[6] K. Chakrabarty and I. Despi, $n^k$*-bags*, Int. J. Intell. Syst. 22(2), 223 - 236 (2007).

[7] K. Chakrabarty, R. Biswas and S. Nanda, *Fuzzy shadows, Fuzzy Sets and Systems* 101(3), 413 - 421 (1999).

[8] K. Chakrabarty, R. Biswas and S. Nanda, *On Yagers theory of bags and fuzzy bags*, Computers and Artificial Intelligence (1999).

[9] G. F. Clements, *On multiset k-families*, Discrete Mathematics 69(2), 153 - 164 (1988).

[10] M. Conder, S. Marshall and Arkadii M. Slinko, *Orders on multisets and discrete cones*, A Journal on The Theory of Ordered Sets and Its Applications 24, 277 - 296 (2007).

[11] K. P. Girish, S. J. John, *Multiset topologies induced by multiset relations*, Information Sciences 188(0), 298 - 313 (2012).

[12] K. P. Girish, S. J. John, *On multiset topologies, Theory and applications of Mathematics and Computer Science*, 2(1) (2012) 37 - 52.

[13] K. P. Girish, S. J. John, *General relations between partially ordered multisets and their chains and antichains*, Mathematical Communications 14(2), 193 - 206 (2009).

[14] K. P. Girish, S. J. John, *Relations and functions in multiset context*, Inf. Sci. 179(6), 758 - 768 (2009).

[15] K. P. Girish, S. J. John, *Rough multisets and information multisystems*, Advances in Decision Sciences p. 17 pages (2011).

[16] S. P. Jena, S.K. Ghosh and B.K. Tripathy, *On the theory of bags and lists*, Information Sciences 132(14), 241 - 254 (2001).

[17] Sk. Nazmul, S. K. Samanta, *On soft multigroups*, 10(2), 271 { 285 (2015).

[18] Sk. Nazmul, S. K. Samanta, *On multisets and multigroups* , 6(30), 643 - 656 (2013).

[19] J. L. Peterson, *Computation sequence sets*, Journal of Computer and System Sciences 13(1), 1 - 24 (1976).

[20] R. Roy, S. Das, S. K. Samanta, *On multi metric spaces*, Communicated.

[21] D. Singh, *A note on the development of multiset theory*, Modern Logic 4(4), 405 - 406 (1994).

[22] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, *Complementation in multiset theory*, International Mathematical Forum 6(38), 1877 - 1884 (2011).

[23] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, *An overview of the applications of multisets*, Novi Sad J. Math 37(2), 73 - 92 (2007).

[24] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, *Some combinatorics of multisets*, International Journal of Mathematical Education in Science and Technology 34(4), 489 - 499 (2003).

[25] D. Tokat, I. Osmanoglu, Y. Ucok, *On Soft Multi Continuous Functions*, The Journal of New Theory, 1(2015) 50 - 58.