An Axiomatic Development of Multi-Natural Numbers

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Abstract — *In this paper we define and study multi-natural number system from axiomatic point of view.*

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1. Introduction

Repeated roots of the polynomial equation $(x-1)^2(x-2)(x-3)^3=0$, although identical in all respects, are treated as multiplicity ([1], [3]). So, it is convenient to accept a collection like $\{1,1,1,2,3,3\}$ of roots rather than a set like $\{1,2,3\}$ of roots. The former if viewed as a set, will be identical to the latter. In the physical world, it is observed that there is enormous repetition ([2], [5], [6], [7], [11]). For example, a carbon atom and a hydrogen atom are obviously distinct whereas two hydrogen atoms are different but identical. So, we can say that two physical objects are the same or identical if they are indistinguishable, but possibly separate, and identical if they physically coincide ([1], [11]). In Cantorian classical set theory, a set is well-defined collection of distinct objects. If repeated occurrences of any object are allowed in a set, then that mathematical structure is called a multiset (mset in short or bag) ([1], [11]). So, a multiset is a collection of objects (called elements) in which elements may occur more than once. The number of times an element occurs in a multiset is called its multiplicity. The cardinality of a multiset is the sum of the multiplicities of its elements. For the shake of convenience, a multiset is written as $\{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ in which the element x_1 occurs k_i times ([11]). Multisets are now of special interest in some area of mathematics, computer science, physics and philosophy ([1], [4], [8], [9], [15], [17] - [21]). There are many situations in the above subjects where it is more convenient to consider a collection like multiset. e.g., the repeated eigen values of a matrix, prime factors of a positive integer, repeated observations in a statistical sample, data structure etc. Although the root of the studies in multiset is in combinatorics from ancient times ([25], [26], [27]), the modern research in this field about the structural development in multiset context is a relatively new concept. Some research works on the relations and functions in multiset context ([13], [14], [22]), multiset topology ([1], [11], [12], [18]), multiset ordering ([13]), multi group theory ([10], [23], [24]) etc. have been done by some researchers. In order of develop various structures on multisets we start from the beginning viz development of multi-number system. In this paper, we introduce a concept multi-natural number system from axiomatic point of view and study its properties related to compositions and order relations.

2. Preliminaries

2.1. Definition

An mset M drawn from a set X is represented by a function count M or C_M defined as $C_M : X \to N$ where N represents the set of nonnegative integers ([11]). Let M be an mset from the set $X = \{x_1, x_2, \ldots, x_n\}$ with x_i appearing k_i times in M. It is denoted by $x_i \in^{k_i} M$. The mset M drawn from the set X is then denoted by $\{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ in which the element x_i occurs k_i times ([11]). Also $C_M(x)$ is the number of occurrences of the element x in the mset M. However, those elements which are not include in the mset M have zero count ([11]).

2.2. Example

Let $X = \{a,b,c,d,e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X ([11]).

2.3. Definition

Let M and N be two msets drawn from a set X. Then the following are defined:

(i) $M = N$ if $C_M(x) = C_N(x)$ $\forall x \in X$.

(ii) $M \subseteq N$ if $C_M(x) \le C_N(x)$ $\forall x \in X$ (then we call N to be a submset of M).

(iii) $P = M \cup N$ if $C_P(x) = Max\{C_M(x), C_N(x)\}$ $\forall x \in X$.

(iv) $P = M \cap N$ if $C_P(x) = Min\{C_M(x), C_N(x)\}$ $\forall x \in X$.

(v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ $\forall x \in X$.

(vi) $P = M \Theta N$ if $C_P(x) = Max\{C_M(x) - C_N(x), 0\}$ $\forall x \in X$.

Where $\oplus$ and Θ represent mset addition and mset subtraction respectively ([11]).

Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set and it is also called the root set. The cardinality of an mset M drawn from a set X denoted by card (M) or $|M|$ and is given by $|M| = \sum_{x \in X} C_M(x)$ ([11]).

2.4. Definition

A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times ([16], [11]).

The element $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset ([16], [11]).

If $X = \{x_1, x_2, \ldots, x_k\}$,

then $[X]^m = \{\{m_1/x_1, m_2/x_2, \ldots, m_k/x_k\}$: for $i = 1,2,\ldots,k; m_i \in \{0,1,2,..m\}\}$ ([16], [11]).

2.5. Definition

Let X be a support set and $[X]^m$ be the mset space defined over X, then the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that

$C_M^c(x) = m - C_M(x)$ $\forall x \in X$ ([16], [11]).

2.6. Definition (Different types of Submsets)

A submset N of a mset M (i.e., $M \subseteq N$) is a whole submset of M with each element in N having full multiplicity as in M. i.e., $C_N(x) = C_M(x)$ for every x in N^* ([11]).

A submset N of M is a partial whole submset of M with at least one element in N having same multiplicity as in M. i.e., $C_N(x) = C_M(x)$ for some x in N^* ([11]).

A submset N of M is a full submset of M if, $M^* = N^*$ and $C_N(x) \le C_M(x)$ for every x in N^* ([11]).

2.6.1. Empty set ϕ is a whole submset of every mset but it is neither a full submset nor a partial whole submset of any nonempty mset M ([11]).

2.6.2. Consider the mset $M = \{2/x, 3/y, 5/z\}$. The following are the some of the submsets of M which are whole submsets, partial whole submsets and full submsets ([11]).

2.6.3. A submset $\{2/x, 3/y\}$ is a whole submset and partial whole submset of M but it is not a full submset of M ([11]).

2.6.4. A submset $\{1/x, 3/y, 2/z\}$ is a partial whole submset and full submset of M but it is not a whole submset of M ([11]).

2.6.5. A submset $\{1/x, 3/y\}$ is a partial whole submset of M which is neither full submset of M nor a whole submset of M ([11]).

2.7. The five axioms of Peano ([28], [29], [30])

If we assume the existence of a set N with the following properties:

N(i): There exist an element $1 \in N$

N(ii): For every $n \in N$, there exist an element $\sigma(n) \in N$ such that $\{(n, \sigma(n)) \mid n \in N\}$ is a function

N(iii): $1 \notin \sigma(n)$

N(iv): σ is one to one

N(v): If P is any subset of N such that $1 \in P$ and $\sigma(n) \in P$ $\forall$ $n \in P$, then $P = N$

then such a set N is defined to be a set of natural numbers and the elements of this set to be natural numbers. $(N, 1, \sigma)$ is called a natural number system and σ is called successor function.

2.7.1. ***Note:*** Any two natural number systems defined by Peano's axiomatic definition are isomorphic ([28], [29], [30]).

2.8. Iteration theorem ([29], [30])

Let $(N, 1, \sigma)$ be any natural number system and X be any non-empty set. Given $x \in X$ and a function $\varphi : X \to X$, then $\exists$ a unique function $\phi : N \to X$ such that $\phi(1) = x$ and $\phi(\sigma(n)) = \varphi(\phi(n))$ $\forall n \in N$.

2.9. Theorem: Addition of two natural numbers ([29], [30])

There exists a unique function $\alpha : N \times N \to N$ with the following properties:

B(i): $\alpha(p, 1) = \sigma(p)$ $\forall$ $p \in N$.

B(ii): $\alpha(p, \sigma(n)) = \sigma(\alpha(p, n))$ $\forall$ $p, n \in N$.

2.9.1. ***Note:*** $\alpha(p, n)$ is denoted by $p + n$ and is defined as the addition of two natural numbers.

2.9.2. ***Note:*** $\sigma(p) = p + 1 \forall p \in N$:

Proof: From B(i) of 2.9, $\forall p \in N$, $\alpha(p, 1) = \sigma(p)$ $\Rightarrow p + 1 = \sigma(p)$ (by Note 2.9.1)

2.10. Theorem: Multiplication of two natural numbers ([29], [30])

$\exists$ a unique function $\beta : N \times N \to N$ with the following properties:

M(i): $\beta(q, 1) = q$ $\forall$ $q \in N$.

M(ii): $\beta(q, \sigma(m)) = \alpha(q, \beta(q, m))$ $\forall$ $q, m \in N$.

2.10.1. ***Note:*** $\beta(q, m)$ is denoted by qm and is defined as the multiplication of two natural numbers.

2.11. *Lemma:* For every natural number p, either $p=1$ or $p=\sigma(q)$ for some natural number q:

Proof: Let $E=\{1\}\cup\{p\in N \mid p=\sigma(q)$ for some $q\in N\}$.

Then $1\in E$.

Let $p\in E$.

If $p=1$ then $\sigma(1)\in E$, by definition of E.

If $p\neq 1$ then by definition of E, $\exists q\in N$ such that $p=\sigma(q)$.

Therefore, $\sigma(p)=\sigma(\sigma(q))$.

Therefore, $\sigma(p)\in E$.

In either case, $\sigma(p)\in E$.

Therefore, $p\in E \Rightarrow \sigma(p)\in E$.

Therefore, by N(v) of 2.7, $E=N$.

Hence the lemma.

2.11.1. Note: So, for any $p(\neq 1)\in N$, there is $q\in N$ such that $\sigma(q)=p$. q is unique since σ is one-one. We define such $q=p-1$.

2.12. *Theorem:* $\forall p\in N-\{1\}$, $p=\sigma^{(p-1)}(1)$.

Proof: Let $T=\{1\}\cup\{p\in N \mid p=\sigma^{(p-1)}(1)\}$.

Then $1\in T$.

Let $p\in T$.

If $p=1$,

then $\sigma(p)=\sigma(1)=\sigma^{(p)}(1)=\sigma^{(\sigma(p)-1)}(1)$

[since $p=1\Rightarrow\sigma(p)\neq 1\Rightarrow\sigma(p)\in N-\{1\}$, so by lemma 2.11, $\exists(\sigma(p)-1)\in N$ such that $\sigma(\sigma(p)-1)=\sigma(p)\Rightarrow\sigma(p)-1=\sigma(p)$, since σ is one-one].

Therefore, $\sigma(p)\in T$ in this case.

If $p\neq 1$ then by definition of T,

$p=\sigma^{(p-1)}(1)$

$\Rightarrow\sigma(p)=\sigma(\sigma^{(p-1)}(1)) \;=\sigma^{(p)}(1)\; =\sigma^{(\sigma(p)-1)}(1)$

[since $\sigma(\sigma(p)-1)=\sigma(p)\Rightarrow\sigma(p)-1=p$ as σ is one-one].

Therefore, $\sigma(p)\in T$ in this case too.

Therefore, $\sigma(p)\in T$ in either case.

Therefore, $p\in T\Rightarrow\sigma(\mathrm{p})\in \mathrm{T}$,

Therefore, by N(v) of 2.7, $T=N$.

Hence the theorem.

2.13. Theorem: α obey commutative property on N ([29], [30]). So,

$B'(i)$: $\alpha(1,p)=\sigma(p)\quad\forall p\in N$.

$B'(ii)$: $\alpha(\sigma(n),p)=\sigma(\alpha(n,p))\quad\forall p,q\in N$.

2.13.1. Note: $\alpha(\sigma^{(n)}(1),p)=\sigma^{(n)}(\alpha(1,p))$ $\forall n\in N$ and $\forall p\in N$:

Proof: The proof is straight forward.

2.14. Theorem: β obey commutative property on N. ([29], [30]).

So,

M'(i): $\beta(1,q)=q\ \forall\ q\in N$.

M'(ii): $\beta(\sigma(m),q)=\alpha(\beta(q,m),q)\ \forall\ q,m\in N$.

3. Axiomatic Definition of Multi-Natural Numbers

Now we shall define here multi-natural numbers axiomatically. Our foundation is obviously ordinary natural number system $(N,1,\sigma)$ which is defined axiomatically by Peano.

3.1. *Definition*

Axiom1: For all $p,q\in N$, there exist a multi-natural number denoted by N_p^q.

Axiom 2: Two multi-natural numbers N_p^q and N_r^s are equal iff $p=r$ and $q=s$.

Axiom 3: For any multi-natural number N_p^q, $p,q\in N$, there exist a multi-natural number $N_{\sigma(p)}^q$ (defined to be support successor of N_p^q) and another multi-natural number $N_p^{\sigma(q)}$ (defined to be multiplicity successor of N_p^q).

Axiom 4: N_1^q, $\forall q\in N$, is not support successor of any multi-natural number, also, N_p^1, $\forall p\in N$, is not multiplicity successor of any multi-natural number.

Axiom 5: Let $P(N_p^q)$ be any proposition involving a multi-natural number N_p^q. Suppose that $P(N_1^1)$ is true. Also suppose that whenever $P(N_p^q)$ is true, then $P(N_{\sigma(p)}^q)$ and $P(N_p^{\sigma(q)})$ both are also true. Then $P(N_p^q)$ is true for every multi-natural number N_p^q.

Let us denote the set of all multi-natural numbers as $m(N)$. Also, let us define $p\in N$ and $q\in N$ as respectively the support and the multiplicity of a multi-natural number N_p^q.

3.2. *Successor Functions*

3.2.1. Support Successor Function: $S: m(N)\to m(N)$ defined by $S(N_p^q) = N_{\sigma(p)}^q$, $N_p^q \in m(N)$ is the support successor function.

3.2.2. ***Multiplicity Successor Function***: $M : m(N) \to m(N)$ defined by $M(N_p^q) = N_p^{\sigma(q)}$, $N_p^q \in m(N)$ is the multiplicity successor function.

3.2.3. ***Note***: S and M both are one to one since σ is one to one. i.e., different multi-natural numbers have different multiplicity successors as well as different support successors.

3.3. Definition of Multi-natural numbers using successor Functions:

We assume the existence of a set $m(N)$ with the following properties:

Axiom 1: For all $p, q \in N$, there exist a member of $m(N)$, denoted by N_p^q.

Axiom 2: Two multi-natural numbers N_p^q and N_r^s are equal iff $p = r$ and $q = s$.

Axiom 3: For every $N_p^q \in m(N)$, there exist an element $S(N_p^q) \in m(N)$ such that $\{(N_p^q, S(N_p^q)) : N_p^q \in m(N)\}$ is a function, also there exist an element $M(N_p^q) \in m(N)$ such that $\{(N_p^q, M(N_p^q)) : N_p^q \in m(N)\}$ is a function.

Axiom 4: $N_1^q \notin S(m(N)) \forall q \in N$ also $N_p^1 \notin M(m(N)) \forall p \in N$.

Axiom 5: If $m(P)$ be any subset of $m(N)$ such that $N_1^1 \in m(P)$ and $S(N_p^q), M(N_p^q) \in m(P)$ for all $N_p^q \in m(P)$. Then $m(P) = m(N)$.

3.4. ***Note:*** $S \circ M = M \circ S$ **:**

Both the functions $S \circ M$ and $M \circ S$ are functions from $m(N)$ into $m(N)$.

Again, $\forall\ N_p^q \in m(N)$, $S \circ M(N_p^q)$

$= S(M(N_p^q)) = S(N_p^{\sigma(q)}) = N_{\sigma(p)}^{\sigma(q)}$

$= M(N_{\sigma(p)}^q) = M(S(N_p^q)) = M \circ S(N_p^q)$.

Therefore, $S \circ M = M \circ S$.

3.5. ***Note:*** Define $S^{(p)} = S \circ S \circ \ldots \circ S$ (p times), $p \in N$.

Also, $S^{(p)}(N_r^s) = N^s_{\sigma^{(p)}(r)}$ (by repeated application of 3.2.1).

3.6. ***Note:*** Define $M^{(q)} = M \circ M \circ \ldots \circ M$ (q times), $q \in N$.

Also $M^{(q)}(N_r^s) = N_r^{\sigma^{(q)}(s)}$ (by repeated application of 3.2.2).

3.7. ***Note:*** $S^{(p)} \circ M^{(q)} = M^{(q)} \circ S^{(p)}$ $\forall\ p, q \in N$ **:** The proof is similar to 3.4.

3.8. ***Elementary Properties of*** $m(N)$

3.8.1. $m(N)$ ***is infinite:***

Proof: By Note 3.2.3, S is one to one.

Furthermore, $N_1^1 \notin S(m(N))$ by axiom 4 of 3.3.

So, S is onto some proper subset of $m(N)$.

Hence $m(N)$ is infinite.

3.8.2. $S(N_p^q) \neq N_p^q$ ***for all*** $N_p^q \in m(N)$ **:**

Proof:

Let $m(P) = \{N_p^q \in m(N) : S(N_p^q) \neq N_p^q\}$.

Since by axiom 4 of 3.3, $N_1^1 \notin S(m(N))$,`

so $N_1^1 \in m(P)$.

Suppose, $N_p^q \in m(P)$.

If $S(S(N_p^q)) = S(N_p^q)$, then $S(N_p^q) = N_p^q$ (since S is one to one from 3.2.3), contradicting $N_p^q \in m(P)$.

So $S(S(N_p^q)) \neq S(N_p^q)$

consequently, $S(N_p^q) \in m(P)$ whenever $N_p^q \in m(P)$.

Again $S(M(N_p^q)) = M(N_p^q)$

$\Rightarrow S \circ M(N_p^q) = M(N_p^q)$

$\Rightarrow M \circ S(N_p^q) = M(N_p^q)$ (by 3.4)

$\Rightarrow M(S(N_p^q)) = M(N_p^q)$

$\Rightarrow S(N_p^q) = N_p^q$ (since M is one to one from 3.2.3), contradicting $N_p^q \in m(P)$.

So $S(M(N_p^q)) \neq M(N_p^q)$, consequently, $M(N_p^q) \in m(P)$ whenever $N_p^q \in m(P)$.

So, by axiom 5 of 3.3, $m(P) = m(N)$.

Therefore, $S(N_p^q) \neq N_p^q \quad \forall N_p^q \in m(N)$.

3.8.3. $M(N_p^q) \neq N_p^q$ ***for all*** $N_p^q \in m(N)$ ***:***
Proof is similar to 3.8.2.

3.8.4. For all $N_p^q \in m(N) - \{N_1^t : t \in N\}$ ***, there exist*** $N_k^q \in m(N)$ ***such that*** $N_p^q = S(N_k^q)$ ***:***

Proof:

Let $m(P) = \{N_1^t : t \in N\} \cup \{N_p^q : \exists N_k^q \in m(N)$ such that $N_p^q = S(N_k^q)\}$.

Then by definition of $m(P)$, $N_1^t \in m(P) \forall t \in N$, so $N_1^1 \in m(P)$.

Suppose, $N_p^q \in m(P)$.

$S(N_p^q)$ is a support successor of $N_p^q \in m(N)$.

Hence, $S(N_p^q) \in m(P)$ whenever $N_p^q \in m(P)$.

Now we shall show that $M(N_p^q) \in m(P)$.

In this connection if $p = 1$,

then $M(N_p^q) = M(N_1^q) = N_1^{\sigma(q)} \in m(P)$ (since by definition of $m(P)$, $N_1^t \in m(P) \forall t \in N$ and $\sigma(q) \in N \quad \forall q \in N$).

If $p \neq 1$ then $M(N_p^q) = M(S(N_k^q))$

[since $N_p^q \in m(P)$ and $p \neq 1$ so by definition of $m(P)$, there exist $N_k^q \in m(N)$ such that $N_p^q = S(N_k^q)$]

$= M \circ S(N_k^q) = S \circ M(N_k^q) = S(M(N_k^q)) \in m(P)$.

Therefore, in either case $M(N_p^q) \in m(P)$ whenever $N_p^q \in m(P)$.

Therefore, by axiom 5 of 3.3, $m(P) = m(N)$.
Hence the result.

3.8.5. For all $N_p^q \in m(N) - \{N_t^1 : t \in N\}$ ***, there exist*** $N_p^k \in m(N)$ ***such that*** $N_p^q = M(N_p^k)$ ***:***
The proof is similar to 3.8.4.

3.8.6. If $N_p^q \in m(N)$ ***, then*** $p = 1$ ***iff*** $N_p^q \notin S(m(N))$ ***:***

$p = 1 \Rightarrow N_p^q \notin S(m(N))$ follows immediately from axiom 4 of 3.3.
Also, $N_p^q \notin S(m(N)) \Rightarrow p = 1$ follows from 3.8.4.

3.8.7. If $N_p^q \in m(N)$ ***, then*** $q = 1$ ***iff*** $N_p^q \notin M(m(N))$ ***:***

$q = 1 \Rightarrow N_p^q \notin M(m(N))$ follows immediately from axiom 4 of 3.3.
Also, $N_p^q \notin M(m(N)) \Rightarrow q = 1$ follows from 3.8.5.

4. AXIOMATIC DEFINITION OF ADDITION ON $m(N)$

4.1. Definition of Addition:

A function $A : m(N) \times m(N) \to m(N)$ with the following properties:

Axiom 1: $A(N_p^q, N_1^1) = S(N_p^q)$,

Axiom 2: $A(N_p^q, S(N_n^m)) = S(A(N_p^q, N_n^m))$,

Axiom 3: $A(N_p^q, M(N_n^m)) = M^{(q)}(A(N_p^q, N_n^m))$,

N_p^q, $N_n^m \in m(N)$, is called addition of two multi-natural numbers.

4.2. Theorem (Existence and uniqueness theorem of addition):

There exists a unique addition function.

Proof:

Let any $p \in N$ be given.

By iteration theorem 2.8 with $X = N$ and $x = \sigma(p)$, $\varphi = \sigma$, $\exists$ a unique function $\alpha_p : N \to N$ such that $\alpha_p(1) = \sigma(p)$ and $\alpha_p \circ \sigma(n) = \sigma \circ \alpha_p(n) \ \forall \ n \in N$.

Similarly, let any $q \in N$ be given.

By iteration theorem 2.8 with $X = N$ and $x = q$, $\varphi = \alpha_q$, $\exists$ a unique function $\beta_q : N \to N$ such that $\beta_q(1) = q$ and $\beta_q \circ \sigma(m) = \alpha_q \circ \beta_q(m) \ \forall \ m \in N$.

Let us define $A : m(N) \times m(N) \to m(N)$ by $A(N_p^q, N_n^m) = N_{\alpha_p(n)}^{\beta_q(m)}$, $N_p^q, N_n^m \in m(N)$.

Then immediately $A : m(N) \times m(N) \to m(N)$ is a function since $\alpha_p : N \to N$ and $\beta_q : N \to N$ both are functions.

Now $A(N_p^q, N_1^1) = N_{\alpha_p(1)}^{\beta_q(1)} = N_{\sigma(p)}^q = S(N_p^q)$.

Therefore, A satisfies Axiom 1 of 4.1.

Also, $A(N_p^q, S(N_n^m)) = A(N_p^q, N_{\sigma(n)}^m)$

$= N_{\alpha_p(\sigma(n))}^{\beta_q(m)} = N_{\sigma(\alpha_p(n))}^{\beta_q(m)}$

$= N_{\sigma(\alpha_p(n))}^{\beta_q(m)} = S(N_{\alpha_p(n)}^{\beta_q(m)})$

$= S(A(N_p^q, N_n^m))$, $\forall\ N_n^m \in m(N)$.

Therefore, A satisfies Axiom 2of 4.1.

Also, $A(N_p^q, M(N_n^m)) = A(N_p^q, N_n^{\sigma(m)})$

$= N_{\alpha_p(n)}^{\beta_q(\sigma(m))} = N_{\alpha_p(n)}^{\alpha_q \circ \beta_q(m)} = N_{\alpha_p(n)}^{\alpha_q \circ \beta_q(m)}$

$= N_{\alpha_p(n)}^{\alpha(q,\beta_q(m))}$

$$= \begin{cases} N_{\alpha_p(n)}^{\alpha(1,\beta_q(m))} if q = 1 \\ N_{\alpha_p(n)}^{\alpha(\sigma^{(q-1)}(1),\beta_q(m))} if q \neq 1 \end{cases}$$

(By theorem 2.12)

$$= \begin{cases} N_{\alpha_p(n)}^{\sigma(\beta_q(m))} if q = 1 \\ N_{\alpha_p(n)}^{\sigma^{(q-1)}(\alpha(1,\beta_q(m)))} if q \neq 1 \end{cases}$$

(By Note 2.13.1)

$$= \begin{cases} N_{\alpha_p(n)}^{\sigma(\beta_q(m))} if q = 1 \\ N_{\alpha_p(n)}^{\sigma^{(q-1)}(\sigma(\beta_q(m)))} if q \neq 1 \end{cases}$$

(By $B'(i)$ of 2.13)

$$= \begin{cases} N_{\alpha_p(n)}^{\sigma(\beta_q(m))} if q = 1 \\ N_{\alpha_p(n)}^{\sigma^{(q)}(\beta_q(m))} if q \neq 1 \end{cases}$$

$$= \begin{cases} M(N_{\alpha_p(n)}^{\beta_q(m)}) if q = 1 \\ M^{(q)}(N_{\alpha_p(n)}^{\beta_q(m)}) if q \neq 1 \end{cases}$$

(By 3.6)

$= M^{(q)}(N_{\alpha_p(n)}^{\beta_q(m)})$

$= M^{(q)}(A(N_p^q, N_n^m))$, $\forall\ N_n^m \in m(N)$.

Therefore, A satisfies Axiom 3 of 4.1.

Therefore, the function A satisfies all the three axioms of the definition given in 4.1.

Let $A': m(N) \times m(N) \to m(N)$ be another function which also satisfies all the three axioms of the definition given in 4.1.

For any $N_p^q \in m(N)$ let $m(P)$

$= \{N_n^m \in m(N) \mid A(N_p^q, N_n^m) = A'(N_p^q, N_n^m)\}$.

Then since $A(N_p^q, N_1^1) = A'(N_p^q, N_1^1)$,

so $N_1^1 \in m(P)$.

Let $N_n^m \in m(P)$.

Then $A(N_p^q, S(N_n^m)) = S(A(N_p^q, N_n^m))$ (by axiom 2) $= S(A'(N_p^q, N_n^m))$ (since $N_n^m \in m(P)$) $= A'(N_p^q, S(N_n^m))$ (by axiom 2).

Therefore, $S(N_n^m)$ $\in m(P)$ whenever $N_n^m \in m(P)$.

Again,

$A(N_p^q, M(N_n^m)) = M^{(q)}(A(N_p^q, N_n^m))$ (by axiom 3) $= M^{(q)}(A'(N_p^q, N_n^m))$ (since $N_n^m \in m(P)$) $= A'(N_p^q, M(N_n^m))$ (by axiom 3).

Therefore, $M(N_n^m)$ $\in m(P)$ whenever $N_n^m \in m(P)$.

Therefore, by axiom 5 of definition of multi-natural number (3.1), $m(P) = m(N)$.

Therefore, $A(N_p^q, N_n^m) = A'(N_p^q, N_n^m)$ $\forall\ N_p^q,\ N_n^m \in m(N)$.

So A exist is uniquely.

4.3. Theorem:

The function $\bar{A}: m(N) \times m(N) \to m(N)$ defined by $\bar{A}(N_p^q, N_n^m) = N_{p+n}^{qm}$, $N_p^q, N_n^m \in m(N)$, satisfies Axiom 1-3 of Addition:

Proof: $\bar{A}(N_p^q, N_1^1) = N_{p+1}^q = N_{\sigma(p)}^q$ (From 2.9.2) $= S(N_p^q)$ (From 3.2.1).

So $\bar{A}$ satisfies Axiom 1 of 4.1.

$\bar{A}(N_p^q, S(N_n^m)) = \bar{A}(N_p^q, N_{\sigma(n)}^m)$ (From 3.2.1)

$= N_{p+(\sigma(n))}^{qm} = N_{\sigma(p+n)}^{qm}$ (By B(ii) of 2.9)

$= S(N_{p+n}^{qm})$ (From 3.2.1) $= S(\bar{A}(N_p^q, N_n^m))$.

So $\bar{A}$ satisfies Axiom 2 of 4.1.

Finally, $\bar{A}(N_p^q, M(N_n^m))$ $= \bar{A}(N_p^q, N_n^{\sigma(m)})$ (From 3.2.2) $= N_{p+n}^{q\sigma(m)} = N_{p+n}^{q+qm}$ (By M(ii) of 2.10) $= \begin{cases} N_{p+n}^{1+m} if q = 1 \\ N_{p+n}^{\sigma^{(q-1)}(1)+qm} if q \neq 1 \end{cases}$

(From theorem 2.12)

$$= \begin{cases} N_{p+n}^{\sigma(m)} if q = 1 \\ N_{p+n}^{\sigma^{(q-1)}(1+qm)} if q \neq 1 \end{cases}$$

(From $B'(i)$ of 2.13 and 2.13.1)

$$= \begin{cases} M(N_{p+n}^m) if q = 1 \\ M^{(q-1)}(N_{p+n}^{1+qm}) if q \neq 1 \end{cases}$$

(From 3.2.2 and 3.6)

$$= \begin{cases} M(N^{1.m}_{p+n}) if q=1 \\ M^{(q-1)}(N^{\sigma(qm)}_{p+n}) if q\neq 1 \end{cases}$$

(From $M'(i)$ of 2.14 and $B'(i)$ of 2.13)

$$= \begin{cases} M(N^{1.m}_{p+n}) if q=1 \\ M^{(q-1)}(M(N^{qm}_{p+n})) if q\neq 1 \end{cases}$$

(From 3.2.2)

$$= \begin{cases} M(N^{1.m}_{p+n}) if q=1 \\ M^{(q)}(N^{qm}_{p+n}) if q\neq 1 \end{cases}$$

(From 3.6)

$$= \begin{cases} M(\bar{A}(N^q_p + N^m_n)) if q=1 \\ M^{(q)}(\bar{A}(N^q_p + N^m_n)) if q\neq 1 \end{cases}$$

$$= M^{(q)}(\bar{A}(N^q_p + N^m_n)).$$

So $\bar{A}$ satisfies Axiom 3 of 4.1.
Hence the theorem.

4.3.1. ***Note:*** The Addition function A defined in 4.1. is unique (By theorem 4.2.), so we can write

$$A(N^q_p, N^m_n) = N^{qm}_{p+n},\ N^q_p, N^m_n \in m(N).$$

4.3.2. ***Note:*** According to Note 4.3.1., Support of $A(N^q_p, N^m_n)$ is $(p+m)$ and multiplicity of $A(N^q_p, N^m_n)$ is qm.

4.3.3. ***Note:*** From now on, we will denote $A(N^q_p, N^m_n)$ as $N^q_p + N^m_n$.

4.3.4. ***Note:*** Combining axiom 2 & 3, we can write,

$$A(N^q_p, S\circ M(N^m_n))$$
$$= S\circ M^{(q)}(S(N^q_p, N^m_n)).$$

4.4. Properties of Addition: Following properties of addition can be deduced:

4.4.1. ***Property:*** $S(N^q_p) = N^q_p + N^1_1$.

4.4.2. ***Property:***

$$N^q_p + (N^t_k + N^1_1) = (N^q_p + N^t_k) + N^1_1$$
$$\forall N^q_p, N^t_k \in m(N)$$

4.4.3. ***Property:***

$$N^1_1 + N^q_p = N^q_p + N^1_1 \forall N^q_p \in m(N)$$

4.4.4. ***Property:***

$$(N^q_p + N^1_1) + N^t_k = (N^q_p + N^t_k) + N^1_1,$$
$$\forall N^q_p, N^t_k \in m(N).$$

4.4.5. ***Property:***

$$N^q_p + N^t_k = N^t_k + N^q_p \forall N^q_p, N^t_k \in m(N)$$

(the commutative law of addition).

4.4.6. ***Property:***

$$(N^q_p + N^t_k) + N^n_m = N^q_p + (N^t_k + N^n_m),$$
$$\forall N^q_p, N^t_k, N^n_m \in m(N)$$

(the associative law of addition.)

4.4.7. ***Property:***

$$N^q_p \neq N^q_p + N^t_k \forall N^q_p, N^t_k \in m(N) \quad :$$

Proof follows from 4.2.2.

4.4.8. ***Property:***

$$N^q_p + N^t_k = N^q_p + N^n_m \Rightarrow N^t_k = N^n_m,$$
$$\forall N^q_p, N^t_k, N^n_m \in m(N)$$

(the cancellation law for addition).

4.5. Counting

4.5.1. ***Examples:*** We know that the actual essence of natural number is counting. But the process of counting of elements of a multiset is not a natural process due the presence of the copies of the elements.

If we consider the copies as different from original, we must use natural numbers for counting, otherwise we must give number of distinct elements and number of copies for each.

But here we introduce a different way to count the multi-number of elements of a multiset by giving the following examples (definition given in 4.3.6):

Example1: The multi-number of elements of the multiset $X = \{a,a,a\}$ is N^3_1. The multi-number of elements of the multiset $Y = \{b,b\}$ is N^2_1 also, the multi-number of elements of $Z = \{c,c\}$ is N^2_1.

Example2: Consider the multiset $A = \{a,a,a,b,b,c,c\}$ which is a multiset with $\{a,b,c\}$ as the support set with 12 multiplicity (by fundamental law of association). In this connection, if we say that the set A has 3 elements (discarding the copies) or if we say that the set has 7 elements (otherwise), then the essence of multiset will drive out.

In this paper, we define the multi-number of elements of A is $(N^3_1 + N^2_1) + N^2_1 = N^6_2 + N^2_1 = N^{12}_3$.

We claim that this type of counting using multi-natural number preserves the essence of counting of the elements of a multiset and is favored by the addition of two multi-natural numbers. It's worth noting that the multi-number of elements of $B = \{a,a,a,a,a,a,a,a,a,a,a,a,b,c\}$ is $N^{12}_1 + N^1_1 + N^1_1 = N^{12}_3$.

4.5.2. Full submsets: Again, we know that full submsets are very important submsets in multiset context. Also, the number of full submsets of $A=\{a,a,a,b,b,c,c\}$ is 12. Also, all the full msubsets of A are such msubsets of A for which support is $\{a,b,c\}$. Therefore, the multiset A has the support set $\{a,b,c\}$ and has 12 full submsets but the converse is not true. The support set $\{a,b,c\}$ generates many multisets except A all of which have 12 full submsets. Let $\rho(p,q)$ be the number of positive integral solutions of the equation $x_1.x_2\ldots x_p=q$, then $\rho(p,q)$ is the number of multisets each having same support set with p elements, same number of full submsets and sane multi-number of elements.

4.5.3. Multiple Roots: The list of roots of the polynomial equation $(x-1)^2(x-3)=0$ is 1,1,3. Therefore, the multiset of roots of $(x-1)^2(x-3)=0$ is $\{1,1,3\}$. But the equation has two distinct roots 1 and 3 giving the set of roots of the equation is $\{1,3\}$. But if about the collection of roots of the equation we represent the set $\{1,3\}$, then it will be very unjustified and the representation will be inadequate. The essence of multiness of the multiset $\{1,1,3\}$ of the roots of the equation $(x-1)^2(x-3)=0$ will be preserved if we consider the set $\{1,3\}$ of the roots with multiplicity 2 which in turn support the multi-number of the roots of the equation is N_2^2.

4.5.4. N_p^q -Count Power msets: Let $q=q_1^{r_1}.q^{r_2}{}_2\ldots q_k{}^{r_k}$ be the canonical form of q where $q_1,q_2,\ldots,q_k$ ($q_1<q_2<\ldots<q_k$) is the complete list of prime factors of q with multiplicities $r_1,r_2,\ldots,r_k$ respectively.

Let us consider the equation $x_1.x_2\ldots x_p=q_1^{r_1}.q_2^{r_2}..q_k^{r_k}$, where $x_i\in N$.

Let us consider the equation $x_1+x_2+\ldots+x_p=r_i$, $i=1,2,\ldots,k$, $x_i\in N\cup\{0\}$. The number of solutions of the second equation is $\binom{r_i+p-1}{p-1}$, $i=1,2,\ldots,k$.

So, the number of solutions of the first equation is $\prod_{i=1}^{k}\binom{r_i+p-1}{p-1}=\rho(N_p^q)$, say.

Let $\mu=(\mu_1,\mu_2,\ldots,\mu_p)$ be one of such solution of first equation. Then for each set $A=\{a_1,a_2,\ldots,a_p\}$ having p elements and for each μ, there exist a multiset $\mu(A)$ (with A as the support set) for which $C_{\mu(A)}(a_i)=\mu_i$, $i=1,2,\ldots,p$. Let us denote the set of all such multisets as $\wp(A,(N_p^q))$ and define it as ' N_p^q -count power mset of A'.

Also in this case, $\sum_{i=1}^{p}N_1^{\mu_i}=N_p^q$.

4.5.5. Single Whole Submsets, Single Submsets: Let us now define a submset N of a mset M drawn from a set X as a 'single whole submset' if $C_{N\cap M}(x)=C_M(x)$ or 0 $\forall x\in X$ and $\{x\in X:C_{N\cap M}(x)=C_M(x)\}$ is a singleton set, say $\{n\}$, then let us denote it as $M_{\{n\}}(=N)$, i.e., a single whole submset is such a submset of a multiset for which exactly one element of the support set belongs to it with the same count as in the mset.

Let us now define a mset as a single mset if it has a singleton support set.

So immediately, each mset can be expressed as a union of all its single whole submsets. Therefore, $M=\bigcup_{n\in X}M_{\{n\}}$.

In this connection, we note that single whole submsets are pairwise disjoint.

4.5.6. Multi Number of Elements in a Multiset: Which we have illustrated in the example1 and example2 of 4.3.1, to represent the concept of multi-number of elements in a multiset, we now defining that as follows:

Let N be a single mset also let x is the only element of N with $C_N(x)=n$.

Let us now define N_1^n as the multi-number of elements in N. Next let us consider an mset M whose support $N^*=\{x_1,x_2,\ldots,x_n\}$ is a finite set and multiplicity of each of its elements is finite and is given by the count function $C_N(x_i)=t_i$, $i=1,2,\ldots,n$. Then we define the multi number of elements in M as the sum of the multi-numbers of the elements in all its single whole submsets i.e., $N_1^{t_1}+N_1^{t_2}+\ldots+N_1^{t_n}=N_n^t$ where $t=t_1t_2\ldots t_n$.

5. AXIOMATIC DEFINITION OF MULTIPLICATION ON $m(N)$

5.1. Definition of Multiplication:

A function $P:m(N)\times m(N)\rightarrow m(N)$ with the following properties:

Axiom 1: $P(N_p^q,N_1^1)=N_p^q$,

Axiom 2: $P(N_p^q,S(N_n^m))=S^{(p)}(P(N_p^q,N_n^m))$,

Axiom3: $P(N_p^q,M(N_n^m))=M^{(q)}(P(N_p^q,N_n^m))$,

is called multiplication of two multi-natural numbers.

5.2. Theorem (Existence and uniqueness theorem of multiplication):

There exists a unique multiplication function.

Proof:

Let any $p \in N$ be given.

By iteration theorem 2.8 with $X = N$ and $x = \sigma(p)$, $\varphi = \sigma$, $\exists$ a unique function $\alpha_p : N \to N$ such that $\alpha_p(1) = \sigma(p)$ and $\alpha_p \circ \sigma(n) = \sigma \circ \alpha_p(n)\ \forall\ n \in N$.

Similarly, let any $q \in N$ be given.

By iteration theorem 2.8 with $X = N$ and $x = q$, $\varphi = \alpha_q$, $\exists$ a unique function $\beta_q : N \to N$ such that $\beta_q(1) = q$ and $\beta_q \circ \sigma(m) = \alpha_q \circ \beta_q(m)\ \forall\ m \in N$.

Let us define $P : m(N) \times m(N) \to m(N)$ by $P(N_p^q, N_n^m) = N_{\beta_p(n)}^{\beta_q(m)}$, $N_p^q, N_n^m \in m(N)$.

Then immediately $P : m(N) \times m(N) \to m(N)$ is a function since $\beta_p : N \to N$ and $\beta_q : N \to N$ both are functions.

Then proceeding in the similar argument as in theorem 4.2 we can show that P satisfies all the three axioms of the definition given in 5.1. also, we can show that P exists uniquely.

5.3. Theorem:

The function $\bar{P} : m(N) \times m(N) \to m(N)$ defined by $\bar{P}(N_p^q, N_n^m) = N_{pn}^{qm}$, $N_p^q, N_n^m \in m(N)$, satisfies Axiom 1-3 of multiplication:

The proof is similar to the proof of the theorem 4.3.

5.3.1. ***Note:*** The multiplication function P defined in 5.1. is unique (By theorem 5.2.), so we can write

$P(N_p^q, N_n^m) = N_{pn}^{qm}$, $N_p^q, N_n^m \in m(N)$.

5.3.2. According to Note 3.3.1., Support of $P(N_p^q, N_n^m)$ is pn and multiplicity of $P(N_p^q, N_n^m)$ is qm.

5.3.3. ***Note:*** From now on, we will denote $P(N_p^q, N_n^m)$ as $N_p^q . N_n^m$.

5.3.4. ***Note:*** Combining axiom 1 and axiom 2 of 5.1., we can write

$P(N_p^q, S \circ M(N_n^m))$

$= S^{(p)} \circ M^{(q)}(P(N_p^q, N_n^m))$.

5.4. Properties of multiplication: Following properties of multiplication can be deduced:

5.4.1. $P(N_1^1, N_p^q) = N_p^q = P(N_p^q, N_1^1)$.

5.4.2. $P(N_p^q, N_n^m) = P(N_n^m, N_p^q)$ (the commutative law of multiplication).

5.4.3. $P(P(N_m^n, N_p^q), N_r^s) = P(N_m^n, P(N_p^q, N_r^s))$ (the associative law of multiplication).

5.4.4. In general,

$P(N_p^q, A(N_m^n, N_r^s))$

$\neq A(P(N_p^q, N_m^n), P(N_p^q, N_r^s))$,

(i.e., P do not obey distributive property over A).

5.4.5. ***Note:*** But in particular,

$P(N_p^1, A(N_m^n, N_r^s))$

$= A(P(N_p^1, N_m^n), P(N_p^1, N_r^s))$.

5.5. Following important results can be deduced:

1) $N_m^n + N_m^n + \ldots$(up to k times) $= N_{km}^{n^k}$.

2) $N_m^n . N_m^n \ldots$(up to k factors) $= N_{m^k}^{n^k}$.

3) $N_m^n + N_m^n + \ldots\ldots$ (up to k times)

$= N_m^1 . N_1^n + N_m^1 . N_1^n + \ldots + N_m^1 . N_1^n$

$= N_m^1 . N_k^{n^k} = N_k^{n^k} . N_m^1$.

6. ORDER ON $m(N)$

Our multi-natural number system seems to be taking shape very nicely. We can add them, multiply them and even take the power of multi-natural numbers in some cases. Now we need to order our multi-natural numbers.

6.1. Definition

We say that for $N_p^q, N_m^n \in m(N)$, $N_p^q = N_m^n$ iff $(p = m$ as well as $q = n)$. Also we say that for $N_p^q, N_m^n \in m(N)$, N_p^q is greater than N_m^n and we write $N_p^q > N_m^n$ if $\exists N_r^s \in m(N)$ such that $N_p^q = N_m^n + N_r^s (= N_{m+r}^{ns})$, i.e., (if $p > m$ as well as $n \mid q$).

We say that N_p^q is greater than or equal to N_m^n and we write $N_p^q \geq N_m^n$ if $N_p^q > N_m^n$ or $N_p^q = N_m^n$, i.e., if $(p > m$ as well as $n \mid q)$ or if $(p = m$ as well as $n = q)$.

6.2. Properties of order on $m(N)$

6.2.1. ***Theorem:*** the relation $\geq$ on $m(N)$ is a partial order relation which is not total.

Proof: Reflexivity is clear since $N_p^q = N_p^q \forall N_p^q \in m(N)$.

Suppose for N_p^q, $N_m^n \in m(N)$ $N_p^q \geq N_m^n$ and $N_m^n \geq N_p^q$.
Therefore, $p = m$ as well as $n = q$.
Therefore, $N_p^q = N_m^n$.
Therefore, $\geq$ is antisymmetric on $m(N)$.
Finally, for $N_p^q, N_m^n, N_r^s \in m(N)$, let $N_p^q \geq N_m^n$ and $N_m^n \geq N_r^s$.

If at least one of the equality hold then immediately $N_p^q \geq N_r^s$. Otherwise, ($p > m$ as well as $n | q$) and ($m > r$ as well as $s | n$). Then ($p > r$ as well as $s | q$).
Therefore, $N_p^q \geq N_r^s$. Therefore, $\geq$ is a transitive relation on $m(N)$. Therefore, $\geq$ is a partial order relation on $m(N)$.

To show that the relation $\geq$ is not total, we note that N_3^2 is not greater than or equal to N_2^3 also N_2^3 is not greater than or equal to N_3^2. Therefore, $\geq$ is not a total order relation on $m(N)$.

6.2.2. Therefore $(m(N), \geq)$ is a poset but not a chain. Immediately $(m(N), \geq)$ do not obey the Law of Trichotomy.

6.2.3. $N_p^q > N_p^q$ is not true $\forall$ $N_p^q \in m(N)$:
Proof: Since p is not greater than p so the result is immediate.

6.2.4. $N_p^q \geq N_1^1$ $\forall N_p^q \in m(N)$:
Proof: Since $p \geq 1$ and $1 | q$ $\forall p \in N$ so the result follows immediately.

6.2.5. $(N_p^q + N_m^n) \geq N_p^q$ $\forall$ $N_m^n \in m(N)$:
Proof: From 5.1.3, $(N_p^q + N_m^n) = N_{pm}^{qn}$ which is immediately greater than or equal to N_p^q since ($pm > p$ as well as $q | qn$ for $p \neq 1$ and $q \neq 1$) or ($pm = p$ as well as $q = qn$ for $p = 1$ and $q = 1$).

6.2.6. $N_p^q > N_m^n \Leftrightarrow N_p^q + N_r^s > N_m^n + N_r^s$ $\forall$ $N_r^s \in m(N)$:
Proof: Let $N_p^q > N_m^n$.
Then $p > m$ as well as $n | q$.
Now $N_p^q + N_r^s = N_{pr}^{qs}$ and $N_m^n + N_r^s = N_{mr}^{ns}$.
Since ($p > m$ and $n | q$) so ($pr > mr$ $\forall r \in N$) as well as ($ns | qs$ $\forall s \in N$).
Therefore,
$N_{pr}^{qs} > N_{mr}^{ns}$ $\forall$ $N_r^s \in m(N)$
i.e., $N_p^q + N_r^s > N_m^n + N_r^s$ $\forall$ $N_r^s \in m(N)$.
Conversely let, $N_p^q + N_r^s > N_m^n + N_r^s$ $\forall$ $N_r^s \in m(N)$.
i.e., $N_{pr}^{qs} > N_{mr}^{ns}$ $\forall$ $N_r^s \in m(N)$.
Then ($pr > mr$ $\forall r \in N$) as well as ($ns | qs$ $\forall s \in N$).
Therefore, ($p > m$ and $n | q$).
So, $N_p^q > N_m^n$.
Hence the result.

6.2.7. $N_p^q > N_m^n$ and $N_r^s > N_t^u$ $\Rightarrow N_p^q + N_r^s > N_m^n + N_t^u$ $\forall$ $N_p^q, N_r^s, N_m^n \in m(N)$:
Proof: Let $N_p^q > N_m^n$ and $N_r^s > N_t^u$.
Then ($p > m$ and $n | q$) also ($r > t$ and $u | s$).
Now $N_p^q + N_r^s = N_{p+r}^{qs}$
and $N_m^n + N_t^u = N_{m+t}^{nu}$.
Since $p > m$ and $r > t$ so $p + r > m + t$
And since $n | q$ and $u | s$ so $nu | qs$.
Therefore, $N_{p+r}^{qs} > N_{m+t}^{nu}$.
Consequently, $N_p^q + N_r^s > N_m^n + N_t^u$.

6.2.8. $N_m^n \geq N_p^q \Leftrightarrow N_m^n + N_1^1 > N_p^q$, $\forall$ $N_p^q, N_m^n \in m(N)$: The proof is straight forward.

6.2.9. $\forall$ $N_m^n, N_p^q \in m(N)$, $N_p^q + N_m^n > N_p^q$:
The proof is straight forward.

6.2.10. $N_p^q + N_r^s = N_m^n + N_r^s$ $\Rightarrow N_p^q = N_m^n$ $\forall N_p^q, N_r^s, N_m^n \in m(N)$:
The proof is straight forward.

6.2.11. $N_m^n > N_p^q \Rightarrow N_m^n . N_r^s > N_p^q . N_r^s$, $\forall N_r^s \in m(N)$: The proof is straight forward.

6.2.12. $N_p^q.N_r^s = N_m^n.N_r^s \Rightarrow N_p^q = N_m^n$, $\forall N_p^q, N_r^s, N_m^n \in m(N)$ (cancellation law for multiplication): The proof is straight forward.

6.2.13. $N_p^q.N_m^n \geq N_p^q$: The proof is straight forward.

6.2.14. $N_m^n > N_p^q$ & $N_t^u > N_k^s$ $\Rightarrow$ $N_m^n.N_t^u > N_p^q.N_k^s$: The proof is straight forward.

6.2.15. Intuitively, we realise that for any multi-natural number, we can make the product 'as big as we please' by multiplying it with another suitable multi-natural number. Like the ordinary natural number system, it is also very fundamental property of the multi-natural number system. In this connection, we give the following theorem:

Theorem: For $N_p^q, N_m^n \in m(N)$, $\exists N_r^s \in m(N)$ such that $N_p^q.N_r^s > N_m^n$.

Proof: For any $N_p^q, N_m^n \in m(N)$ we take $N_r^s = M^{(n)} \circ S(N_m^n)$.

Then

$$N_p^q.(M^{(n)} \circ S(N_m^n)) = N_p^q.N_{(m+1)}^{2n} = N_{(m+1)}^{2qn} \rangle N_m^n,$$

(since $p(m+1)\rangle m$ and $n \mid 2qn$)

CONCLUSIONS

In this paper, we have defined and studied multi-natural number system from axiomatic point of view.

There is a lot of scope of future research work in the field of multi set. Specially, further study can be carried out in the following directions:

To study the possible extension process of Multi Natural Number System towards Multi Integer System, Multi Rational Number System, Multi Real Number System etc.

Also, to study thoroughly the properties of algebraic operations and order relations defined on them.

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