Investigation for Stability of Fractional Explicit Method for pricing option

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Abstract— *In numerical analysis, explicit and implicit approaches are used to obtain numerical approximations of time dependent ordinary and partial differential equations. Fractional order differential equations are used widely for finance market analysis. Implicit solution methods require more computational efforts and are complex to program. In order to overcome these difficulties, explicit method for fractional order differential equation has been introduced which is one of the most recently developed areas in the world of finance. The main aim of this paper is to investigate stability of Fractional Explicit method for q^{th} order time fractional Black-Schols equation by the well known Fourier analysis method and a numerical experiment is presented for comparison of European call option prices for different values of 'q'.*

Keywords— *Fractional calculus; Fractional Explicit Method; stability; European call options; time fractional Black-Schols equation; Fourier analysis.*

I. INTRODUCTION

In Numerical analysis, the use of Fractional calculus is increasing day by day. The field of fractional calculus is not new for mathematicians. It is as old as in the year 1695 , when L'Hopital sent a letter to Leibniz asking him an important question about the order of the derivative, " What would be the result if order of derivative is ½?". Leibniz replied in a predictive way , "An apparent paradox, from which one day useful consequences will be drawn". In these words Leibniz was farseeing the beginning of the area which is now called as Fractional Calculus. Fractional calculus is the branch of mathematics which is felicitous for non integer powers of the differentiation operator. The arbitrary order derivatives are called as differ-integrals. A number of textbooks [15], [18], [22] have been published in this field with various aspects in various ways. The non integer order of differential operator was first introduced by [12]. Also, [7], [19],[1], [13], made important contributions in the field. They defined and developed fractional integral and differentiation. Podlubny[18] shows that the geometric interpretation of fractional integration is "shadows on the walls" and its physical interpretation is "shadows of the past." [3].

Numerical approaches to different kinds of fractional diffusion models have been increasingly noticeable in literature. Huang [10], investigate the time fractional diffusion equation in whole space and also in half space. Zhuang[30] approximated the time fractional diffusion equation by implicit finite difference method. Sousa [2] derived a second order numerical method for the fractional advection diffusion equation which is explicit and also analyzed the convergence of the numerical method through the consistency and the stability. Meerchaert [14] considered the stochastic solution of space-time fractional diffusion equation. Acedo [2] and Yuste [27] introduced an explicit scheme and weighted average finite difference methods for the fractional diffusion equation and analyzed these two scheme's stability by Von Neumann method. Zhuang [30] presented a new way for solving sub-diffusion equation by integration of the equation on both sides to obtain an implicit finite difference method. Stability and convergence of the scheme were proved by the energy method. Riesz[21], Riesz[20] proved the mean value theorem for fractional integrals and introduced another formulation that is associated with the Fourier transform. Murio [16] established implicit finite difference approximation for time fractional diffusion equations but in showing stability of the method by Fourier method he made some aw. The stability of implicit finite difference approximation for time fractional diffusion equations were shown by Ding[6]. Chen[4] constructed the difference scheme for fractional sub diffusion equation based on Grunwald-Letnikov formula and showed the stability and convergence of the difference scheme using the Fourier method. Zhang[28] presented the unconditionally stable finite difference method for fractional partial differential equation. Chen[5] showed finite difference method for the fractional reaction-sub-diffusion equation. Singh[23] solved the bioheat equations by finite difference method and homotopy perturbation method.

This paper is arranged in different sections. After introduction 1, the next section 2, will review the working of Fractional Explicit method. Section 3 is based on the stability analysis of the method. In section 4, there is a numerical experiment analyzing the performance of Fractional Explicit method for different values of 'q'. Data for this experiment is taken from historical data section of NSE website of jet airways of the period from 1st November 2016 to 30th November 2016. Graphical representation is given for the more precise comparison. Finally in section 5 there is concluding remarks for the study.

II. FRACTIONAL EXPLICIT METHOD

Fractional Explicit method was proposed by [9]. It is observed by adjusting method and Fractional Explicit method both agree. Fractional explicit method is derived using explicit method which is the most popular one within finite difference methods. Basic idea behind each finite difference method is to replace partial derivatives in the PDE by finite difference approximations and solving the resulting system of equations from Riez[21], Lateef[11] and UĞur [26]. All finite difference method involves similar four step process:

- Discretize the appropriate differential equation.
- Specify a grid of stock price and time.
- Calculate the payoff of the option at specific *boundaries* of the grid of underlying prices.
- Iteratively determine the option price at all other grid points, including the point for the current time and underlying price (i.e. the option price today).

Now For Fractional Explicit method consider the qth order time fractional Black Scholes Equation:

$$\frac{\partial^q V}{\partial t^q} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad \ldots (1)$$

The boundary conditions for European call option can be given as:

Final condition:

V(S, t) = max(S$-E$, 0), S > 0

Boundary condition as:

V (S_{min} , t) = 0,

V (S_{max} , t) = S$-$E$e^{-r(T-t)}$,

Where K is a strike price and S_{min} and S_{max} represents the minimum and maximum values of stock price. Now first we will divide the domain S and t into N and M parts from [9].

$$\Delta t = \frac{T-t_0}{M} \qquad \text{for } t_0 \le t \le T$$

$$\Delta S = \frac{S_{max} - S_{min}}{N} \qquad \text{for } S_{min} \le S \le S_{max}$$

The computational domain is discretized by a uniform grid (S_k, t_i), with $S_k = kh(\, k = 0,1,,\ldots,N)$ and $t_i = i\,\tau\,(i = 0,1,\ldots.,M)$. On making some arrangements S and t are obtained as

$$t_i = t_0 + i\Delta t \qquad \text{for } i = 0,1,\ldots,M$$

$$S_k = S_{min} + k\Delta S \qquad \text{for } k = 0,1,\ldots,N$$

For the notation of points (S_K, t_i), we denote the approximation of option price as

$$\text{V}\,(S_K, t_i) \approx \omega_k^i$$

Also the final and boundary conditions for European call option in terms of $\omega_{k,i}$, M and N are given by [9],

$$\omega_k^M \approx \max(S_k - E, 0),\ \text{S} > 0 \qquad \omega_0^i \approx 0 \qquad \ldots(2)$$

$$\omega_N^i \approx S_N - \text{E}e^{-r(t_M - t_i)}$$

Next we have to discretize the qth order time fractional Black Scholes Equation. For which we have to replace all the partial derivatives as follows:

$$\frac{\partial V(S_K, t_i)}{\partial S_k} \approx \frac{\omega_{k+1}^i - \omega_{k-1}^i}{2\Delta S}$$

$$\frac{\partial^2 V(S_K, t_i)}{\partial {S_k}^2} \approx \frac{\omega_{k+1}^i - 2\omega_k^i + \omega_{k+1}^i}{(\Delta S)^2}$$

The approximation for q-th order time fractional derivative of V (S_K, t_i) can be stated as the sum differences with the coefficients g_j as given in [9].

$$\frac{\partial^q V}{\partial t^q} = \frac{1}{(\Delta t)^q}\sum_{j=0}^{i} g_j\, \omega_k^{i-j} \qquad \ldots(3)$$

Where g_j is the function of gamma functions of q and j,

$$g_j = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = (-1)^j \binom{q}{j}, \quad j = 0,1,2,..$$

On substituting all the derivatives in (1) and making some arrangements and simplification we will get,

$$\sum_{j=1}^{i} g_j \omega_k^{i-j} = \propto_k \omega_{k-1}^i + \beta_k \omega_k^i + \gamma_k \omega_{k+1}^i \quad ...(4)$$

for i = M, M-1,.....,1 and k = 1,.....N-1 and the terms $\propto_k$ β_k and γ_k
are as follows:

$$\propto_k = -\frac{1}{2}(\Delta t)^q \left\{ \sigma^2 \left(\frac{S_k}{\Delta S}\right)^2 - r\frac{S_k}{\Delta S} \right\}$$

$$\beta_k = (\Delta t)^q \left\{ \sigma^2 \left(\frac{S_k}{\Delta S}\right)^2 + r \right\} - 1$$

$$\gamma_k = -\frac{1}{2}(\Delta t)^q \left\{ r\frac{S_k}{\Delta S} + \sigma^2 \left(\frac{S_k}{\Delta S}\right)^2 \right\} \quad ...(5)$$

For q =1, it can be recursively evaluated from [8]

$$g_0 = 1, \quad g_j = \left(1 - \frac{q+1}{j}\right) g_{j-1}, j = 1,2,..$$

Consequently $g_1 = -1$ and $g_j = 0$ for $j =$ 2,3, ...(6)

Now for i =1, equation (4) can be written as,

$$g_1 \omega_k^0 = \propto_k \omega_{k-1}^1 + \beta_k \omega_k^1 + \gamma_k \omega_{k+1}^1$$

For i≥ 2,

$$\sum_{j=1}^{i} g_j \omega_k^{i-j} = \propto_k \omega_{k-1}^i + \beta_k \omega_k^i + \gamma_k \omega_{k+1}^i \quad ...(7)$$

Now we are going to check the stability and convergence of these two equations in the next sections from [9] and [24].

III. STABILITY ANALYSIS

A finite difference approximation is said to be stable if the errors (truncation, round-off etc) decay as the computation proceeds from one marching step to the next. Stability of a finite difference approximation is assessed using Von-Neumann stability analysis. To analyse stability, first we have to analyse a generic component of the solution. For this, let μ_k^i is the approximate solution of (7), then we define the round off error as defined in [24].

$$\epsilon_k^i = \omega_k^i - \mu_k^i$$

Which satisfies (7) as follows,

$$g_1 \epsilon_k^0 = \propto_k \epsilon_{k-1}^1 + \beta_k \epsilon_k^1 + \gamma_k \epsilon_{k+1}^1$$

$$\sum_{j=1}^{i} g_j \epsilon_k^{i-j} = \propto_k \epsilon_{k-1}^i + \beta_k \epsilon_k^i + \gamma_k \epsilon_{k+1}^i$$

….(8)

Where $\propto_k$ β_k and γ_k are given in (5)

$$i = M, M-1, \ldots, 1 \quad k = 1,2, \ldots, N-1$$

$$\epsilon_0^i = \epsilon_N^i = 0, \qquad i = 0, \ldots, M$$

Now we define grid function as, in [24]

$$\epsilon^i(S) = \begin{cases} \epsilon_k^i & when\ s_k - \frac{h}{2} < S \le S_k + \frac{h}{2},\ k = 1, \ldots, N-1 \\ 0 & when\ 0 \le S \le \frac{h}{2}\ or\ S_{max} - \frac{h}{2} < S \le S_{max} \end{cases}$$

Then we can define $\epsilon^i(S)$ in a fourier series as:

$$\epsilon^i(S) = \sum_{l=-\infty}^{l=\infty} \rho_i(l)\, e^{I2\pi lS/S_{max}}, \text{ i= 1, 2, …., M}$$

Where $\rho_i(l) = \frac{1}{S_{max}} \int_0^{S_{max}} \epsilon^i(S)\, e^{-I2\pi lS/S_{max}}$ ds

Let , $\epsilon^i = \left[\epsilon_1^i, \epsilon_2^i, \ldots, \epsilon_{N-1}^i\right]^T$

$$\|\epsilon^i\|_2 = \left(\sum_{k=1}^{N-1} h|\epsilon_k^i|^2\right)^{1/2} = \left(\int_0^{S_{max}} |\epsilon^i(S)|^2 dS\right)^{1/2}$$

Using Parseval's equality:

$$\int_0^{S_{max}} |\epsilon^i(S)|^2 dS = \sum_{l=-\infty}^{\infty} |\rho_i(l)|^2,$$

From above results we can write,

$$\|\epsilon^i\|_2^{\ 2} = \sum_{l=-\infty}^{\infty} |\rho_i(l)|^2 \quad ….(9)$$

Based on the above analysis, we can suppose that the solution has the following form:

$$\epsilon_k^i = \rho_i e^{I\vartheta kh},$$

where $\vartheta = 2\pi l/S_{max}$, I= $\sqrt{-1}$

on substituting above result in (8),

$$g_1 \rho_0 e^{I\vartheta kh} = \propto_k \rho_1 e^{I\vartheta(k-1)h} + \beta_k \rho_1 e^{I\vartheta kh} + \gamma_k \rho_1 e^{I\vartheta(k+1)h}$$

$$\sum_{j=1}^{i} g_j \rho_{i-j} e^{I\vartheta kh} = \propto_k \rho_i e^{I\vartheta(k-1)h} + \beta_k \rho_i e^{I\vartheta kh} + \gamma_k \rho_i e^{I\vartheta(k+1)h}$$

On simplifying above equations we obtain,

$$g_1 \rho_0 = \rho_1 \varphi_{k,1}$$

$$\sum_{j=1}^{i} g_j \rho_{i-j} = \rho_i \varphi_{k,i} \qquad \text{....(10)}$$

Where

$$\varphi_{k,1} = -1 + (\Delta t)^q \left[\sigma^2 \left(\frac{S_k}{\Delta S}\right)^2 (1-\cos(\vartheta h)) + r\left(1-\left(\frac{S_k}{\Delta S}\right) I\sin(\vartheta h)\right)\right]$$

$$\varphi_{k,i} = -1 + (\Delta t)^q \left[\sigma^2 \left(\frac{S_k}{\Delta S}\right)^2 (1-\cos(\vartheta h)) + r\left(1-\left(\frac{S_k}{\Delta S}\right) I\sin(\vartheta h)\right)\right]$$

Proposition 1: Supposing that ρ_i $(i = 1,2 \ldots\ldots, M)$ be the solution of (8) then we have $|\rho_i| \le |\rho_0|$ (i = 1,2,…,M).

***Proof* : .** We will use mathematical induction to complete the proof.

For k=1, using results given in (6) from first equality of (10) gives

$$|\rho_1| = \frac{|g_1|}{|\varphi_{k,i}|}|\rho_0| \le |\rho_0|$$

[Using (6)]

If $|\rho_{i-1}| \le |\rho_i|$,

Then from the second equality of (8), we obtain

$$|\rho_i| \le \frac{\left|\sum_{j=1}^{i} g_j\right|}{|\varphi_{k,i}|}|\rho_{i-j}| \le \frac{|g_1\rho_0| + \left|\sum_{j=2}^{i} g_j\rho_{i-j}\right|}{|\varphi_{k,i}|} \le |\rho_0|$$

[Again using (6)]

This completes the proof.

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Theorem 1: The fractional explicit scheme given in (4) is unconditionally stable.

***Proof*: .** By applying proposition 1 and observing (9), we obtain

$$\|\epsilon^i\|_2 \le \|\epsilon^0\|_2 \qquad \text{i= 1, 2... M}$$

which proves that Fractional Explicit scheme is unconditionally stable.

IV. NUMERICAL EXPERIMENT

In this section we are going to compare European call option prices for different values of "q" by fractional explicit method and Black schols method. Results are compared with the help of graphs. For this comparison we have taken real market data of jet airways from 1st November 2016 to 30th November 2016 of National Stock Exchange (NSE).We have chosen data in such a way that the number of contracts at each point are near to 100 or more than 100 from [17].We have consider a European call option with exercise price E= 500, risk free rate =0.1, volatility= 0.18.Variation of Stock price and time to maturity is given above each table.

TABLE 1

Results for different values of q for Stock price S=456.7 & time to maturity t =0.055556 by Fractional Explicit Method

q	Fractional Explicit Method
1.05	1.5835
1.1	1.5808
1.15	1.5787
1.2	1.5770
1.25	1.5756
1.3	1.5746
1.35	1.5738
1.4	1.5731
1.45	1.5726
1.5	1.5722

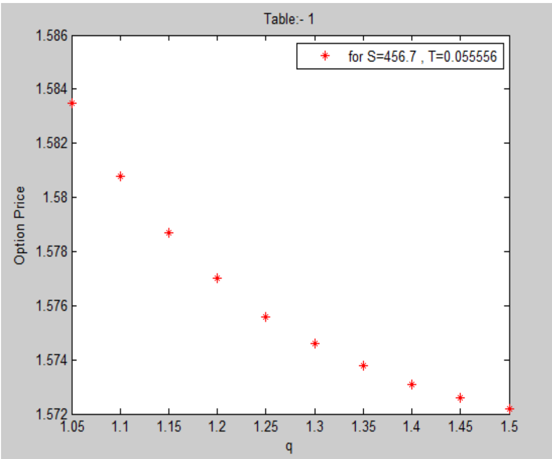

Fig 1: Representation of Table 1

TABLE 2

Results for different values of q for Stock price S=435.8 & time to maturity t =0.05158 by Fractional Explicit Method

q	Fractional Explicit Method
1.05	0.9374
1.1	0.9339
1.15	0.9312
1.2	0.9290
1.25	0.9273
1.3	0.9260
1.35	0.9249
1.4	0.9241
1.45	0.9234
1.5	0.9229

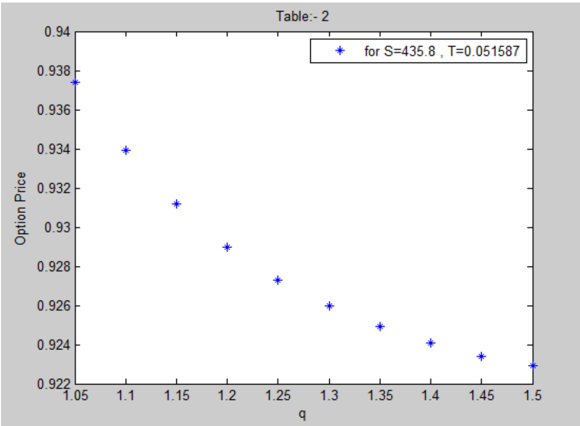

Fig **2**: Representation of Table 2

TABLE 3

Results for different values of q for Stock price S=435.15 & time to maturity t =0.04761 by Fractional Explicit Method

Q	Fractional Explicit Method
1.05	0.8490
1.1	0.8459
1.15	0.8435
1.2	0.8417
1.25	0.8402
1.3	0.8391
1.35	0.8382
1.4	0.8375
1.45	0.8369
1.5	0.8365

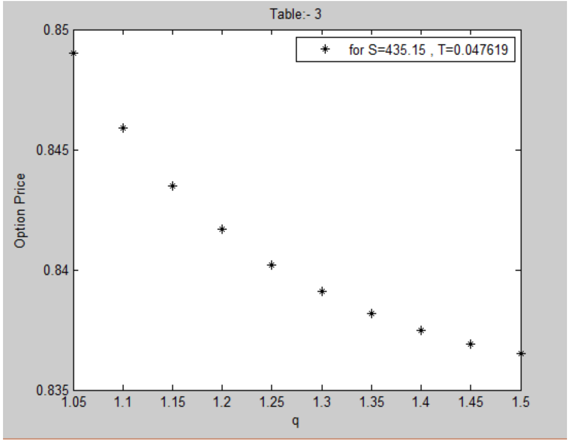

TABLE 4

Results for different values of q for Stock price S=437.2 & time to maturity t =0.043651 by Fractional Explicit Method

q	Fractional Explicit Method
1.05	0.8215
1.1	0.8190
1.15	0.8171
1.2	0.8155
1.25	0.8144
1.3	0.8134
1.35	0.8127
1.4	0.8121
1.45	0.8117
1.5	0.8114

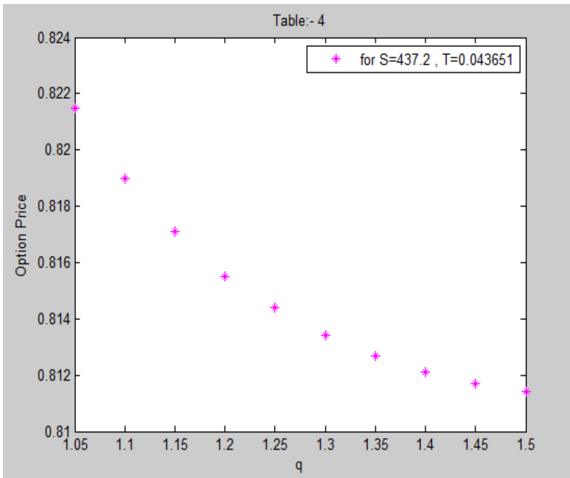

Fig 4: Representation of Table 4

TABLE 5

Results for different values of q for Stock price S=424.95 & time to maturity t =0.031746 by Fractional Explicit Method

q	Fractional Explicit Method
1.05	0.4025
1.1	0.4008
1.15	0.3996
1.2	0.3986
1.25	0.3979
1.3	0.3973
1.35	0.3969
1.4	0.3965
1.45	0.3963
1.5	0.3961

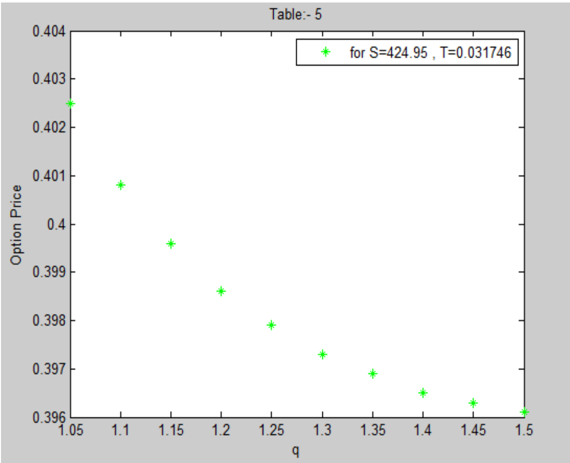

Fig 5: Representation of Table 5

Discussion of results:

In this experimental study we have taken real market data of jet airways from 1st November to 30th November 2016 from NSE website[17].We have five stock prices and for each stock price, we have ten different values of q. We have calculated european call option prices with the help of the software MATLAB for Fractional Explicit Method. Results are given in following tables .Graphical representation of each table is given in opposite space of each table.

V. CONCLUSION

Stability of Fractional Explicit method is investigated in this study. Several authors have discussed about the stability of implicit finite difference method. But due to huge amount of computation and complex coding, interest is moving towards explicit method. Fourier analysis is employed to investigate the stability of Fractional Explicit method and we found that the method is unconditionally stable. Also, we have taken a numerical data from NSE to compare the European call option prices for different values of q and found that there is not much difference in the results by change in q. Comparison is shown with the help of graph .For more precise comparison ,we can make a single graph for the data then we found that there is no difference among all the values due to overlapping in the graph. Also in future fractional explicit method may be applied to other fractional PDEs and can be used to find solutions of other problems. Also one can derive fractional version of Implicit and Crank Nicolson method for pricing Option.

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