On the convergence of a finite family of contractive type mappings in CAT (0) space

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Abstract — *In this paper, we analyze the Mann and Ishikawa type iteration schemes for a finite family of uniformly L-Lipschizian asymptotically demicontractive mappings in CAT(0) space. Our results are the generalization of several recent results in the current literature.*

Keywords — Iteration schemes, CAT (0) space, Uniformly L-Lipschizian asymptotically demi-contractive mappings.

I. INTRODUCTION

More recently, many of the standard ideas of nonlinear analysis have been extended to the class of so-called CAT(0) spaces, [So named by Gromov[1] in honor of Cartan, Alexandrov, and Toponogov]. First time, W.A. Kirk[4] developed the fixed point theory for CAT(0) spaces and proved an interesting fact about the fixed point set. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed. In 2008, Dhompongsa and Panyanak[7] used the concept of Δ-convergence introduced by Lim to prove the CAT(0) space analogs and obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in the CAT(0) space setting.

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane [2]. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Complete CAT(0) spaces are often called Hadamard spaces[4]. For a thorough discussion of these space and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [2] or Burago et al.[3].

Liu [8] has proved the convergence of Mann and Ishikawa iterative sequence for uniformly L-Lipschitzian asymptotically demicontractive and hemicontractive mappings in Hilbert space. The approximation of fixed points of one or more nonexpansive, asymptotically nonexpansive, or asymptotically quasi-nonexpansive mappings by various iterations have been extensively studied in Banach spaces, convex metric spaces, $CAT(0)$ spaces, and so on [4, 8, 10-25].

In this paper, we establish theorem of strong convergence for the Mann iteration scheme to a fixed point of a finite family of a uniformly L-Lipschitzian asymptotically demicontractive mapping in $CAT(0)$ space.

II. PRELIMINARIES

Let (X, d) be a metric space. A geodesic path joining x ∈ X to y ∈ X (or, more briefly, a geodesic from x to y) is a mapping c from a closed interval $[0,l] \subseteq R$ to X such that c(0) = x ; c(l) = y, and d(c(t), c(t$_0$)) = $|t-t_0|$ for all t, $t_0 \in [0,l]$. In particular, c is an isometry and d(x, y) = l . The image of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by [x, y]. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, y ∈ X. A subset Y $\subseteq$ X is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane R^2 such that $d_{R^2}(\bar{x}_i, \bar{y}_j) = d(x_i, y_j)$ for i , j ∈ $\{1,2,3\}$. Such a triangle always exists [2].

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subseteq R^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all x, y $\in \Delta$ and all the comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x,y) \le d(\bar{x},\bar{y})$$

If x_1, y_1, z_1 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \le \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2)$$

This is the (CN) inequality of Bruhat and Tits [5]. In fact, a geodesic space is CAT(0) space if and only if it satisfies the (CN) inequality [2]. The previous inequality has been extended by Khamsi and Kirk [6] as

$$d^2(z, \alpha x \oplus (1-\alpha) y) \le \alpha d^2(z,x) + (1-\alpha) d^2(z,y) - \alpha(1-\alpha) d^2(x,y) \qquad (CN^*)$$

for any $\alpha \in [0,1]$ and x, y, z $\in$ X. The inequality (CN^*) also appeared in [7].

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [2, page 163]). Moreover, if X is a CAT(0) metric space and x, y ∈ X, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1 - \alpha) y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha) y) \le \alpha d(z,x) + (1-\alpha) d(z,y)$$

for any z ∈ X and

$[x, y] = \{ \alpha x \oplus (1 - \alpha) y : \alpha \in [0, 1]\}$.

Now we introduce some important definitions as follows:

Definitions: Let C be a nonempty subset of a metric space (X, d). Let $F(T)$ denote the fixed point set of T. Let $F(T) \ne \phi$.

(1) A mapping $T : C \to C$ is said to be k-strict asymptotically pseudocontractive with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ for some constant k, $0 \le k < 1$ and

$$d^2(T^n x, T^n y) \le a_n^2 d^2(x,y) + k\left(d(x, T^n x) - d(y, T^n y)\right)^2$$

for all $x, y \in C$, $n \in \mathbb{N}$.

If $k = 0$, then T is said to be asymptotically nonexpansive with sequence $\{a_n\}$, that is,

$$d(T^n x, T^n y) \le a_n d(x, y), \ \forall\, x, y \in C.$$

(2) A mapping $T : C \to C$ is said to be asymptotically demicontractive with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ for some constant k, $0 \le k < 1$, and

$$d^2(T^n x, p) \le a_n^2 d^2(x,p) + k \cdot d^2(x, T^n x),$$

$\forall\, p \in F(T)$, $x \in C$, $n \in \mathbb{N}$.

If $k = 0$, then T is said to be asymptotically quasi nonexpansive with sequence $\{a_n\}$, that is,

$$d(T^n x, p) \le a_n d(x, p), \ \forall\, x \in C, \forall\, p \in F(T).$$

(3) A mapping $T : C \to C$ is said to be asymptotically pseudocontractive with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ and

$$d^2(T^n x, T^n y) \le a_n d^2(x,y) + \left(d(x, T^n x) - d(y, T^n y)\right)^2$$

for all $x, y \in C$, $n \in \mathbb{N}$.

(4) A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if for some constant $L > 0$,

$$d\left(T^{n}x,T^{n}y\right)\le L\cdot d\left(x,y\right),\ \forall\ x,y\in C,$$

for all $n \in$ N.

Let C be a nonempty convex subset of a $CAT(0)$ space (X, d) and let $T : C \to C$ be a given mapping. Let $x_1 \in C$ be a given point.

The sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative process

$$x_{n+1} = (1 - \alpha_n)\, x_n \oplus \alpha_n T^n y_n,$$

$$y_n = (1 - \beta_n)\, x_n \oplus \beta_n T^n x_n, \quad n \ge 1, \tag{2.1}$$

is called an Ishikawa-type iterative sequence [26].

If $\beta_n \equiv 0$, then (2.1) reduces to the sequence $\{x_n\}$ defined by the iterative process

$$x_{n+1} = (1 - \alpha_n)\, x_n \oplus \alpha_n T^n x_n, \quad n \ge 1, \tag{2.2}$$

which is called a Mann-type iterative sequence [27].

Lemma 2.1 [8]. Let sequences $\{a_n\}$, $\{b_n\}$ satisfy that

$$a_{n+1} \le a_n + b_n,\ \ a_n \ge 0,$$

for all $n \in$ N, $\sum_{n=1}^{\infty} b_n$ is convergent, and $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to 0. Then, we must have $\lim_{n\to\infty} a_n = 0$

III. Main Results

First we prove a lemma as follows:

Lemma 3.1 Let (X, d) be a CAT(0) space and let C be a nonempty convex subset of X. Let T_i : C→ C, $i = 1,2,\ldots\ldots,n$ be a finite family of uniformly L-Lipschitzian mapping and let $\{\alpha_n\},\{\beta_n\}$ be sequences in [0,1]. Define the iteration scheme $\{x_n\}$ as (2.1). Then

$$d\left(x_n,T_i x_n\right)\le d\left(x_n,T_i^{n}x_n\right) + L(1+2\mathrm{L}+\mathrm{L}^2)\,\mathrm{d}\left(x_{n-1},T_i^{n-1}x_{n-1}\right)$$

for all n ≥ 1.

Proof. Let $D_n = d\left(x_n,T_i^{n}x_n\right)$, We have

$$d\left(x_{n-1},y_{n-1}\right) = d\left(x_{n-1},(1-\beta_{n-1})x_{n-1}\oplus\beta_{n-1}T_i^{n-1}x_{n-1}\right)$$

$$\le \beta_{n-1}\cdot d\left(x_{n-1},T_i^{n-1}x_{n-1}\right) = \beta_{n-1}D_{n-1} \tag{3.1}$$

From (3.1), we get

$$\begin{aligned} d\left(x_{n-1},T_i^{n-1}y_{n-1}\right) &\le d\left(x_{n-1},T_i^{n-1}x_{n-1}\right) + d\left(T_i^{n-1}x_{n-1},T_i^{n-1}y_{n-1}\right)\\ &\le D_{n-1} + L\cdot d\left(x_{n-1},y_{n-1}\right)\\ &\le D_{n-1} + \beta_{n-1}\cdot L\cdot D_{n-1} \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} & d\left(x_n,T_i x_n\right)\le d\left(x_n,T_i^{n}x_n\right)+d\left(T_i^{n}x_n,T_i x_n\right)\\ &\le D_n + L\cdot d\left(T_i^{n-1}x_n,x_n\right)\\ &\le D_n + L\cdot\left\{d\left(T_i^{n-1}x_n,T_i^{n-1}x_{n-1}\right)+d\left(T_i^{n-1}x_{n-1},x_n\right)\right\}\\ &\le D_n + L^2\cdot d\left(x_n,x_{n-1}\right)+L\cdot d\left(T_i^{n-1}x_{n-1},x_n\right)\\ &\le D_n + L^2\cdot d\left(\left(1-\alpha_{n-1}\right)x_{n-1}\oplus\alpha_{n-1}T_i^{n-1}y_{n-1},x_{n-1}\right)\\ &\quad + L\cdot d\left(T_i^{n-1}x_{n-1},\left(1-\alpha_{n-1}\right)x_{n-1}\oplus\alpha_{n-1}T_i^{n-1}y_{n-1}\right)\\ &\le D_n + L^2\cdot\alpha_{n-1}d\left(T_i^{n-1}y_{n-1},x_{n-1}\right)\\ &\quad + L\cdot\left\{\begin{array}{l}\left(1-\alpha_{n-1}\right)d\left(T_i^{n-1}x_{n-1},x_{n-1}\right)\\ \qquad +\alpha_{n-1}d\left(T_i^{n-1}x_{n-1},T_i^{n-1}y_{n-1}\right)\end{array}\right\}\\ &\le D_n + L^2\cdot\alpha_{n-1}\left(D_{n-1}+\beta_{n-1}\cdot L\cdot D_{n-1}\right)\\ &\quad + L\cdot\left(1-\alpha_{n-1}\right)D_{n-1}+L^2\cdot\alpha_{n-1}\cdot\beta_{n-1}\cdot D_{n-1}\\ &\le D_{n-1}+L(1+2\mathrm{L}+\mathrm{L}^2)D_{n-1},\quad n\ge 1 \end{aligned} \tag{3.3}$$

This completes the proof of lemma.

Theorem 3.1 Let (X, d) be a complete CAT(0) space, let C ⊆X be a nonempty bounded closed convex set. Let $T_i : C \to C, i = 1,2,3,\ldots\ldots,n$ be a family of complete continuous and uniformly L-Lipschitzian and asymptotically demicontractive with sequence $\{a_n\}$, $a_n \in [1,\infty)$,

$\sum_{n=1}^{\infty}\left(a_n^2-1\right)<\infty,\ \varepsilon\le\alpha_n\le 1-k-\varepsilon$, for all n ∈ N and some $\varepsilon > 0$. Given $x_0 \in C$, define the iteration scheme $\{x_n\}$ as

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_i^n x_n$$

Then $\{x_n\}$ converges strongly to some fixed point of $\{T_i\}$.

Proof. Since T_i be a family of a completely continuous mapping in a bounded closed convex subset C of complete metric space, from Schauder's theorem, (T_i) is nonempty. It follows from (CN^*) inequality that

$$d^2(x_{n+1}, p) = d^2((1-\alpha_n)x_n \oplus \alpha_n T_i^n x_n, p)$$

$$\le (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(T_i^n x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, T_i^n x_n) \tag{3.4}$$

for all $p \in F(T_i)$.

Since T_i be a family of asymptotically demicontractive mappings, we get

$$\begin{aligned} d^2(x_{n+1}, p) &\le (1-\alpha_n)d^2(x_n, p) \\ &\quad + \alpha_n\left\{a_n^2 d^2(x_n, p) + k \cdot d^2(x_n, T_i^n x_n)\right\} \\ &\quad - \alpha_n(1-\alpha_n)d^2(x_n, T_i^n x_n) \\ &\le d^2(x_n, p) + \alpha_n(a_n^2 - 1)d^2(x_n, p) \\ &\quad - \alpha_n(1-\alpha_n-k)d^2(x_n, T_i^n x_n) \end{aligned} \tag{3.5}$$

for all $p \in F(T_i)$.

Since $0 < \varepsilon \le \alpha_n \le 1-k-\varepsilon$, we have $1-k-\alpha_n \ge \varepsilon$. Thus,

$$\alpha_n(1-k-\alpha_n) \ge \varepsilon^2 \tag{3.6}$$

Now (3.5) and (3.6) implies that

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) + \alpha_n(a_n^2 - 1)d^2(x_n, p) \\ &\quad - \varepsilon^2 \cdot d^2(x_n, T_i^n x_n) \end{aligned} \tag{3.7}$$

for all $p \in F(T_i)$.

Since C is bounded and T_i's are self-mapping in C, there exists some M >0 so that $d^2(x_n, p) \le M$, for all n∈N. Since $0 < \alpha_n \le 1$, it follows from (3.7) that

$$\begin{aligned} d^2(x_{n+1}, p) &\le d^2(x_n, p) + (a_n^2 - 1)M \\ &\quad - \varepsilon^2 \cdot d^2(x_n, T_i^n x_n) \end{aligned} \tag{3.8}$$

for all $p \in F(T_i)$.

Therefore,

$$\begin{aligned} \varepsilon^2 \cdot d^2(x_n, T_i^n x_n) &\le d^2(x_n, p) - d^2(x_{n+1}, p) \\ &\quad + (a_n^2 - 1)M \end{aligned} \tag{3.9}$$

So,

$$\begin{aligned} \sum_{n=1}^{m} \varepsilon^2 \cdot d^2(x_n, T_i^n x_n) &\le d^2(x_1, p) - d^2(x_{m+1}, p) \\ &\quad + \sum_{n=1}^{m}(a_n^2 - 1)M \\ &\le 2M + \sum_{n=1}^{m}(a_n^2 - 1)M \end{aligned}$$

for all m ∈ N. Since $\sum_{n=1}^{m}(a_n^2 - 1) < \infty$, we get

$$\sum_{n=1}^{m} \varepsilon^2 \cdot d^2(x_n, T_i^n x_n) < \infty \tag{3.10}$$

Therefore,

$$\lim_{n\to\infty} d^2(x_n, T_i^n x_n) = 0,$$

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0 \tag{3.11}$$

Since T_i be a family of uniformly L-Lipschitzian mappings, it follows from lemma 3.1 that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \tag{3.12}$$

Since $\{x_n\}$ is a bounded sequence and T_i's are completely continuous, there exist a convergent subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$. Therefore, from (3.12), $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $\lim_{n\to\infty} x_{n_k} = q$. It follows from the continuity of T_i's and (3.12), we have $q = T_i q$. Therefore, {x} has a subsequence which converges to the fixed point q of T_i.

Let $p=q$ in the inequality (3.8). Since $\sum_{n=1}^{m}\left(a_n^2-1\right)<\infty$ and $\sum_{n=1}^{m}\varepsilon^2\cdot d^2\left(x_n,T_i^n x_n\right)<\infty$, from (3.8) and Lemma 2.1, we have

$$\lim_{n\to\infty} d^2\left(x_n,q\right)=0,$$

Therefore, $\lim_{n\to\infty} x_n = q,$

This completes the proof of Theorem 3.1.

Corollary 3.2 Let (X, d) be a complete CAT(0) space, let $C \subseteq X$ be a nonempty bounded closed convex set. Let $T_i : C \to C, i = 1,2,3,.......,n$ be a family of complete continuous and uniformly L-Lipschitzian and k-strict asymptotically pseudocontractive with sequence $\{a_n\}$, $a_n \in [1,\infty)$, $\sum_{n=1}^{\infty}\left(a_n^2-1\right)<\infty$, $\varepsilon\le\alpha_n\le1-k-\varepsilon$, for all n ∈ N and some $\varepsilon>0$. Given $x_0 \in C$, define the iteration scheme $\{x_n\}$ as

$$x_{n+1}=\left(1-\alpha_n\right)x_n\oplus\alpha_n T_i^n x_n$$

Then $\{x_n\}$ converges strongly to some fixed point of $\{T_i\}$.

Proof. Since T_i's are k-strict asymptotically pseudocontractive; then T_i's must be asymptotically demicontractive (by definition). Therefore, Corollary 3.2 can be proved by using theorem 3.1.

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