Fixed Point Theorems for Digital Contractive Type Mappings in Digital Metric Spaces

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Abstract**—***In this paper, we introduce a class Φ and define φ −contractive type mappings for digital metric spaces. We prove a crucial Lemma in digital metric spaces. Using this Lemma we prove existence and uniqueness of fixed point theorems in digital metric spaces. And we obtain Banach contraction principle in digital metric spaces as a corollary. We also give examples to illustrate our result.*

Keywords:

Digital image, Digital metric space, Banach contractive principle, $-contraction$, $\varphi - contractive$, Finite sequence, increasing and strictly decreasing sequence, $\alpha - admissible$.

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I. INTRODUCTION

Fixed point theory plays an important role in functional analysis, and it has wider applications in differential and integral equations. Fixed point theory, broadly speaking, demonstrates the existence, uniquenessand construction of fixed points of a function or a family of functions under diverse assumptions about the structure of the domain **X** (such as a metric space or normed linear space or a topological space) of the concerned functions.

The concept of a metric space was introduced by **M. Ferchet** [18] in **1906**.

Fixed point theory beginning from Banach contraction principle of **Banach** [1] **(1922)** with complete metric spaces as a background and went back to Brouwer fixed point theorem of **Brouwer** [7, 8] **(1910)** with $\mathbb{R}^n$ as background. It begins with some literature in **1960**'s goes up to **1990**'s which includes variants and generalizations of Banach contraction principle [9, 13, 20, 21, 35].

Extension and development of this fixed point theory other than metric spaces, which are generalizations of metric spaces, such as statistical metric spaces, Menger spaces, d – complete topological spaces, F – complete metric spaces, G – Metric spaces, Fuzzy Metric spaces was carried by several authors [10, 11, 12, 14, 19, 32, 33, 34].

Digital topology is the study of the topological properties of images arrays. The results provide a sound mathematical basis for image processing operations such as image thinning, border following, contour filling and object counting.

Digital topology is a developing area on general topology and functional analysis which studies feature of 2D and 3D digital image. **Rosenfeld** [24, 25], first to consider digital topology as a tool to study digital images. **Kong** [22], then introduced the digital fundamental group of a discrete object. The digital version of the topological concept was given by **Boxer** [2, 3, 4].

A. Rosenfeld [25] first studied the almost fixed point property of digital images. **Ege and Karaca** [16, 17] gave relative and reduced Lefschetz fixed point theorem for digital images. They also calculated the degree of antipodal map for the sphere like digital images using fixed point properties. **Ege and Karaca** [15] defined a digital metric space and proved the famous Banach Contraction Principle for digital images. But this paper has many slips and was refined and corrected by **S. E. Han** [31].

In this paper, we introduce $\varphi -$ contractions and $\varphi -$ contractive mappings on digital metric spaces. We prove an important Lemma and use it to prove the existence and uniqueness of fixed points in digital metrics spaces.

II. PRELIMINARIES

Let X be a subset of $\mathbb{Z}^n$ for a positive integer n where $\mathbb{Z}^n$ is the set of lattice points in the $n-$ dimensional Euclidean Space and ℓ represent an adjacency relation for the members of X. A digital image consists of (X, ℓ).

2.1 Definition (Boxer [3]): Let ℓ, n be positive integers, $1 \le \ell \le n$ and p, q be two distinct points

$$p = (p_1, p_2, \dots, p_n), \quad q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$$

p and q are $\ell -$ adjacent if there are at most ℓ indices i such that $|p_i - q_i| = 1$ and for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

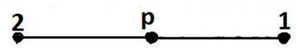
The following statements can be obtained from definition 2.1

For a given $p \in \mathbb{Z}^n$, the number of points $q \in \mathbb{Z}^n$ which are $\ell -$ adjacenct to p is denoted by $k(\ell, n)$. It may be noted that $k(\ell, n)$ is independent of p.

In practice we write $k = k(\ell, n)$.

1. If $p \in \mathbb{Z}$ (i.e., $n = 1$) then ℓ can take only one value $\ell = 1$. In this case, $k(1,1) = 2$, since $p - 1$ & $p + 1$ are the only points $1 -$ adjacent to p in $\mathbb{Z}$.

   Thus, $k = k(1,1) = 2$ and q is $1 -$ adjacent to p if and only if $|p - q| = 1$.



1 - adjacency

2. If $p \in \mathbb{Z}^2$ (i.e., $n = 2$) then ℓ can take values $\ell = 1, 2$.

   When $\ell = 2$,

   the points $2 -$ adjacent to $p = (p_1, p_2)$ are

   $(p_1 \pm 1, p_2), (p_1, p_2 \pm 1), (p_1 \pm 1, p_2 \pm 1)$.

   Thus,

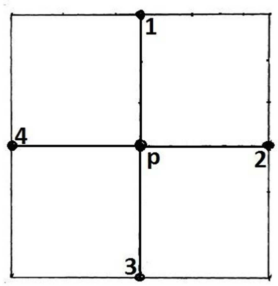
   the number of points $2 -$ adjacent to p is 8, so that $k = k(2,2) = 8$. (fig. (b))

   When $\ell = 1$,

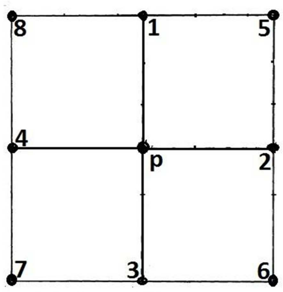
   the points $1 -$ adjacent to $p = (p_1, p_2)$ are

   $(p_1 \pm 1, p_2), (p_1, p_2 \pm 1)$.

   Thus, the number of points $1 -$ adjacent to p is 4, so that $k = k(1,2) = 4$. (fig. (a))



(a) 1 - adjacency



(b) 2 - adjacency

3. If $p \in \mathbb{Z}^3$ (i.e., $n = 3$) then ℓ can take values $\ell = 1, 2, 3$.

   When $\ell = 3$,

   the points $3-$adjacent to $p = (p_1, p_2, p_3)$ are

$$(p_1 \pm 1, p_2, p_3), (p_1, p_2 \pm 1, p_3), (p_1, p_2, p_3 \pm 1),$$

$$(p_1 \pm 1, p_2 \pm 1, p_3), (p_1 \pm 1, p_2, p_3 \pm 1),$$

$$(p_1, p_2 \pm 1, p_3 \pm 1), (p_1 \pm 1, p_2 \pm 1, p_3 \pm 1).$$

   Thus, number of points $3-$adjacent to p is 26,

   so that $k = k(3,3) = 26$. (fig. (c))

   When $\ell = 2$,

   the points $2-$adjacent to $p = (p_1, p_2, p_3)$ are

$$(p_1 \pm 1, p_2, p_3), (p_1, p_2 \pm 1, p_3), (p_1, p_2, p_3 \pm 1),$$

$$(p_1 \pm 1, p_2 \pm 1, p_3), (p_1 \pm 1, p_2, p_3 \pm 1),$$

$$(p_1, p_2 \pm 1, p_3 \pm 1).$$

   Thus,

   number of points $2-$adjacent to p is 18,

   so that $k = k(2,3) = 18$. (fig. (b))

   When $\ell = 1$,

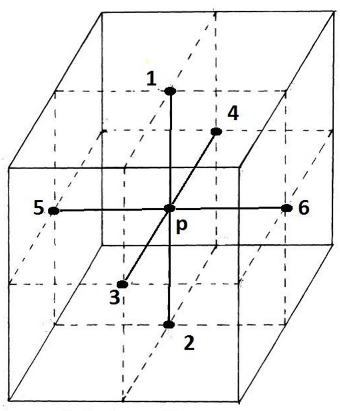
   the points $1-$adjacent to $p = (p_1, p_2, p_3)$ are

$$(p_1 \pm 1, p_2, p_3), (p_1, p_2 \pm 1, p_3), (p_1, p_2, p_3 \pm 1).$$

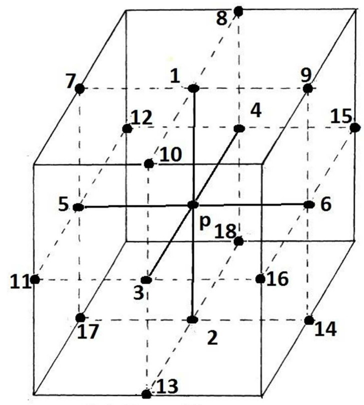
   Thus,

   the number of points $1-$adjacent to p is 6,

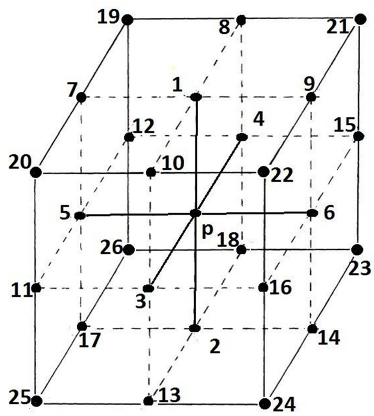
   so that $k = k(1,3) = 6$. (fig. (a))



(a) 1 - adjacency



(b) 2 - adjacency



(c) 3 - adjacency

In general to study nD digital image, if $1 \le \ell \le n$ then $k = k(\ell, n)$ is given by the following formula [28] (see also [29, 30]).

$$k(\ell, n) = \sum_{i=n-\ell}^{n-1} 2^{n-i} C_i^n \quad --- (2.1.2)$$

where $C_i^n = \frac{n!}{(n-i)!i!}$

Suppose X is a non-empty subset of $\mathbb{Z}^n$, $1 \le \ell \le n$, $k = k(\ell, n)$. Then (X, ℓ) is called a digital image with $\ell-$adjacency (Rosenfeld [24]). We also say that (X, ℓ) is called nD digital image [25, 26, 27].

Suppose $p \in \mathbb{Z}^n$ and $1 \le \ell \le n$. Then the digital $\ell-$neighborhood of p in $\mathbb{Z}^n$ (See [24]) is the set

$$N_\ell(p) = \{\, q \mid q \text{ is } \ell - \text{adjacent to } p\}$$

If q is $\ell-$adjacent to p then we say that p and q are $\ell-$neighbours.

Further, we write (See [15])

$$N_\ell^*(p) = N_\ell(p) \cup \{p\}$$

Suppose $p, q \in \mathbb{Z}$ and $p \leq q$. Then the digital interval [23] is defined as

$$[p, q]_{\mathbb{Z}} = \{r \in \mathbb{Z} \mid p \leq r \leq q\}$$

A digital image $X \subset \mathbb{Z}^n$ is said to be $\ell -$ connected [23] if for every two points $u, v \in X$, there is a set $\{u_0, u_1, \dots, u_r\}$ of points of digital image X such that $u = u_0$, $v = u_r$ and u_i and u_{i+1} are $\ell -$ neighbours for $i = 0, 1, \dots, r-1$.

Suppose (X, ℓ_0) is a digital image of $\mathbb{Z}^{n_0}$, (Y, ℓ_1) is digital image of $\mathbb{Z}^{n_1}$ and $T : X \longrightarrow Y$ is a function. Then

- T is said to be $(\ell_0, \ell_1) -$ continuous [3], if $\ell_0 -$ connected subsets E of X are mapped into $\ell -$ connected subsets of Y. i.e., E is $\ell_0 -$ connected in X implies T(E) is $\ell_1 -$ connected in Y.

- T is $(\ell_0, \ell_1) -$ continuous if and only if the image of $\ell_0 -$ adjacent points of X are either coincident or $\ell_1 -$ adjacent in Y. i.e., u_0, u_1 are $\ell_0 -$ adjacent points of X then either $T(u_0) = T(u_1)$ or $T(u_0)$ and $T(u_1)$ are $\ell_1 -$ adjacent in Y.

- T is called $(\ell_0, \ell_1) -$ isomorphism [5], if T is $(\ell_0, \ell_1) -$ continuous, onto and T^{-1} is $(\ell_1, \ell_0) -$ continuous. In this case we write $X \cong_{(\ell_0, \ell_1)} Y$.

2.2 Definition:

Suppose $m \in \mathbb{Z}^+$, (X, ℓ) is a digital image in $\mathbb{Z}^n$ and $T : [0, m]_{\mathbb{Z}} \longrightarrow X$ is $(1, \ell) -$ continuous.

Suppose $u, v \in \mathbb{Z}$ are such that $T(0) = u$ and $T(m) = v$. Then we say that T is a digital $\ell -$ path [3] from u to v.

Suppose $m \geq 4$, $T : [0, m-1]_{\mathbb{Z}} \longrightarrow X$ is a $\ell -$ path and the sequence $\{\mathbf{T}(\mathbf{0}), \mathbf{T}(\mathbf{1}), \dots, \mathbf{T}(\mathbf{m-1})\}$ of images of the $\ell -$ path is such that $T(i)$ and $T(j)$ are $\ell -$ adjacent if and only if $i = j \pm 1 (\text{mod } m)$.

Then we say that T is a simple closed $\ell -$ curve of m points in the digital image (X, ℓ) [6].

2.3 Definition (Han [31]): Let $X \subset \mathbb{Z}^n$, d be the Euclidean metric on $\mathbb{Z}^n$. (X, d) is a metric space. Suppose (X, ℓ) is a digital image with $\ell -$ adjacency then (X, d, ℓ) is called a digital metric space.

2.4 Definition (Han [31]): We say that a sequence $\{x_n\}$ of points of the digital metric space (X, d, ℓ) is a Cauchy sequence if there is $M \in \mathbb{N}$ such that,

$$d(x_n, x_m) < 1 \quad \text{for all } n, m > M.$$

2.5 Note:

Since x_n, x_m are lattice points of $\mathbb{Z}^n$, $\left(d(x_n, x_m)\right)^2$ is a positive integer if $x_n \neq x_m$, and $d(x_n, x_m) = 0$ if $x_n = x_m$.

Consequently,

$$d(x_n, x_m) < 1 \Longrightarrow d(x_n, x_m) = 0 \Longrightarrow x_n = x_m.$$

2.6 Theorem (Han [31]): For a digital metric space (X, d, ℓ), if a sequence $\{x_n\} \subset X \subset \mathbb{Z}^n$ is a Cauchy sequence, there is $M \in \mathbb{N}$ such that for all $n, m > M$, we have $x_n = x_m$.

2.7 Definition (Han [31]): A sequence $\{x_n\}$ of points of a digital metric space (X, d, ℓ) converges to a limit $L \in X$ if for all $\epsilon > 0$, there is $M \in \mathbb{N}$ such that

$$d(x_n, L) < \epsilon \quad \text{for all } n > M,$$

2.8 Proposition (Han [31]): A sequence $\{x_n\}$ of points of a digital metric space (X, d, ℓ) converges to a limit $L \in X$ if there is $M \in \mathbb{N}$ such that $x_n = L$ for all $n > M$.

$$(\text{i.e., } x_n = x_{n+1} = x_{n+2} = \dots = L).$$

2.9 Definition (Han [31]): A digital metric space (X, d, ℓ) is complete if any Cauchy sequence $\{x_n\}$ converges to a point L of (X, d, ℓ).

2.10 Theorem (Han [31]): A digital metric space (X, d, ℓ) is complete.

2.11 Definition (Han [31]): Let (X, d, ℓ) be a digital metric space and $T : (X, d, \ell) \longrightarrow (X, d, \ell)$ be a self-map. If there exists $\lambda \in [0, 1)$ such that,

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all } x, y \in X,$$

then T is called a contraction map.

2.12 Proposition (Han [31]):

Every digital contraction map $T : (X, d, \ell) \longrightarrow (X, d, \ell)$ is $\ell -$ continuous (Digital continuous).

III. MAIN RESULTS

First we introduce a notation.

3.1 Notation: Let $\Phi = \{\varphi : [0, \infty) \longrightarrow [0, \infty)\}$ be such that φ is increasing,

$\varphi(t) < t$ for $t > 0$ and $\varphi(t) = 0$ iff $t = 0$.

Now we prove a Lemma, which plays an important role in our further development.

3.2 Lemma: Let $X \subseteq \mathbb{Z}^n$ and (X, d, ℓ) be a digital metric space. Then there does not exist a sequence $\{x_m\}$ of distinct elements in X, such that

$$d(x_{m+1}, x_m) < d(x_m, x_{m-1}) \quad \text{for } m = 1, 2, \ldots --- (3.2.1)$$

Proof: We know that,

$$d(x_1, x_0) = \sqrt{N} \text{ for some positive integer N},$$

since d is Euclidean metric in X and $X \subseteq \mathbb{Z}^n$ for some positive integer n.

Further, $d(x_{m+1}, x_m) \geq 1$ or 0

Lemma 3.2, (from (3.2.1))

$$d(x_{m+1}, x_m) < 1 \text{ after N steps},$$

so that, $d(x_{m+1}, x_m) = 0$ for $m > N$.

Thus, $x_{m+1} = x_m$ for $m > N$.

Consequently, the sequence $\{x_m\}$ is a finite sequence.

3.3 Definition: Suppose (X, d, ℓ) is a digital metric space, $T : X \longrightarrow X$ and $\varphi \in \Phi$. Suppose

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big) \quad \text{for all } x, y \in X.$$

Then, T is called a digital $\varphi -$ contraction.

Now we prove a fixed point Theorem on $\varphi -$ contraction.

3.4 Theorem: Suppose (X, d, ℓ) is a digital metric space, $T : X \longrightarrow X$ and $\varphi \in \Phi$. Suppose

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big) \quad \text{for all } x, y \in X.$$

T is digital $\varphi -$ contraction. Then, T has unique fixed point.

Proof: Let $x_0 \in X$.

Write $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$.

We may suppose that $x_n \neq x_{n+1}$ for $n = 0, 1, 2, \ldots$.

for otherwise x_n is a fixed point.

Now,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \varphi\big(d(x_n, x_{n-1})\big) \\ &< d(x_{n-1}, x_n) \end{aligned}$$

Therefore, $d(x_{n+1}, x_n) < d(x_{n-1}, x_n) \quad (\because x_n \neq x_{n-1})$

Therefore,

$\{d(x_{n+1}, x_n)\}$ is a strictly decreasing sequence.

Therefore, $x_n = x_{n+1}$ for large n **(by Lemma 3.2)**

Therefore, x_n is a fixed point of T for large n.

Suppose, x and y are fixed points of T.

Then,

$d(x, y) = d(Tx, Ty) \leq \varphi\big(d(x, y)\big) < d(x, y)$ if $x \neq y$

a contradiction.

Therefore, $x = y$. Thus T has unique fixed point.

3.5 Corollary:

(Banach Contraction Principle in Digital Metric Spaces)

Let (X, d, ℓ) be a digital metric space and $T : X \longrightarrow X$ be such that

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all } x, y \in X$$

and for some $\lambda \in [0, 1)$. Then, T has unique fixed point.

Proof: Take $\varphi(t) = \lambda t$ in the above **Theorem 3.4**. Then we get the result.

3.6 Definition:

Let (X, d, ℓ) be a digital metric space, $T : X \longrightarrow X$ and $\varphi \in \Phi$. We say that T is $\varphi-$contractive if

$$\varphi\big(d(Tx, Ty)\big) < \varphi\big(d(x, y)\big) \text{ for all } x, y \in X, x \neq y.$$

3.7 Theorem: Suppose (X, d, ℓ) is a digital metric space, $T : X \longrightarrow X$ and $\varphi \in \Phi$. If T is $\varphi-$contractive then T has unique fixed point.

Proof: Let $x_0 \in X$.

Write $x_{n+1} = Tx_n$ for $n = 0, 1, 2 \ldots.$

We may suppose that $x_{n+1} \neq x_n$ for $n = 0, 1, 2 \ldots.$

for otherwise x_n is a fixed point.

Now,

$$\varphi\big(d(x_{n+1}, x_n)\big) = \varphi\big(d(Tx_n, Tx_{n-1})\big) < \varphi\big(d(x_n, x_{n-1})\big) \quad (\because x_n \neq x_{n-1})$$

Thus, $\{d(x_{n+1}, x_n)\}$ is a strictly decreasing sequence.

Therefore, $x_n = x_{n+1}$ for large n **(by Lemma 3.2)**

Therefore, x_n is a fixed point of T for large n.

Suppose, x and y are fixed points of T.

Then,

$$\varphi\big(d(x, y)\big) = \varphi\big(d(Tx, Ty)\big) < \varphi\big(d(x, y)\big) \quad \text{if } x \neq y$$

so that, $d(x, y) < d(x, y)$ a contradiction.

Therefore, $x = y$.

Thus, T has unique fixed point.

3.8 Definition: Let X be a nonempty set, $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$. We say that T is $\alpha-$admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1$$

3.9 Definition:

Let (X, d, ℓ) be a digital metric space and let $T: X \longrightarrow X$ be a digital $-\alpha-\psi-\varphi-$contractive type mapping if there exist three functions $\alpha : X \times X \longrightarrow [0, \infty)$ and $\psi, \varphi \in \Phi$ such that

$$\alpha(x, y)\psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big) \quad \text{for all } x, y \in X.$$

3.10 Theorem:

Let (X, d, ℓ) be a digital metric space and let $T: X \longrightarrow X$ be a digital $-\alpha-\psi-\varphi-$ contractive type mapping. Suppose T satisfies the following conditions

(i) T is $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

Then

(iii) T has a fixed point.

(iv) If further u , v are fixed points of T with

$$\alpha(u, v) \geq 1 - - - - (3.10.1)$$

then $u = v$. It is in this sense T has unique fixed point.

Proof: Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$.

Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \text{for all } n \geq 0.$$

Since T is $\alpha-$admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$$

$$\Longrightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$$

Inductively, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n = 0, 1, 2, \ldots.$$

If $x_0 = x_1$ then $x_0 = x_1 = Tx_0$ so that x_0 is a fixed point of T.

Hence, we may suppose that $x_0 \neq x_1$.

Inductively assume that

$$x_{n-1} \neq x_n \quad \text{for } n = 1, 2, \ldots$$

Then,
$$\psi\big(d(x_n, x_{n+1})\big) = \psi\big(d(Tx_{n-1}, Tx_n)\big)$$
$$\leq \alpha(x_{n-1}, x_n)\psi\big(d(Tx_{n-1}, Tx_n)\big)$$
$$\leq \psi\big(d(x_{n-1}, x_n)\big) - \varphi\big(d(x_{n-1}, x_n)\big)$$
$$< \psi\big(d(x_{n-1}, x_n)\big) \quad (\because x_{n-1} \neq x_n)$$

Thus,

$\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)) \quad (\because\ x_{n-1} \neq x_n)$

Hence,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) ----(3.10.2)$$

$(\because\ \psi$ is increasing$)$

This is true for $n = 1, 2, \ldots$

Thus, $\{x_n\}$ is a sequence of distinct elements satisfying (3.10.2). This contradicts **Lemma 3.2**.

Hence $\{x_n\}$ is finite sequence, say $x_0, x_1, \ldots, x_N$.

Then, $x_{N+1} = x_N$, so that x_N is a fixed point.

Suppose,

u and v are fixed points of T with $\alpha(u, v) \geq 1$.

Suppose $u \neq v$.

Then, $\psi(d(u, v)) = \psi(d(Tu, Tv))$

$$\leq \alpha(u, v)\, \psi(d(Tu, Tv))$$

$$\leq \psi(d(u, v)) - \varphi(d(u, v))$$

$$< \psi(d(u, v)) \qquad \text{if } u \neq v$$

a contradiction.

Therefore, $u = v$.

Therefore, T has unique fixed point.

3.11 Example: Let $X = [0, \infty)$ be the digital metric space with $1-$adjacency,Thus $X = \{0, 1, 2, \ldots\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$.

Consider the self mapping $T : X \longrightarrow X$ given

$$T(x) = x(x - 1).$$

Define $\alpha : X \times X \longrightarrow [0, \infty)$ as $\alpha = \begin{cases} 2, & \text{if } x, y \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$

Then T has $\alpha-$admissible.

Define the mapping

$\psi, \varphi : [0, \infty) \longrightarrow [0, \infty)$ by $\psi(t) = t,\ \varphi(t) = \frac{1}{2}t$

T is a $\alpha - \psi - \varphi -$ contractive type mapping with

$$\psi(t) = t,\ \varphi(t) = \tfrac{1}{2}t \quad \forall\ t \geq 0\,.$$

$$\alpha(x, y)\, \psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$$

$$\text{for } x, y \in X$$

Therefore, T has two fixed points 0 and 2.

T Satisfies all the hypothesis of **Theorem 3.10**.

This example shows that, if (3.10.1) is violated. T may not have unique fixed point.

Since in this case $u = 0$ and $v = 2$ are two fixed points of T, but $\alpha(u, v) \geq 1$.

3.12 Theorem: Let (X, d, ℓ) be a digital metric space, $T : X \longrightarrow X$ be a mapping and $\alpha : [0, \infty) \longrightarrow [0, \infty)$. Suppose

1) T is a $\alpha-$admissible,
2) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,
3) T is digital contiuous,
4) $\alpha(x, y)\, \psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y))$

where $M(x, y) =$

$$\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$$

$$\text{for all } x, y \in X --- (3.12.1)$$

Then T has a fixed point. If further u, v are two fixed points of T with $\alpha(u, v) \geq 1$ then $u = v$. It is in this sense, T has unique fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Define the sequence $\{x_n\}$ in Xby

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \text{for all } n = 0, 1, 2, \ldots$$

Since T is a $\alpha-$admissible, we have

$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$

$$\Longrightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$$

Inductively, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n = 0, 1, 2, \ldots.$$

If $x_0 = x_1$ then $x_0 = x_1 = Tx_0$ so that x_0 is a fixed point of T.

Hence, we may suppose that $x_0 \neq x_1$.

Inductively, assume that $x_{n-1} \neq x_n$ for $n = 1, 2, \ldots$

Then,

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\, \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)) \\ &\leq \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\ &\quad - \varphi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})\end{aligned}$$

Let $d(x_{n-1}, x_n) = A$ and $d(x_n, x_{n+1}) = B$

Therefore, $\psi(B) \leq \psi(\max\{A, B\}) - \varphi(\max\{A, B\})$

Suppose, $A \leq B$ then

$$\psi(B) \leq \psi(B) - \varphi(B) \Longrightarrow \varphi(B) \leq 0$$
$$\Longrightarrow \varphi(B) = 0$$
$$\Longrightarrow B = 0$$
$$\Longrightarrow d(x_n, x_{n+1}) = 0$$
$$\Longrightarrow x_n = x_{n+1}$$

a contradiction.

Therefore, $B < A$

Therefore, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$.

This is true for $n = 1, 2, \ldots$

Therefore, by **Lemma 3.2,** $\{x_n\}$ is a finite sequence.

Thus, there exist N such that $x_{N+1} = x_N$

Therefore, x_N is a fixed point of T.

Suppose,

u and v are fixed points of T with $\alpha(u, v) \geq 1$.

Suppose $u \neq v$.

Then, $\psi(d(u, v)) = \psi(d(Tu, Tv))$

$$\leq \alpha(u, v)\, \psi(d(Tu, Tv))$$
$$\leq \psi(M(u, v)) - \varphi(M(u, v))$$
$$\leq \psi(d(u, v)) - \varphi(d(u, v)) \text{ if } u \neq v$$

Therefore,

$$\psi(d(u, v)) \leq \psi(d(u, v)) - \varphi(d(u, v))$$

Therefore, $\varphi(d(u, v)) < 0$

a contradiction.

Therefore, $u = v$.

Therefore, T has unique fixed point.

3.13 Theorem: Let (X, d, ℓ) be a digital metric space, $T : X \longrightarrow X$ such that

$$d(Tx, Ty) < M(x, y) \quad \text{for all } x, y \in X, \quad x \neq y \; --(3.13.1)$$

where $M(x, y) = \max\left\{ \begin{matrix} d(x, y), d(x, Tx), d(y, Ty), \\ \frac{1}{2}[d(x, Ty) + d(y, Tx)] \end{matrix} \right\}$ for all $x, y \in X$.

Then, T has unique fixed point.

Proof: Let $x_0 \in X$.

Write $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$.

We may suppose that $x_n \neq x_{n+1}$ for $n = 0, 1, 2, \ldots$.

for otherwise x_n is a fixed point.

Now, $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$$< M(x_{n-1}, x_n)$$
$$= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

Hence, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n = 1, 2, \ldots$.

Hence, by **Lemma 3.2,**

T has a fixed point. Clearly, fixed point is unique.

3.14 Corollary: Let (X, d, ℓ) be a digital metric space, $T : X \longrightarrow X$ be a mapping and $\varphi \in \Phi$. Suppose

$$\psi(d(Tx, Ty)) < \varphi(M(x, y)) \quad \text{for all } x, y \in X, \quad x \neq y$$

$$M(x, y) = \max\left\{ \begin{matrix} d(x, y), d(x, Tx), d(y, Ty), \\ \frac{1}{2}[d(x, Ty) + d(y, Tx)] \end{matrix} \right\} \quad \text{for all } x, y \in X.$$

Then, T has unique fixed point.

3.15 Example: Let $X = \{0, 1, 2, \ldots\}$, $d(x, y) = |y - x|$ and $(X, d, 1)$ is a digital metric space in $\mathbb{Z}$ with $1-$adjacency. Define $T : X \longrightarrow X$ by $Tx = x + 1$. Then,

$$d(Tx, Ty) < \max\left\{ \begin{matrix} d(x, y), d(x, Tx), d(y, Ty), \\ d(x, Ty), d(y, Tx) \end{matrix} \right\}, x \neq y$$
$$\text{for all } x, y \in X \; --- (3.14.1)$$

Solution: Given $X = \{0, 1, 2, \ldots\}$ with $d(x, y) = |y - x|$ $(X, d, 1)$ is a digital metric space in $\mathbb{Z}$ with $1-$adjacency and $Tx = x + 1$ and $Ty = y + 1$

Therefore, $d(Tx, Ty) = |y + 1 - x - 1| = |y - x|$

$\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$

$$= \max\left\{ \begin{matrix} |y - x|, |x + 1 - x|, |y + 1 - y|, \\ |y + 1 - x|, |x + 1 - y| \end{matrix} \right\}$$
$$= \max\{|y - x|, 1, 1, |y - x + 1|, |x - y + 1|\}$$
$$= y - x + 1 \quad \text{if } x < y$$

Therefore, (3.14.1) holds.

But, T dose not have any fixed point.

Note: This example shows that Theorem 3.13 may not hold if we replace (3.13.1) by (3.14.1)

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