Construction of Zero Divisor Graphs of Rings

D. Baruah

Research Scholar, Department of Mathematics, Dibrugarh University, Dibrgarh, Assam, India

Abstract: *If R is a commutative ring, $Z(R)$ is the set of zero-divisor of R and $Z^*(R) = Z(R) - \{0\}$, then the zero-divisor graph of R, $\Gamma(Z^*(R))$ usually written as $\Gamma(R)$, is the graph in which each element of $Z^*(R)$ is a vertex and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this paper we present a construction of zero divisor graphs of rings. In particular we consider ring of Gaussian integers modulo n, i.e. $\Gamma(\mathbb{Z}_n)[i]$.*

Keywords: *Bipartite graph, Complete bipartite graph, Girth of a graph, Ring of Gaussian integers modulo n, Zero-divisor.*

I. Introduction

The zero-divisor graph was first introduced by Beck (1988)[1] in the study of commutative rings. According to him, given a ring R, $G(R)$ denote the graph whose vertex set is R, such that distinct vertices r and s are *adjacent* provided that $rs = 0$. By Beck's definition, the zero vertex is adjacent to every other element of the ring, so the graph $G(R)$ is connected with diameter at most 2. He conjectured that the chromatic number of a ring i.e., the minimal number of colours necessary to colour the ring's graph such that no two adjacent elements have the same colour, is equal to the size of the largest complete subgraph of the graph i.e., the largest subgraph G such that for all vertices *a*, *b* in *G*, *a* is adjacent to *b*.

After that Anderson and Livingston (1999) [2] studied on zero divisor graphs on commutative ring. According to them, the vertices of zero divisor graph are the non-zero zero-divisor of the ring. That is if R is a commutative ring, $Z(R)$ is the set of zero-divisor of R and $Z^*(R) = Z(R) - \{0\}$, then the zero-divisor graph of R, Γ(Z*(R)), usually written as Γ(R), is the graph in which each element of $Z^*(R)$ is a vertex and two distinct vertices x and y are adjacent if and only if $xy = 0$. They prove that if R is a commutative ring, then the zero divisor graph, $\Gamma(R)$ is connected and $\mathrm{diam}(\Gamma(R)) \leq 3$, where $diam(\Gamma(R)) = sup\{d(x,y): x, y \text{ are vertices of } \Gamma\}$ is the diameter of a graph G and $d(x,y)$ is the length of the shortest path from x to y.

S. Akbari, H.R. Maimani and S. Yassemi [3] conducted a study "When a zero-divisor graph is planar or a complete r-partite graph" (An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset). They give a result that if R is a local ring with at least 33 elements and $\Gamma(R) \neq \emptyset$, then $\Gamma(R)$ is not planar.

J. Warfel (2004) [4] conducted study on zero divisor graphs for direct products of commutative rings. In his study, he examines the preservation of the diameter of the zero divisor graph of polynomial and power series rings.

F. DeMeyer, L. DeMeyer (2004) [5] studied on the zero divisor graph of a commutative semigroup with zero is a graph whose vertices are the non-zero zero divisors of the semigroup, with two distinct vertices joined by an edge if their product in the semigroup is zero.

S. Akbari, A. Mohammadian (2007) [6] gave a simple proof of the statement that for any finite ring R, Γ(R) has an even number of edges. They also gave some properties of zero divisor graphs of matrix rings and group rings.

E. A. Osba, S. Al-Addasi and N. A. Jaradeh (2008) [7] conducted a study on "Zero divisor graph for the ring of Gaussian integers modulo n". They give the number of vertices, the diameter, the girth of zero divisor graph for each positive integer n.

L. Demeyer, L. Greve, A. Sabbaghi and J. Wang [8] conducted a study on "The zero-divisor graph associated to a semigroup". They give the lowest bound on the number of edges necessary to a graph is a zero-divisor graph. They give the result that all graphs in $K_{n,p}$ for $p \leq \lceil n/2 \rceil + 1$ are zero-divisor graphs.

J. Coykendall, S. Sather-Wagstaff, L. Sheppardson, S. Spiroff (2012) [9] survey the research conducted on zero divisor graphs. They also introduced a graph for modules that is useful for studying zero divisor graphs of trivial extensions.

In this paper we present the basic properties of the zero-divisor graph for the ring of Gaussian integers modulo n, i.e. $\Gamma(\mathbb{Z}_n)[i]$.

The rest of this paper is organized as follows. In section II, we present the concept of ring of Gausssian integer. In section III, we present few definitions related to this study. In section IV, we study Zero-divisor graph for $\mathbb{Z}_{t^n}[i]$. Zero divisor graphs for $\mathbb{Z}_n[i]$ is presented in section V. Conclusions are made in section VI.

II. Ring of Gaussian integer modulo n, $\mathbb{Z}_n[i]$

Ring of Gaussian integers is a set of complex numbers whose real and imaginary parts are both integer, written as $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$ and it's norm is $N(a + ib) = a^2 + b^2$. In $\mathbb{Z}[i]$, an element $a + ib$ is said to be a *unit* if and only if $N(a + ib) = 1$. So the only units of $\mathbb{Z}[i]$ are $1, -1, i$ and $-i$.

The prime elements of $\mathbb{Z}[i]$ are known as Gaussian primes. Gaussian primes can be described as follows:

1. $1 + i$ and $1 - i$ are Gaussian primes.
2. If q is a prime integer with $q \equiv 3(mod 4)$, then q is a Gaussian prime.
3. If p is a prime integer with $p \equiv 1(mod 4)$ and $p = a^2 + b^2$ for some integer a and b, then $a + ib$ and $a - ib$ are Gaussian primes.

Let n be a natural number and let $\langle n \rangle$ be the principal ideal generated by n in $\mathbb{Z}[i]$ and let $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\}$ be the ring of integers modulo n. Then the factor ring $\mathbb{Z}[i]/\langle n \rangle$ is isomorphic to $\mathbb{Z}_n[i] = \{\bar{a} + i\bar{b} : \bar{a}, \bar{b} \in \mathbb{Z}_n\}$, i.e. $\mathbb{Z}_n[i]$ is a principal ideal ring [10]. The ring $\mathbb{Z}_n[i]$ is called *the ring of Gaussian integers modulo n.*

$\bar{a} + i\bar{b}$ is a unit in $\mathbb{Z}_n[i]$ if and only if $\bar{a}^2 + \bar{b}^2$ is a unit in $\mathbb{Z}_n$. If $n = \prod_{j=1}^{s} a_j^{k_j}$ is the prime power decomposition of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{a_j^{k_j}}[i]$. [7]

III. Some Definitions

***Definition 1* [7]:** If R is a commutative ring, $Z(R)$ is the set of zero-divisor of R and $Z^*(R) = Z(R) - \{0\}$, then the *zero-divisor graph* of R, $\Gamma(Z^*(R))$ usually written as $\Gamma(R)$, is the graph in which each element of $Z^*(R)$ is a vertex and two distinct vertices x and y are adjacent if and only if $xy = 0$.

***Definition 2* [7]:** The *diameter of Γ* is defined as $diam(\Gamma) = sup\{d(x, y) : x, y \text{ are vertices of } \Gamma\}$, where $d(x, y)$ is the length of the shortest path from x to y.

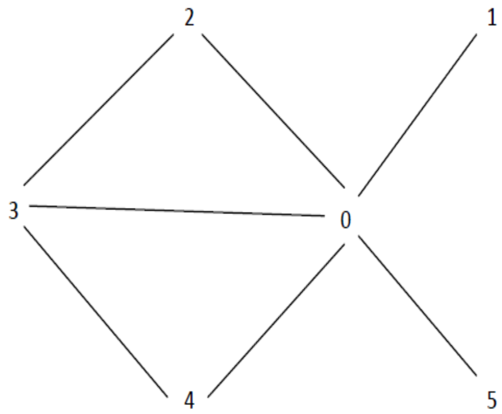

Fig 1: Beck's zero divisor graph of $\mathbb{Z}_6$

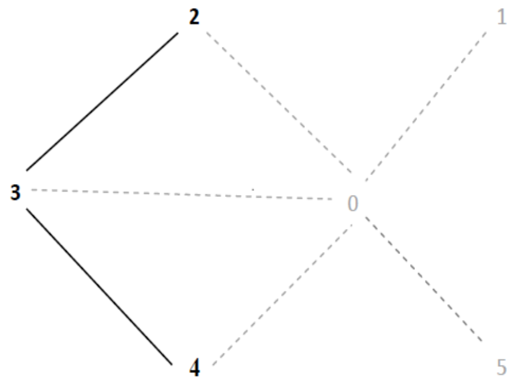

Fig 2: Anderson and Livingston's zero divisor graph of $\mathbb{Z}_6$

***Definition 3* [7]:** The *girth of Γ*, denoted by g(Γ), is the length of a shortest cycle in Γ. If Γ contains no cycle, then g(Γ)=∞.

***Definition 4* [7]:** If the vertex set of a graph Γ can be split into two disjoint sets A and B so that each edge of Γ joins a vertex of A to a vertex of B, then Γ is called a *bipartite graph*. If each vertex in A is joined to each vertex in B by a edge, then the graph is called a *complete bipartite graph*. If A contains n vertices and B contains m vertices, then Γ is denoted by $K_{n,m}$.

IV. Zero-divisor graph for $\mathbb{Z}_{t^n}[i]$

Here we consider three cases: $t = 2, t \equiv 3(mod 4) or\ t \equiv 1(mod 4)$.

4.1. *Zero-divisor graph for $\mathbb{Z}_{2^n}[i]$*

Since $2 = (1 + i)(1 - i)$, so 2 is not a Gaussian prime. Again $2 = -i(1 + i)^2$, so $\mathbb{Z}_2[i]$ is isomorphic to the local ring $\mathbb{Z}[i]/\langle(1 + i)^2\rangle$ with its only maximal ideal $\{\bar{0}, \bar{1} + i\bar{1}\}$.

Moreover, in $\mathbb{Z}_2$[i] we have $\bar{1} + i\bar{1} = \bar{1} - i\bar{1}$.

Thus we have $V\big(\Gamma(\mathbb{Z}_2[i])\big) = \{\bar{1} + i\bar{1}\}$, which implies that $\Gamma(\mathbb{Z}_2[i])$ is the null graph N_1, i.e., a graph with one vertex and no edges.

Now let n be an integer greater than 1.Then $2^n = (-i)^n(1 + i)^{2n}$ and so $\mathbb{Z}_{2^n}[i] \cong \frac{\mathbb{Z}[i]}{\langle 2^n \rangle} = \frac{\mathbb{Z}[i]}{\langle(1+i)^{2n}\rangle}$.

Hence $\mathbb{Z}_{2^n}[i]$ is local with its only maximal ideal $M = \langle \bar{1} + i\bar{1} \rangle$ and so $V\big(\Gamma(\mathbb{Z}_{2^n}[i])\big) = \langle \bar{1} + i\bar{1} \rangle \backslash \{\bar{0}\}$.

***Lemma 4.1.1* [7]:** The only maximal ideal in $\mathbb{Z}_{2^n}$[i] is $\{\bar{a} + i\bar{b} : a \text{ and } b \text{ are both even or odd }\}$.

***Theorem 4.1.1* [7]:** Let $n > 1$. Then for all $\alpha \in \mathbb{Z}_{2^n}$[i], $\alpha\bar{2}^{n-1}(\bar{1} + i\bar{1}) = \bar{0}$ or $\alpha\bar{2}^{n-1}(\bar{1} + i\bar{1}) = \bar{2}^{n-1}(\bar{1} + i\bar{1})$.

Proof: If α is not a unit, then $\alpha = (\bar{a} + i\bar{b})(\bar{1} + i\bar{1}) \in \langle \bar{1} + i\bar{1} \rangle$

$$\Rightarrow \alpha \bar{2}^{n-1}(\bar{1} + i\bar{1}) = (\bar{a} + i\bar{b})(\bar{1} + i\bar{1})\bar{2}^{n-1}(\bar{1} + i\bar{1})$$

$$= (\bar{a} + i\bar{b})(\bar{1} + i\bar{1})^2\bar{2}^{n-1} = \bar{0}$$

If α is a unit, then $\alpha = \bar{a} + i\bar{b}$, where a and b are neither both even nor odd.

So, $\bar{2}^{n-1}(\bar{1} + i\bar{1})(\alpha - \bar{1}) = \bar{2}^{n-1}(\bar{1} + i\bar{1})(\bar{a} + i\bar{b} - \bar{1})$

Since in this case $a - 1$ and b are both even or both odd, and so in this case $\alpha - \bar{1} \in \langle \bar{1} + i\bar{1} \rangle$ and so $\bar{2}^{n-1}(\bar{1} + i\bar{1})(\alpha - \bar{1}) = 0$

$$\Rightarrow \alpha \bar{2}^{n-1}(\bar{1} + i\bar{1}) = \bar{2}^{n-1}(\bar{1} + i\bar{1}).\blacksquare$$

Theorem 4.1.2: For $n \geq 1$, $|V(\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}]))| = 2^{2n-1} - 1$.

Proof: The total number of elements in $\mathbb{Z}_{2^n} = 2^n$, where the number of even elements $= 2^n/2$ and the number of odd elements$= 2^n/2$.

Since the only maximal ideal in $\mathbb{Z}_{2^n}[\mathrm{i}]$ is $\{\bar{a} + i\bar{b}: a \text{ and } b \text{ are both even or odd}, a, b \in \mathbb{Z}_{2^n}\}$, so the number of elements in the maximal ideal of $\mathbb{Z}_{2^n}[\mathrm{i}]$ is $= \frac{2^n}{2} \times \frac{2^n}{2} + \frac{2^n}{2} \times \frac{2^n}{2} = \frac{2^{2n} + 2^{2n}}{4} = 2.2^{2n}/4 = 2^{2n-1}$.

Since, $V(\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])) = \langle \bar{1} + i\bar{1} \rangle - \{\bar{0}\}$, so $|V(\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}]))| = 2^{2n-1} - 1$, for $n \geq 1$. $\blacksquare$

Theorem 4.1.3 **[7]:** For $n \geq 1$, $diam(\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])) = 2$.

Proof: The graph $\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])$ is not complete, since $\bar{2}$ and $\bar{1} + i\bar{1}$ are vertices in $\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])$ but $\bar{2}\ (\bar{1} + i\bar{1}) \neq 0$.

So, let $\alpha(\bar{1} + i\bar{1})$ and $\beta(\bar{1} + i\bar{1})$ be vertices in $\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])$ with $\alpha, \beta \in \mathbb{Z}_{2^n}[\mathrm{i}]$ and $\alpha(\bar{1} + i\bar{1}).\beta(\bar{1} + i\bar{1}) \neq 0$.

Then we have $\alpha(\bar{1} + i\bar{1}) \neq (\bar{1} + i\bar{1})^{n-1}$ and $\beta(\bar{1} + i\bar{1}) \neq (\bar{1} + i\bar{1})^{n-1}$.

Thus we have a path $\alpha(\bar{1} + i\bar{1})\_\_\_(\bar{1} + i\bar{1})^{n-1}\_\_\_\beta(\bar{1} + i\bar{1})$.

Hence $diam(\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])) = 2, n > 1$. $\blacksquare$

Theorem 4.1.4 **[7]:** For $n > 1$, $\mathrm{g}(\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])) = 3$.

Proof: For n=2, we have the cycle $\bar{2}\_\_\_i\bar{2}\_\_\_\bar{2} + i\bar{2}\_\_\_\bar{2}$.

For $n > 2$, we have the cycle $\bar{2}^{n-1}\_\_\_\bar{2}\_\_\_i\bar{2}^{n-1}\_\_\_\bar{2}^{n-1}$.

Hence, $\mathrm{g}(\Gamma(\mathbb{Z}_{2^n}[\mathrm{i}])) = 3, n > 1$. $\blacksquare$

Example: Consider the graph for n=2 i.e $\Gamma(\mathbb{Z}_4[\mathrm{i}])$.

Here the set of vertices is, $V(\Gamma(\mathbb{Z}_4[\mathrm{i}]))$=$\{\bar{2}, \bar{2}i, \bar{2} + \bar{2}i, \bar{1} + \bar{1}i, \bar{1} + \bar{3}i, \bar{3} + \bar{1}i, \bar{3} + \bar{3}i\}$.

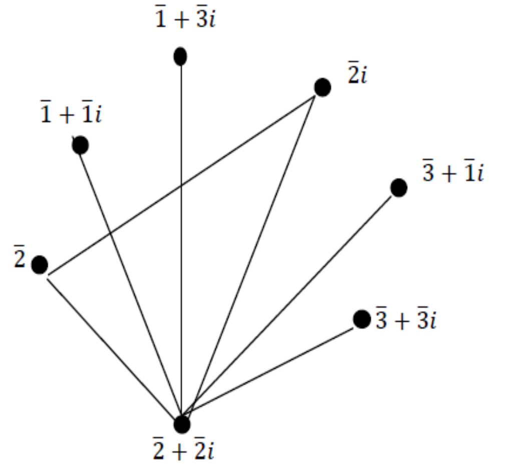

Fig 3: $\Gamma(\mathbb{Z}_4[\mathrm{i}])$

4.2. Zero-divisor graph for $\mathbb{Z}_{q^n}[i]$, q $\equiv$ 3(mod 4)

If q $\equiv$ 3(mod 4), then q is a Gaussian prime and so $\mathbb{Z}_q[i]$ is a splitting field for the polynomial $g(x) = x^2 + 1$ over the field $\mathbb{Z}_q$ and $\mathbb{Z}_q[i]$ is isomorphic to the field $\mathbb{Z}[i]/\langle q \rangle$. So in this case $\mathbb{Z}_q[i]$ has no nonzero zero divisors.

If $n > 1$, then $\mathbb{Z}_{q^n}[i] \cong \mathbb{Z}[i]/\langle q^n \rangle$ is a local ring with maximal ideal $\langle \bar{q} \rangle$. Hence $V\left(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])\right) = \langle \bar{q} \rangle \backslash \{\bar{0}\}$.

For any vertex $\alpha\bar{q}$ in $\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])$, $\alpha\bar{q}$ is adjacent to $\bar{q}^{n-1}$. In this case if $\alpha\bar{q}$ is a vertex in $\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])$, then $\alpha\bar{q}$ is adjacent to every element in $\langle \bar{q}^{n-1} \rangle \backslash \{\bar{0}\}$.

Theorem 4.2.1 **[7]:** For $n > 1$, $|V(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}]))| = q^{2n-2} - 1$.

Proof: We know that the number of units in $\mathbb{Z}_{q^n}[\mathrm{i}]$ is $q^{2n} - q^{2n-2}$.[7]

$$\therefore \left|V\left(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])\right)\right| = |\langle \bar{q} \rangle| - 1 = q^{2n} - (q^{2n} - q^{2n-2}) - 1 = q^{2n-2} - 1.\blacksquare$$

Theorem 4.2.2 **[7]:** For $n > 2$, $diam(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])) = 2$.

Proof: $\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])$ is not complete, since if $\alpha\bar{q}$ and $\beta\bar{q}$ be vertices in $\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])$ with $\alpha, \beta \in \mathbb{Z}_{q^n}[\mathrm{i}]$ and $\alpha\bar{q}.\beta\bar{q} \neq 0$, then we have $\alpha\bar{q} \neq \bar{q}^{n-1}$ and $\beta\bar{q} \neq \bar{q}^{n-1}$.

Thus we have a path $\alpha\bar{q}\_\_\_(\bar{q})^{n-1}\_\_\_\beta\bar{q}$. Hence $diam\left(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])\right) = 2, n > 2$. $\blacksquare$

But if $n=2$, then $\Gamma(\mathbb{Z}_{q^2}[\mathrm{i}])$ is the complete graph K_{q^2-1} and so $diam\left(\Gamma(\mathbb{Z}_{q^2}[\mathrm{i}])\right)=1$.

***Theorem 4.2.3* [7]:** For $n>1$, $\mathrm{g}(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}]))=3$.

Proof: If $n=2$, then $\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}])$ is complete with more than 3 vertices and so $\mathrm{g}(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}]))=3$.

If $n>2$, then we have the cycle $\bar{q}^{n-1}\_\_\_\bar{q}\_\_\_i\bar{q}^{n-1}\_\_\_\bar{q}^{n-1}$.

Hence $\mathrm{g}(\Gamma(\mathbb{Z}_{q^n}[\mathrm{i}]))=3$. ■

For example, consider the graph $\Gamma(\mathbb{Z}_{3^2}[\mathrm{i}])$. The set of vertices of the graph is, $V(\Gamma(\mathbb{Z}_9[\mathrm{i}]))$={$\bar{3},\bar{3}i,\bar{6},\bar{6}i,\bar{3}+\bar{3}i,\bar{3}+\bar{6}i,\bar{6}+\bar{3}i,\bar{6}+\bar{6}i$}.

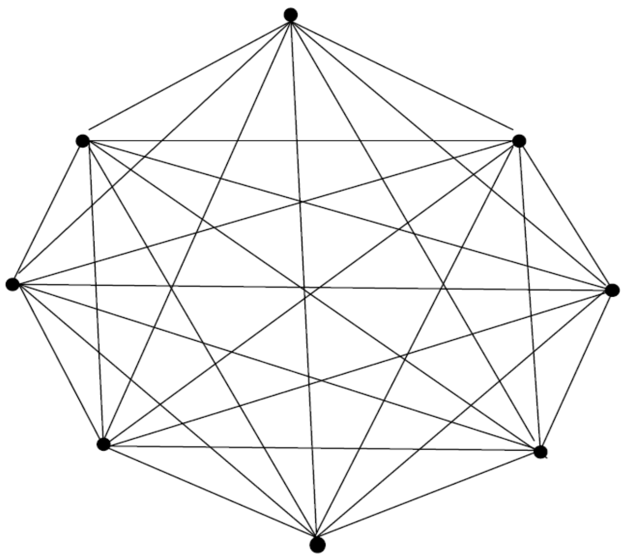

Fig 4: $\Gamma(\mathbb{Z}_9[\mathrm{i}])$

4.3 *Zero divisor graphs for* $\mathbb{Z}_{p^n}[i], p\equiv 1(mod\ 4)$

Let $\mathrm{p}\equiv 1(\mathrm{mod}\ 4)$, then there is an integer c such that $\mathrm{c}^2\equiv -1(\mathrm{mod}\ \mathrm{p})$ and there exist $a,b\in\mathbb{Z}$ such that $p=a^2+b^2=(a+ib)(a-ib)$. Also $a+ib$ and $a-ib$ are Gaussian primes in $\mathbb{Z}[i]$.

Thus the ideals $\langle a+ib\rangle$and $\langle a-ib\rangle$ are the only maximal ideals in $\mathbb{Z}[i]$ containing p, since $\mathbb{Z}[i]$ is a unique factorization domain.

Hence, $\mathbb{Z}_p[i]\cong\mathbb{Z}[i]/\langle p\rangle\cong(\mathbb{Z}[i]/\langle a+ib\rangle)(\mathbb{Z}[i]/\langle a-ib\rangle)$.

We know, the number of units in $\mathbb{Z}_p[i]$ is $(p-1)^2$. [7]

$\therefore\left|V(\Gamma(\mathbb{Z}_{\mathrm{p}}[\mathrm{i}]))\right|=p^2-(p-1)^2-1=2p-2$.

Also, $\Gamma(\mathbb{Z}_{\mathrm{p}}[\mathrm{i}])$ is the complete bipartite graph $K_{p-1,p-1}$ and so $diam\left(\Gamma(\mathbb{Z}_{\mathrm{p}}[\mathrm{i}])\right)=2$ and $\mathrm{g}\left(\Gamma(\mathbb{Z}_{\mathrm{p}}[\mathrm{i}])\right)=4$.

Example: Consider the graph $\Gamma(\mathbb{Z}_5[\mathrm{i}])$.

Here the set of vertices of the graph,

$V(\Gamma(\mathbb{Z}_5[\mathrm{i}]))=\{1+2i,1-2i,2+4i,\ 2-4i,3+i,3-i,4+3i,4-3i\}$.

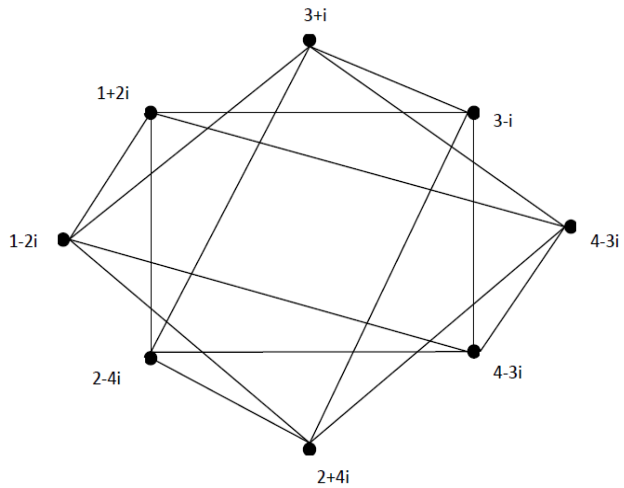

Fig 5: $\Gamma(\mathbb{Z}_5[\mathrm{i}])$

Now for $n>1$, $p^n=(a^2+b^2)^n=(a+ib)^n(a-ib)^n$.

Hence $\mathbb{Z}_{p^n}[i]\cong\mathbb{Z}[i]/\langle p^n\rangle\cong(\mathbb{Z}[i]/\langle(a+ib)^n\rangle)\times(\mathbb{Z}[i]/\langle(a-ib)^n\rangle)$.

In this case, $V\left(\Gamma(\mathbb{Z}_{\mathrm{p}^n}[\mathrm{i}])\right)=(\langle\bar{\mathrm{a}}+\mathrm{i}\bar{\mathrm{b}}\rangle\cup\langle\bar{\mathrm{a}}-\mathrm{i}\bar{\mathrm{b}}\rangle)\backslash\{\bar{0}\}$.

***Theorem 4.3.1* [7]:** For $n>1$, $\left|V(\Gamma(\mathbb{Z}_{\mathrm{p}^n}[\mathrm{i}]))\right|=2p^{2n-1}-p^{2n-2}-1$.

Proof: the number of units in $\mathbb{Z}_{\mathrm{p}^n}[\mathrm{i}]$ is $(p^n-p^{n-1})^2$. [7]

$$\therefore\left|V\left(\Gamma(\mathbb{Z}_{\mathrm{p}^n}[\mathrm{i}])\right)\right|=p^{2n}-(p^n-p^{n-1})^2-1$$
$$=2p^{2n-1}-p^{2n-2}-1. ■$$

***Theorem 4.3.2* [7]:** For $n>1$, $diam(\Gamma(\mathbb{Z}_{\mathrm{p}^n}[\mathrm{i}]))=3$.

Proof: Let $p=a^2+b^2$. Then $d(\bar{a}+i\bar{b},\bar{a}-i\bar{b})>1$.

If there exists $\bar{x}+i\bar{y}$ such that $(\bar{a}+i\bar{b})(\bar{x}+i\bar{y})=\bar{0}=(\bar{a}-i\bar{b})(\bar{x}+i\bar{y})$, then p^n divides $(ax+by)$, $(ay-bx)$, $(ax-by)$and $(ay+bx)$.

So p^n divides $2ax$ and $2by$ and hence p^n divides x and y i.e., $\bar{x}+i\bar{y}=0$.

Thus $d(\bar{a}+i\bar{b},\bar{a}-i\bar{b})>2$. Thus, we have the path $(\bar{a}+i\bar{b})\_\_\_(\bar{a}+i\bar{b})^{n-1}(\bar{a}-i\bar{b})^{n}\_\_\_(\bar{a}+i\bar{b})^{n}(\bar{a}-i\bar{b})^{n-1}\_\_\_(\bar{a}-i\bar{b})$, and since the diameter of a zero divisor graph of a finite commutative ring with unity is always less than or equal to 3, (Anderson et al. 1999), so $diam(\Gamma(\mathbb{Z}_{p^n}[i]))=3$.■

Theorem 4.3.3 **[7]:** For $n>1$, $g(\Gamma(\mathbb{Z}_{p^n}[i]))=3$.

Proof: For $n=2$, we have the cycle $\bar{p}\_\_\_p+i\bar{p}\_\_\_i\bar{p}\_\_\_\bar{p}$.

For $n>2$, we have the cycle $(\bar{p})^{n-1}\_\_\_\bar{p}\_\_\_i(\bar{p})^{n-1}\_\_\_(\bar{p})^{n-1}$.

Hence, $g\left(\Gamma(\mathbb{Z}_{p^n}[i])\right)=3, n>1$. ■

V. Zero divisor graphs for $\mathbb{Z}_n[i]$

In this section we consider the general case by assuming $n=\prod_{j=1}^{m}t_j^{\,n_j}$.

The function θ: $\mathbb{Z}_n[i]\rightarrow\prod_{j=1}^{m}\mathbb{Z}_{t_j^{\,n_j}}[i]$ such that $\theta(\bar{x}+i\bar{y})=((x\ mod\ (t_j)^{n_j})+i(y\ mod(t_j)^{n_j}))_{j=1}^{m}$ is an isomorphism.

Now let $n=2^k\times\prod_{j=1}^{m}q_j^{\,\alpha_j}\times\prod_{s=1}^{l}p_s^{\,\beta_s}$, then the number of units in $\mathbb{Z}_n[i]$ is $2^{2k-1}\times\prod_{j=1}^{m}(q_j^{\,2\alpha_j}-q_j^{\,2\alpha_j-2})\times\prod_{s=1}^{l}(p_s^{\,\beta_s}-p_s^{\,\beta_s-1})^2$.

Thus we get, $|V(\Gamma(\mathbb{Z}_n[i]))|=n-\left(2^{2k-1}\times\prod_{j=1}^{m}(q_j^{\,2\alpha_j}-q_j^{\,2\alpha_j-2})\times\prod_{s=1}^{l}(p_s^{\,\beta_s}-p_s^{\,\beta_s-1})^2\right)-1$.

Proposition 5.1 **[11]:** If R_1 and R_2 are commutative rings with identity and nonzero zero divisors, then $diam(\Gamma(R_1\times R_2))$= 3.

Using this result and the above theorems we get the following theorem:

Theorem 5.1 **[7]:** Let n be a positive integer greater than 1. Then,

(1) $diam(\Gamma(\mathbb{Z}_n[i]))$= 1 if and only if $n=q^2$.
(2) $diam(\Gamma(\mathbb{Z}_n[i]))$= 2 if and only if $n=p$ or $n=2^m$ with m ≥ 2 $or\ n=q^m$ with m ≥ 3.
(3) $diam(\Gamma(\mathbb{Z}_n[i]))$= 3 if and only if $n=p^m$ with $m\geq 2$ or n is divisible at least by two distinct primes.

Theorem 5.2 **[7]:** Let $n=\prod_{j=1}^{m}t_j^{\,n_j}$ be the prime factorization of n. Then:

(1) If $n_k>1$ for some k, then $g(\Gamma(\mathbb{Z}_n[i]))$= 3;
(2) If n_k= 1 for all k and m ≥ 3, then $g(\Gamma(\mathbb{Z}_n[i]))$= 3;
(3) If $n=p_1\times p_2$ or $n=p_1\times q$ or $n=p_1\times 2$, then $g(\Gamma(\mathbb{Z}_n[i]))$= 3;
(4) If $n=q_1\times q_2$, then $g(\Gamma(\mathbb{Z}_n[i]))$= 4;
(5) If $n=2\times q$, then $g(\Gamma(\mathbb{Z}_n[i]))$= 4.

Proposition 5.2 **[2]:** For a commutative ring R, Γ(R) is complete if and only if $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\ or\ xy=0$ for all $x,y\in\Gamma(R)$.

Theorem 5.3 **[7]:** The graph $\Gamma(\mathbb{Z}_n[i])$ is complete if and only if $n=q^2$.

Proof: It was shown earlier that if $n=q^2$, then $\Gamma(\mathbb{Z}_n[i])$ is a complete graph.

So assume that $\Gamma(\mathbb{Z}_n[i])$ is complete. Then n is a composite number not divisible by a^3 for any prime number a, since in this case $\bar{a}$ is not adjacent to $i\bar{a}$. Moreover, n is not divisible by two distinct primes a, b, since in this case, $\bar{a}$ is a vertex in $\Gamma(\mathbb{Z}_n[i])$, but $\bar{a}$ is not adjacent to $i\bar{a}$. Clearly $p^2\nmid n$, since if $p=a^2+b^2$, then $\bar{p}$ is not adjacent to $\bar{a}+i\bar{b}$, we have also $2\nmid n$, since $\bar{1}+i\bar{1}$ is not adjacent to $\bar{2}$. Thus, $n=q^2$.■

It is clear that if Γ is a complete bipartite graph $K_{m,n}$ with $\min\{m,n\}\geq 2$, then g(Γ)=4, so if Γ contains a cycle with length 3, it could not be a complete bipartite graph or even bipartite.

***Lemma 5.1*:** Let $R=R_1\times R_2$. Then Γ(R) is a complete bipartite graph if and only if R_1 and R_2 are integral domains.

Proof: If $R=R_1\times R_2$, where R_1 and R_2 are integral domains, then Γ(R) is a complete bipartite graph with $A=\{(x,0):x\in R_1\backslash\{0\}\}$ and $B=\{(x,0):x\in R_1\backslash\{0\}\}$ as the two disjoint sets of vertices such that every vertex in A is adjacent to every vertex in B, and we have no other adjacency. Now if R_1 is not an integral domain, with $x,y\in R_1\backslash\{0\}$ and $xy=0$, then we have the 3-cycle (x, 0) _ _ _ (y, 0) _ _ _ (0, 1) _ _ _ (x, 0), so Γ(R) is not a complete bipartite graph.■

If R is a direct product of more than two nontrivial integral domains, then R is reduced, and the intersection of any two prime ideals is nontrivial, so *Γ(R)* is not complete bipartite graph. So if p ≡ 1(mod4), with $p=a^2+b^2$, then $\Gamma(\mathbb{Z}_p[i])$ is a complete bipartite graph, since $\mathbb{Z}_p[i]\cong\mathbb{Z}[i]/\langle p\rangle\cong\mathbb{Z}[i]/\langle a+ib\rangle\cong\mathbb{Z}[i]/\langle a-ib\rangle$. And if q_1 and q_2 are two primes such that $q_j\equiv 3(mod4)$, for each j, then $\Gamma(\mathbb{Z}_{q_1q_2}[i])$ is a complete bipartite graph, since $\mathbb{Z}_{q_1q_2}[i]\cong\mathbb{Z}_{q_1}[i]\times\mathbb{Z}_{q_2}[i]$, a direct product of two fields. It is clear that $\Gamma(\mathbb{Z}_4[i])$ is not a complete bipartite graph, similarly $\Gamma(\mathbb{Z}_{q^2}[i])$ is not a complete bipartite graph, since it is complete on more than two vertices. $\Gamma(\mathbb{Z}_{p^2}[i])$ is not a complete bipartite graph, since if $p=a^2+b^2$, then we have the 3-cycle $\bar{p}(\bar{a}+i\bar{b})\_\_\_(\bar{a}-i\bar{b})\_\_\_i\bar{p}(\bar{a}+i\bar{b})\_\_\_\bar{p}(\bar{a}+i\bar{b})$. If a is a prime number, then $\Gamma(\mathbb{Z}_{a^3}[i])$ is not a complete bipartite graph, since we have the 3-cycle $\bar{a}\_\_\_\bar{a}\_\_\_i\bar{a}\_\_\_\bar{a}$. From this we can conclude that the graph $\Gamma(\mathbb{Z}_n[i])$ is complete bipartite if and only if $n=p$ or $n=q_1q_2$.

VI. CONCLUSIONS

In this paper we study Zero-divisor graph of ring of Gaussian integer. Here we discussed the number of vertex, diameter, and girth of the graphs. Also we found that the graph $\Gamma(\mathbb{Z}_n[i])$ is complete bipartite if and only if $n = p$ or $n = q_1 q_2$, where p, q_1, q_2 are prime numbers.

REFERENCES

[1] I. Beck, Coloring of commutative rings. *Journal of Algebra 116*, 1988, 208-226.

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring. *Journal of Algebra 217*, 1999, 434-447.

[3] S. Akbari, H. R. Maimani and S. Yassemi, When a zero-divisor graph is planar or a complete r-partite graph. *Journal of Algebra 270*, 2003, 169-180.

[4] J. Warfel, Zero-Divisor Graphs for Direct Products of Commutative rings. *Huston J. Mathematics*, 2004.

[5] F. DeMeyer and L. DeMeyer, Zero divisor graphs of semigroups, *Journal of Algebra 283*, 2005, 190-198.

[6] S. Akbari and A. Mohammadian, On zero-divisor graphs of finite rings. *Journal of Algebra 314*, 2007, 168-184.

[7] E. A. Osba, S. Al-Addasi and N. AJaradeh, Zero Divisor Graph for the Ring of Gaussian integers modulo n. *Communications in Algebra 36*, 2008, 3865-3877.

[8] L. Demeyer, L. Greve, A. Sabbaghi and J. Wang, The Zero-Divisor Graph Associated To A Semigroup, *Comm. Algebra 38*, 2010, 3370-3391.

[9] J. Coykendall, S. Sather-Wagstaff, L. Sheppardson and S. Spiroff, On Zero Divisor Graphs, *Progress in Commutative Algebra 2*, 2012, 241-299.

[10] G. Dresden and W. M. Dymacek, Finding Factors of Factor rings over the Gaussian Integers, *The Mathematical Association of America*, 2005.

[11] M. Axtell, J. Stickles and J. Warfel, Zero divisor graph for direct products of commutative rings, *Houston J. Math, 32(4)*, 2006, 985-994.