

# Discussion on Group of Inner Automorphisms of Some Groups

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**Abstract:** Let  $G$  be an arbitrary group with  $n$  number of centralizers &  $n$  is any finite number. In this article, we have proved that the group of inner automorphisms of  $G$  is isomorphic to some other groups depending upon  $n$ . Moreover if for some group  $G$ , the group of inner automorphisms  $\text{Inn}(G)$  has order 6 or 9 then  $G$  will be a group with 5 centralizers and if for some group  $H$ , the group of inner automorphisms  $\text{Inn}(H)$  has order 4 then  $H$  will be a group with 4 centralizers & conversely.

## Notations:

- (i)  $C_n$ -groups: The groups with  $n$  number of centralizers.
- (ii)  $\text{Inn}(G)$ : The group of inner automorphisms of any group  $G$ .

## Introduction:

All groups mentioned in this paper are finite group. Thus, one expects group structure to become increasingly complex with decreasing 'abelianness.' Indeed, the basic classification scheme for groups reflects the importance of the notion of commutativity. Beginning abstract algebra students tend to ignore the subtleties of the commutativity issue  $xy = yx$  as far as they are concerned. An effective way to deal with this misconception is to address it directly by asking 'How many pairs of elements of a group commute?' or 'What is the probability that two group elements commute?' The formal answers are  $\#\text{Com}(G) = \text{card}(\{(x, y) : xy = yx\})$  and  $\text{PrCom}(G) = \frac{\#\text{com}(G)}{(\text{o}(G))^2}$  respectively. These

questions and their (formal) answers put the notion of commutativity on a numerical basis, which students enjoy, and provide motivation for a delightful excursion through some nice elementary group theory. Indeed, the fact that  $\#\text{Com}(G) = k \cdot \text{o}(G)$ , where  $k$  is the number of conjugacy classes in  $G$ , is woven from elementary results on subgroups, centralizers, Lagrange's theorem, and conjugacy classes [3]. The equivalent probabilistic statement  $\text{PrCom}(G) = \frac{k}{\text{o}(G)}$ , leads unsurprisingly to a reassuring result,  $G$  is abelian if and only if  $\text{PrCom}(G) = 1$

And pleasantly to a surprising result,  $G$  is nonabelian if and only if  $\text{PrCom}(G) < 5/8$ . Another, less precise, way to say this is that either all of the elements commute or atmost  $5/8$  of the elements commute.

Here's another question relating numbers and commutativity: How many distinct centralizers can a group have? Recall that the centralizer of  $x$  in  $G$ , denoted by  $C(x)$ , is the subgroup of  $G$  consisting of all elements that commute with  $x$ ; i.e.,  $C(x) = \{y \in G : xy = yx\}$ . If we denote the number of distinct centralizers in  $G$  by  $\#\text{Cent}(G)$ , then  $\#\text{Cent}(G) = \text{card}(\{C(x) : x \in G\})$  and our question becomes 'What can we say about  $\#\text{Cent}(G)$ ?' This paper is an itinerary for an excursion in elementary group theory motivated by this question. Our goal is to provide some answers and some more questions. We think these are interesting in their own right and useful to those who teach abstract algebra. We may begin by stating the result that  $G$  is abelian if and only if  $\#\text{Cent}(G) = 1$ .

**Theorem 1:** Let  $G$  be any group then  $\frac{G}{Z(G)}$  is isomorphic to  $\text{Inn}(G)$ .

**Proof:** Let us define a map  $\phi : G \rightarrow \text{Inn}(G)$ , Such that  $\phi(a) = \varphi_a$ , where  $\varphi_a : G \rightarrow G$  is defined as  $\varphi_a(x) = a \cdot x \cdot a^{-1}$

Since

$$\phi(a \cdot b) = \varphi_{a \cdot b} = \varphi_a \circ \varphi_b = \phi(a) \cdot \phi(b)$$

Hence  $\phi$  is homomorphism.

And clearly  $\phi$  is onto.

Now consider  $\ker \phi$ ,

$$\ker \phi = \{a \in G : \phi(a) = \varphi_e\} \\ \Rightarrow \ker \phi = \{a \in G : a \in Z(G)\} = Z(G)$$

Then by fundamental theorem of homomorphism  $\frac{G}{Z(G)}$  is isomorphic to  $\text{Inn}(G)$ .

Hence the theorem.

**Theorem 2:** Let  $G$  be an arbitrary  $C_n$ -group and  $\text{Inn}(G)$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  if and only if  $n = 4$ .

**Proof:** We will start the proof by proving the result for any arbitrary group  $G$  which states that  $\frac{G}{Z}$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  if and only if  $n = 4$ . Where  $Z$  is center of group  $G$ .

If  $\frac{G}{Z}$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,

Then there are non-central elements,  $p, r$ , and  $s$  of  $G$  such that  $= Z \cup Zp \cup Zq \cup Zr$ .

It follows that the three proper subgroups of  $G$  containing  $Z$  are  $P = Z \cup Zp$ ,  $R = Z \cup Zr$ , and  $S = Z \cup Zs$ .

Let  $x$  be one of  $p, r$  or  $s$  and let  $X$  be the corresponding subgroup.

Notice that for  $zx \in Zx$ ,  $G \supset C(zx) \supset X$ . So,  $[G : X] = [G : C(zx)] \cdot [C(zx) : X] = 2$  and  $[G : C(zx)] \neq 1$  thus  $C(zx) = X$ .

Therefore the proper centralizers of  $G$  are precisely  $P$ ,  $R$ , and  $S$ ;

i.e. number of centralizer = 4.

For the converse, it is sufficient to show that  $[G : Z] = 4$

Because then either  $\frac{G}{Z}$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$

or  $\frac{G}{Z}$  is isomorphic to  $\mathbb{Z}_4$

Since,  $G$  is non-abelian,  $\frac{G}{Z}$  cannot be cyclic which means the latter case is impossible.

So, suppose number of centralizer = 4 and let  $P = C(p)$ ,  $R = C(r)$  and  $S = C(s)$  be the three proper centralizers of  $G$ .

Since  $G$  cannot be written as the union of two proper subgroups and since an element must belong to its centralizer,

We may choose  $p$ ,  $r$ , and  $s$  in  $G - (R \cup S)$ ,  $G - (P \cup S)$  and  $G - (P \cup R)$  respectively.

Moreover, at least one of the proper centralizers, say  $P$ , has index two in  $G$ .

For otherwise,

$$o(G) < o(P) + o(R) + o(S) - 2 \cdot o(Z) < \frac{o(G)}{3} + \frac{o(G)}{3} + \frac{o(G)}{3} - 2 < o(G)$$

Further,  $P \cap R = P \cap R \cap S = Z$

Because if  $x \in (P \cap R) - Z$ ,

Then

i)  $C(x) \neq G$  because  $x \notin Z$

$C(x) \neq P$  &  $C(x) \neq$

ii)  $R$  because  $p, r \in C(x)$

iii)  $C(x) \neq S$  because  $x \notin S$ ,

This means that number of centralizer must be at least 5.

Now we can compute  $o(Z)$  using the fact that for subgroups  $X$  and  $Y$  of  $G$ ,

$$o(X \cap Y) = \frac{o(X) \cdot o(Y)}{\text{card}(XY)} \geq \frac{o(X) \cdot o(Y)}{o(G)}$$

$$\text{Indeed, } o(Z) = o(P \cap R) \geq \frac{o(P) \cdot o(R)}{o(G)} = \frac{o(R)}{2}$$

$$\text{since } o(P) = \frac{o(G)}{2}$$

$$\text{But } Z \neq R, \text{ so } o(Z) = \frac{o(R)}{2}$$

$$\text{Similarly, } o(Z) = \frac{o(S)}{2}$$

Thus,

$$o(G) = o(P) + o(R) + o(S) - 2 \cdot o(Z) = \frac{o(G)}{2} + 2 \cdot o(Z) + 2 \cdot o(Z) - 2 \cdot o(Z)$$

y,

$$\text{Which implies that } \frac{o(G)}{2} = 2 \cdot o(H)$$

i.e.  $[G : Z] = 4$ , as desired.

Hence  $\frac{G}{Z}$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

Now, it is proved that  $\frac{G}{Z}$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  if and only if  $n = 4$ .

And since by theorem 1, one can easily conclude that  $\text{Inn}(G)$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  if and only if  $n = 4$ .

**Theorem 3:** Let  $G$  be an arbitrary  $C_n$ -group and  $\text{Inn}(G)$  is isomorphic to  $S_3$  or  $\mathbb{Z}_3 \times \mathbb{Z}_3$  if and only if  $n = 5$ .

**Proof:** Let us proceed by the same approach which was used in proving the theorem 2. Once the result,

which states that “  $\frac{G}{Z}$  is isomorphic to  $S_3$  or  $\mathbb{Z}_3 \times \mathbb{Z}_3$  if and only if  $n = 5$  where  $G$  is any arbitrary  $C_n$ -group” is proved then proving this theorem will be a direct application of theorem 1. And this result is proved by Belcastro and Sherman in [6]

**Conclusion:** For an arbitrary  $C_4$ -group  $G$ , In this article, we have proved that the group of inner automorphisms of  $G$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and conversely. Similarly  $G$  be an arbitrary  $C_5$ -group and  $\text{Inn}(G)$  is isomorphic to  $S_3$  or  $\mathbb{Z}_3 \times \mathbb{Z}_3$  if and only if  $n = 5$ .

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