CRITERIA FOR BOUNDED (UNBOUNDED) OSCILLATIONS OF NEUTRAL IMPULSIVE DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH VARIABLE COEFFICIENTS

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Abstract: In this paper, we obtain conditions ensuring the oscillation of all bounded (unbounded) solutions of a class of second-order linear neutral ordinary differential equations with impulses, variable coefficients and constant delays. Examples are given for clarity.

Keywords *Bounded, Unbounded, Differential equations, Impulsive, neutral, delay.*

1. INTRODUCTION

The theory of impulsive differential equations is emerging as an important area of investigation since such equations appear to represent a natural framework for mathematical modeling of several real phenomena. There have been intensive studies on the qualitative behaviour of solutions of impulsive differential equations.

We must note here that applications of impulsive differential equations cut across neural networks, percussive systems with vibrations to population dynamics, and is even seen in intervention models and interrupted time series analysis ([15], [16], [17]).

Neutral impulsive differential equations are part of impulsive differential equations with deviating arguments (IDEDA). Generally speaking, IDEDA are a very interesting mixture between impulsive differential equations (see [1] and [4]) and differential equations with deviating argument (see [3] and [12]). We note here that [5] is the first work where IDEDA were considered and where an oscillation theory of such equation was studied. Among the numerous publications concerning the oscillation properties of IDEDA – with delayed or advanced arguments, we choose to refer to [2], [6] and [11]. Lately, the pioneering efforts of Isaac and Lipcsey ([7], [8], [9], [13]) in identifying some of the essential oscillatory and non-oscillatory conditions of neutral impulsive differential equations of the first order is also worth commending. However, relatively less attention has been given to oscillations of second-order neutral delay differential equations with impulses.

In this study we seek conditions under which all bounded (unbounded) solutions of a class of second order neutral delay impulsive differential equations with constant retarded arguments and variable coefficients are oscillatory.

A neutral impulsive differential equation (1.1) with constant delay is a system consisting of a differential equation together with an impulsive condition in which the second order derivative of the unknown function appears in the equation both with and without delay.

The above definition becomes more meaningful if we define other related terms and concepts that will continue to be useful as we progress.

In ordinary differential equations, the solutions are continuously differentiable sometimes at least once, whereas the impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play an important role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differential equations are fundamentally different including the definitions of some of the basic terms. In this section, we examine some of these changes.

Notation 1.1: Let $J=(\alpha,\beta)\subset R,\ -\infty<\alpha<\beta<+\infty$ is our domain of investigation

Definition 1.1: Let $S:=\{t_k\}_{k\in E}\subset J$ be a strictly ascending sequence of the time moments of impulse effects and let E be a subscript set which can be the set of natural numbers N or the set of integers Z such that

- $t_k\to\infty$ if $k\to\infty$ and if $E=Z$, then $t_k\to-\infty$ if $k\to-\infty$;
- $t_k\ge 0$ if $k\ge 0$.

Our equation under consideration then has the form

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)\,y(t-\sigma)=0, \\ \qquad t\ge t_0,\ t\in J\setminus S \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k\,y(t_k-\sigma)=0, \\ \qquad t_k\ge t_0,\ \forall\, t_k\in S, \end{cases} \tag{1.1}$$

where $1\le k\le\infty$.

In order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows: Let $D:=[T,\overline{T})\subset J\subset R$ and let the set of impulse points S be fixed.

Definition 1.2: Let

$$PC(D,\mathrm{R}):=\{\varphi\,|\,\varphi:D\to R,\ \varphi\in C(D\setminus S),\ \exists\ \varphi(t-0),\ \varphi(t+0),\ \forall\ t\in D\}.$$

From the studies in Bainov and Simeonov (1998), Lakshmikantham *et al.* (1989) and Isaac *et al.*(2011) ([1], [4], [8]), we define the function space $\forall\ r\in N$:

Definition 1.3: Let

$$PC^r(D,\mathrm{R}):=\left\{\varphi\,|\,\varphi\in PC(D,\mathrm{R}),\frac{d^j\varphi}{dt^j}\in PC(D,\mathrm{R}),\ \forall\ 1\le j\le r\right\}$$
.

To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S)$, $r\in N$.

Definition 1.4 The solution $y(t)$ of an impulsive differential equation is said to be

i) finally positive (finally negative) if there exist $T\ge 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t\ge T$ ([8]);

ii) non-oscillatory, if it is either finally positive or finally negative; and

iii) oscillatory, if it is neither finally positive nor finally negative ([1], [9]).

In the sequel, all functional inequalities that we write are assumed to hold finally, that is, for all sufficiently large t.

2. STATEMENT OF THE PROBLEM

We are concerned with the oscillatory properties of the second order linear neutral delay impulsive differential equation with variable coefficients and constant deviating arguments of the form

$$\begin{cases}\left[y(t)+p(t)y(t-\tau)\right]''+q(t)\,y(t-\sigma)=0,\\ \qquad t\ge t_0,\ t\in J\setminus S\\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k\,y(t_k-\sigma)=0,\\ \qquad t_k\ge t_0,\ \forall\ t_k\in S,\end{cases}\tag{2.1}$$

where $p(t),\ q(t)\in C\left([t_0,\infty),\mathrm{R}\right)$ and τ and σ are non-negative real numbers.

Our aim is to establish some sufficient conditions for every bounded (unbounded) solution of equation (2.1) to be oscillatory. Throughout this study, we shall assume the following:

C2.1: $q_k\ge 0\ \forall\ k\in N$;

C2.2: $p(t)\in PC([t_0,\infty),\mathrm{R}),\ p_1\le p(t)\le p_2$ for $t\in[t_0,\infty)$, where $p_1,\ p_2\in R$;

C2.3: $q(t)\in PC([t_0,\infty),\mathrm{R}),\ q(t)\ge q_1>0\ \ for\ \ t\in[t_0,\infty)$.

The following Lemma 2.1 and Lemma 2.2, which are essential in carrying out our investigation are impulsive extensions of the work done by Grammatikopoulos *et al* [14] and Ladas and Stavroulakis [10], respectively, in their quest to find sufficient conditions for oscillation of all solutions of a type of neutral delay ordinary differential equations.

Here, we demonstrate how well-known mathematical techniques and methods, after suitable modifications, is extended in proving an oscillation theorem for impulsive delay differential equations. We shall restrict ourselves to the study of impulsive differential equations for which the impulse effects take place at fixed moments $\{t_k\}$.

Lemma 2.1: Assume conditions C2.1—C2.3 satisfied and let $y(t)$ be a finally positive solution of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau).\tag{2.2}$$

Then the following statements are true:

a) The functions $z(t)$ and $z'(t)$ are strictly monotone and either

$$\lim_{t\to\infty}z(t)=\lim_{t\to\infty}z'(t)=-\infty\tag{2.3}$$

or

$$\lim_{t\to\infty}z(t)=\lim_{t\to\infty}z'(t)=0;\ z(t)<0 \text{ and } z'(t)>0.\tag{2.4}$$

In particular, $z(t)$ is finally negative.

b) Assume that $p_1\ge -1$, then condition (2.4) holds. In particular, $z(t)$ is bounded.

Proof: **(a)** From equation (2.1), we have that

$$z''(t)=-q(t)y(t-\sigma)\le -q_1y(t-\sigma)<0$$

and (2.5)

$$\Delta z'(t_k)=-q_ky(t_k-\sigma)\le -q_1y(t_k-\sigma)<0$$

which implies that $z'(t)$ is a strictly decreasing function of t and so $z(t)$ is a strictly monotone function. From the above observations it follows that either

$$\lim_{t\to\infty}z(t)=\lim_{t\to\infty}z'(t)=-\infty$$

or

$$\lim_{t\to\infty}z'(t)=\ell \text{ is finite.}\tag{2.6}$$

Let us assume that condition (2.6) holds. Integrating both sides of equation (2.5) from T to t with T sufficiently large, and letting $t\to\infty$, we obtain

$$\int_{t_0}^{\infty}q_1y(t-\sigma)ds+\sum_{t_0\le t_k<\infty}q_1y(t_k-\sigma)\le z'(\mathrm{T})-\ell\ ,$$

which implies that $y(t)\in L_1[T,\infty)$ and so $z(t)\in L_1[T,\infty)$. Since $z(t)$ is monotone, it follows that

$$\lim_{t\to\infty} z(t)=0 \qquad (2.7)$$

And therefore $\ell=0$. Finally, by equations (2.7) and (2.6) with $\ell=0$ and the decreasing nature of $z'(t)$, we conclude that $z(t)<0$ and $z'(t)>0$.

(b) By contradiction, we assume condition (2.4) was false, then from condition (2.3), it would follow that

$$\lim_{t\to\infty} z(t)=-\infty. \qquad (2.8)$$

Using the fact that $p_1\geq -1$ and $z(t)<0$, we obtain

$$y(t)<p(t)y(t-\tau)\leq -p_1 y(t-\tau)\leq y(t-\tau),$$

which implies that $y(t)$ is bounded, contradicting condition (2.8) and proving that condition (2.4) is fulfilled. This, therefore, completes the proof of Lemma 2.1.

Lemma 2.2: Assume that r and μ are positive constants such that

$$r^{\frac{1}{2}}\frac{\mu}{2}>\frac{1}{e}.$$

Then the inequality

$$\begin{cases} x''(t)-rx(t-\mu)\leq 0, & t\notin S \\ \Delta x'(t_k)-rx(t_k-\mu)\leq 0, & \forall\, t_k\in S \end{cases}$$

has no finally negative bounded solution.

3. MAIN RESULTS

The following theorems extend Theorem 3.1.4 and Theorem 3.1.5 of the monograph by Bainov and Mishev [3] by imposing impulsive perturbations as appropriate.

Theorem 3.1: Assume that conditions C2.1—C2.3 are satisfied with

$$p_2<0. \qquad (3.1)$$

Suppose also that there exists a positive constant r such that

$$\begin{cases} \dfrac{q(t)}{p(t+\tau-\sigma)}\leq -r \\ \dfrac{q_k}{p(t_k+\tau-\sigma)}\leq -r \end{cases} \qquad (3.2)$$

and

$$r^{\frac{1}{2}}\frac{\sigma-\tau}{2}>\frac{1}{e}. \qquad (3.3)$$

Then every bounded solution of equation (2.1) is oscillatory.

Proof: By contradiction, we assume that $y(t)$ is a finally bounded positive solution of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau).$$

Then a direct substitution shows that $z(t)$ is a twice piece-wise continuously differentiable solution of the neutral delay impulsive differential equation

$$\begin{cases} z''(t)+R(t)z''(t-\tau)+q(t)z(t-\sigma)=0, \\ \qquad t\geq t_0,\ t\notin S \\ \Delta z'(t_k)+R(t_k)\Delta z'(t_k-\tau)+q_k z(t_k-\sigma)=0, \\ \qquad t_k\geq t_0,\ t_k\in S, \end{cases} \qquad (3.4)$$

where

$$R(t_k)=p(t_k-\sigma)\frac{q_k}{q(t_k-\tau)}.$$

From equation (2.1) we have that

$$z''(t),\ \Delta z'(t_k)<0, \qquad (3.5)$$

and from Lemma 2.1(a) it is known that $z(t)$ is a finally negative function. From condition C2.2 and the fact that $y(t)$ is bounded, it follows that $z(t)$ is bounded. Using condition (3.5) equation (3.4) yields

$$\begin{cases} R(t)z''(t-\tau)+q(t)z(t-\sigma)>0, & t\notin S \\ R(t_k)\Delta z'(t_k-\tau)+q_k z(t_k-\sigma)>0, & \forall\, t_k\in S, \end{cases}$$

and, in view of inequality (3.2), we obtain

$$\begin{cases} z''(t)-\mathrm{r}\, z(t-(\sigma-\tau))<0, & t\notin S \\ \Delta z'(t_k)-\mathrm{r}\, z(t_k-(\sigma-\tau))<0, & \forall\, t_k\in S. \end{cases} \qquad (3.6)$$

But due to condition (3.3), Lemma 2.2 implies that it is impossible for inequality (3.6) to have a finally negative bounded solution. This is a contradiction, and thus completes the proof of Theorem 3.2.

An illustration is given here:

Example 3.1: The neutral delay impulsive differential equation

$$\begin{cases} \left[y(t)-\dfrac{\sqrt{2}}{2}y(t-\dfrac{\pi}{4})\right]''+\dfrac{\sqrt{2}}{2}y(t-\dfrac{7\pi}{4})=0, \\ \qquad t\geq 0,\ t\notin S \\ \Delta\left[y(t_k)-\dfrac{\sqrt{2}}{2}y(t_k-\dfrac{\pi}{4})\right]'+\dfrac{\sqrt{2}}{2}y(t_k-\dfrac{7\pi}{4})=0, \\ \qquad t_k\geq 0,\ t_k\in S \end{cases} \qquad (3.7)$$

satisfies all conditions of Theorem 3.1. Therefore, every bounded solution of equation (3.7) oscillates. For instance, $y(t)=\sin t$ is one of such solutions.

Theorem 3.2: Consider the neutral delay impulsive differential equation (2.1) and assume conditions C2.1—C2.3 are finally fulfilled. Further assume that

$$q(t)\geq 0,\ -1\leq p(t)\leq 0 \qquad (3.8)$$

and

$$\int_{t_0}^{\infty} q(s)ds + \sum_{t_0 \le t_k < \infty} q_k = \infty .$$

Then every unbounded solution of equation (2.1) is oscillatory.

Proof: By contradiction, we assume that $y(t)$ is a finally unbounded solution of equation (2.1). Set

$$z(t) = y(t) + p(t)y(t-\tau) .$$

Then

$$z''(t) = -q(t)\,y(t-\sigma) \le 0$$

and (3.9)

$$\Delta z'(t_k) = -q_k y(t_k - \sigma) \le 0$$

Implying that $z'(t)$ is a decreasing function. We claim that

$$z(t) > 0 . \qquad (3.10)$$

Otherwise, $y(t) \le -p(t)y(t-\tau) \le y(t-\tau)$ which implies that $y(t)$ is bounded. This contradiction shows that condition (3.10) is satisfied. We also claim that $z'(t), \Delta z(t_k) > 0$ finally. Otherwise, $z'(t), \Delta z(t_k) < 0$ and $z''(t), \Delta z'(t_k) \le 0$ finally. This leads to a contradiction as $t \to +\infty$.

Again, the fact that $\lim_{t\to\infty} z(t) = -\infty$ contradicts condition (3.10). Clearly, $y(t) > z(t)$ and so inequality (3.9) yields

$$\begin{cases} z''(t) - q(t)z(t-\sigma) \le 0, \ t \notin S \\ \Delta z'(t_k) - q_k z(t_k - \sigma) \le 0, \ \forall\, t_k \in S. \end{cases} \qquad (3.11)$$

Integrating inequality (3.11) from T to t with T sufficiently large, we get

$$z'(t) - z'(T) + z(T-\sigma)\int_T^t q(s)ds + z(T-\sigma)\sum_{T \le t_k < t} q_k \le 0 .$$

This again leads to a contradiction as $t \to +\infty$, thus completing the proof of Theorem 3.2.

Now, the following illustration shows that under the conditions of Theorem 3.2, the bounded solutions of equation (2.1) need not oscillate.

Example 3.2: Consider the neutral delay impulsive differential equation

$$\begin{cases} \left[y(t) - \left(\dfrac{2e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{e^{\frac{3\pi}{2}} + 2e^{-\frac{3\pi}{2}}} \right) y(t-2\pi) \right]'' + \left(\dfrac{2\left(e^{2\pi} - e^{-2\pi}\right)}{e^{\frac{3\pi}{2}} + 2e^{-\frac{3\pi}{2}}} \right) \\ \qquad \times\, y(t - \frac{\pi}{2}) = 0, \, t \ge 0, \ t \notin S \\ \Delta\left[y(t_k) - \left(\dfrac{2e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{e^{\frac{3\pi}{2}} + 2e^{-\frac{3\pi}{2}}} \right) y(t_k - 2\pi) \right]' + \left(\dfrac{2\left(e^{2\pi} - e^{-2\pi}\right)}{e^{\frac{3\pi}{2}} + 2e^{-\frac{3\pi}{2}}} \right) \\ \qquad \times\, y(t_k - \frac{\pi}{2}) = 0, t_k \ge 0, \ t_k \in S. \end{cases} \qquad (3.12)$$

We observe that all conditions of Theorem 3.2 are satisfied. Therefore, each unbounded solution of equation (3.12) is oscillatory. For instance, $y(t) = e^t \sin t$ is one of such solutions. On the other hand, the bounded solutions of equation (3.12) do not have to oscillate. $y(t) = e^t$ is one such non-oscillatory solution.

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