On Topological ψ*α -Quotient Mappings

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Abstract: *Only a few class of generalized closed sets form a topology. The class of $\psi^*\alpha$-closed set is one among them. In this paper we introduce $\psi^*\alpha$ -quotient maps using $\psi^*\alpha$ -closed sets and study their properties. Also we obtain the relations between weak and strong form of $\psi^*\alpha$ -quotient maps. We also study the relationship between $\psi^*\alpha$ -quotient maps and already existing quotient maps.*

Keywords: ψg-closed sets, ψg-open sets, $\psi^*\alpha$-closed sets, $\psi^*\alpha$-open sets and $\psi^*\alpha$-quotient map

1.Introduction

Njastad [10] introduced the concept of an α -open sets. Lellis Thivagar [6] introduced the concept of α -quotient mappings and α^* -quotient mappings in topological spaces. Balamani and Parvathi [1] introduced the notion of $\psi^*\alpha$-closed sets using ψg-open sets. In this paper we introduce and study $\psi^*\alpha$ -quotient mappings.

2.Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The interior and closure of a subset A of a space (X, τ) are denoted by int(A) and cl(A) respectively.

Definition 2.1 A subset A of a topological space (X, τ) is called
1) generalized closed set (briefly g-closed)[7] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
2) semi-generalized closed set (briefly sg-closed)[5] if scl(A) ⊆ U whenever A ⊆ U and U is semi- open in (X, τ).
3) ψ-closed set [12] if scl(A) ⊆ U whenever A ⊆ U and U is sg-open in (X, τ).
4) ψg-closed set [11] if ψcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
5) $\psi^*\alpha$ -closed set [1] if αcl(A) ⊆ U whenever A ⊆ U and U is ψg -open in (X, τ).
6)The $\psi^*\alpha$ -closure of a set A is defined as $\psi^*\alpha cl(A) = \cap \{F \subseteq X: A \subseteq F$ and F is $\psi^*\alpha$ -closed in (X, τ)}[1]

Definition 2.2 A topological space (X, τ) is said to be a $_{\psi^*\alpha}T_c$-space if every $\psi^*\alpha$ -closed subset of (X, τ) is closed in (X, τ).[2]

Definition 2.3 A map f : (X, τ) → (Y, σ) is called

(i)Continuous [7] if $f^{-1}(V)$ is closed in (X, τ) for each closed set V of (Y, σ).

(ii) α - continuous [8] if $f^{-1}(V)$ is α -closed in (X, τ) for every closed set V of (Y, σ).

(iii) $\psi^*\alpha$ -continuous [3] if $f^{-1}(V)$ is $\psi^*\alpha$ - closed in (X, τ) for each closed set V of (Y, σ).

(iv) quasi $\psi^*\alpha$ -continuous [4] if $f^{-1}(V)$ is closed in (X, τ) for each $\psi^*\alpha$ - closed set V of (Y, σ).

Definition 2.4 A map f : (X, τ) → (Y, σ) is called $\psi^*\alpha$ -irresolute [4] if $f^{-1}(V)$ is $\psi^*\alpha$ -closed in (X, τ) for every $\psi^*\alpha$ -closed set V of (Y, σ).

Definition 2.5A surjective map f : (X, τ) → (Y, σ) is called

(i) a quotient map [9], provided a subset U of (Y, σ) is open in (Y, σ) if and only if $f^{-1}(U)$ is open in (X, τ).

(ii) an α- quotient map [6] if f is α -continuous and $f^{-1}(V)$ is open in (X, τ) implies V is an α -open set in (Y, σ).

(iii) an α^*- quotient map [6] if f is α -irresolute and $f^{-1}(V)$ is an α -open set in (X, τ) implies V is an open set in (Y, σ).

3 ψ^* α -quotient maps, strongly $\psi^*\alpha$ - quotient maps and $(\psi^*\alpha)^*$ - quotient maps

Definition 3.1 A surjective map f : (X, τ) → (Y, σ) is called a **$\psi^*\alpha$ - quotient map** if f is $\psi^*\alpha$ -continuous and f^{-1} (V) is open in (X, τ) implies V is a $\psi^*\alpha$ -open set in (Y, σ)

Example 3.2 Let X = {a, b, c}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y},$\psi^*\alpha O(X)$ = {ϕ, {a}, {a, b}, {a, c}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {m}, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = n. Then f is a $\psi^*\alpha$ -quotient map.

Definition 3.3 A map f : (X, τ) → (Y, σ) is called $\psi^*\alpha$ -open (resp. strongly $\psi^*\alpha$ -open) if f(U) is $\psi^*\alpha$

-open in (Y, σ) for every open set (resp. $\psi^*\alpha$ -open set) U in (X, τ).

Proposition 3.4 If a map f : (X, τ) → (Y, σ) is surjective, $\psi^*\alpha$ -continuous and $\psi^*\alpha$ -open, then f is a $\psi^*\alpha$ -quotient map.

Proof: It is enough to prove that $f^{-1}(V)$ is open in (X, τ) implies V is a $\psi^*\alpha$ -open set in (Y, σ). Let $f^{-1}(V)$ is open in (X, τ). Then $f(f^{-1}(V))$ is $\psi^*\alpha$ -open, since f is $\psi^*\alpha$ -open. As f is surjective $f(f^{-1}(V))$ = V and so V is a $\psi^*\alpha$ -open set in (Y, σ). Hence f is a $\psi^*\alpha$ -quotient map.

Definition 3.5 Let f : (X, τ) → (Y, σ) be a surjective map. Then f is called **strongly $\psi^*\alpha$ - quotient map** provided a set U of (Y, σ) is open in Y if and only if $f^{-1}(U)$ is a $\psi^*\alpha$ -open set in (X, τ).

Example 3.6 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, {m}, {l, m}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {m}, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l = f(b), f(c) = m, f(d) = n. Then f is a strongly $\psi^*\alpha$ - quotient map.

Proposition 3.7 Every strongly $\psi^*\alpha$ -quotient map is a $\psi^*\alpha$ -open map but not conversely.

Proof: f : (X, τ) → (Y, σ) be a strongly $\psi^*\alpha$ - quotient map. Let V be an open set in (X, τ).Since every open set is $\psi^*\alpha$ -open and hence V is $\psi^*\alpha$ - open in (X, τ). That is $f^{-1}(f(V))$ is $\psi^*\alpha$ -open in (X, τ). Since f is strongly $\psi^*\alpha$ -quotient, f(V) is open and hence f(V) is $\psi^*\alpha$ -open in (Y, σ).Therefore f is a $\psi^*\alpha$ -open map.

Example 3.8 Let X = {a, b, c, d}, τ = {ϕ, {a}, X}, Y = {l, m, n}, σ = { ϕ, {l}, {m}, {l, m}, {l, n}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {m}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l = f(b), f(c) = n, f(d) = m. Then f is $\psi^*\alpha$ -open but not strongly $\psi^*\alpha$ - quotient, since f^{-1}({m}) = {d} is not $\psi^*\alpha$ -open in (X, τ)

Proposition 3.9 Every strongly $\psi^*\alpha$ -quotient map is a strongly $\psi^*\alpha$ -open map but not conversely.

Proof: f : (X, τ) → (Y, σ) be a strongly $\psi^*\alpha$ - quotient map. Let V be a $\psi^*\alpha$ -open set in (X, τ). That is $f^{-1}(f(V))$ is $\psi^*\alpha$ -open in (X, τ). Since f is strongly $\psi^*\alpha$ -quotient, f(V) is open and hence f(V) is $\psi^*\alpha$ -open in (Y, σ).Therefore f is a strongly $\psi^*\alpha$ -open map.

Example 3.10 Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {b}, {a, b}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {m}, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = n. Then f is strongly $\psi^*\alpha$ -open but not strongly $\psi^*\alpha$ - quotient, since f^{-1}({l}) = {a} is $\psi^*\alpha$ -open in (X, τ) but the set {l} is not open in (Y, σ).

Definition 3.11 Let f : (X, τ) → (Y, σ) be a surjective map. Then f is called a **$(\psi^*\alpha)^*$ - quotient map** if f is $\psi^*\alpha$ -irresolute and $f^{-1}(U)$ is $\psi^*\alpha$ -open set in (X, τ) implies U is open in (Y, σ).

Example 3.12 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, {l, m}, {l, n}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) =l, f(b) = n, f(c) = f(d) = m. Then f is a strongly $(\psi^*\alpha)^*$ -quotient map.

Proposition 3.13 Every $(\psi^*\alpha)^*$ -quotient map is a $\psi^*\alpha$ - irresolute map but not conversely.

Proof: Follows from the definition.

Example 3.14 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, {l, m}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n, f(c) = f(d) = m. Then f is a $\psi^*\alpha$ - irresolute map but not strongly $(\psi^*\alpha)^*$ -quotient map, since f^{-1}({l, n}) = {a, b} is $\psi^*\alpha$ -open in (X, τ) but the set {l, n} is not open in (Y, σ).

Proposition 3.15 Every $(\psi^*\alpha)^*$ -quotient map is a strongly $\psi^*\alpha$ -open map but not conversely.

Proof: f : (X, τ) → (Y, σ) be a $(\psi^*\alpha)^*$ - quotient map. Let V be a $\psi^*\alpha$ -open set in (X, τ). That is $f^{-1}(f(V))$ is $\psi^*\alpha$ -open in (X, τ). Since f is $(\psi^*\alpha)^*$ - quotient, f(V) is open and hence f(V) is $\psi^*\alpha$ -open in (Y, σ). Therefore f is a strongly $\psi^*\alpha$ -open map.

Example 3.16 Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {b}, {a, b}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {m}, {l, m}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = n. Then f is strongly $\psi^*\alpha$ -open but not $(\psi^*\alpha)^*$ - quotient, since f^{-1}({l}) = {a} is $\psi^*\alpha$ -open in (X, τ) but the set {l} is not open in (Y, σ).

Proposition 3.17

(i) Every quotient map is a $\psi^*\alpha$ - quotient map.
(ii) Every α - quotient map is a $\psi^*\alpha$ - quotient map.

Proof: (i) Since every continuous map is a $\psi^*\alpha$ -continuous map and every open set is $\psi^*\alpha$ -open, the proof follows from the definition.

(ii)Since every α -continuous map is a $\psi^*\alpha$ -continuous map and every α -open set is $\psi^*\alpha$ -open, the proof follows from the definition.

The converse of the statements in the above proposition need not be true as seen from the following examples.

Example 3.18 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n, f(c) = f(d) = m. Then the map f is $\psi^*\alpha$ - quotien but not quotient, since for the $\psi^*\alpha$ -open set {l, m} in (Y, σ). (f^{-1}({l, m}) = {a, c, d} is open in (X, τ) but the set {l, m} is not open in (Y, σ).

Example 3.19 Let X = {a, b, c, d}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l, m}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {a, b}, {a, c}, {a, d},{a, b, c}, {a, b, d}, {a, c, d}, X}, $\psi^*\alpha O(Y)$ = { ϕ, {l}, {m}, {l, m}, Y} and $\alpha O(Y)$ = { ϕ, {l, m}, Y} Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = m, f(c) = f(d) = n. Then the map f is $\psi^*\alpha$ - quotient but not α - quotient, since (f^{-1}({l}) = {a} is open in (X, τ) but the set {l} is not α - open in (Y, σ).

Proposition 3.20 Every strongly $\psi^*\alpha$ -quotient map is a $\psi^*\alpha$ - quotient map but not conversely.

Proof: Let V be an open set in (Y, σ). Since f is strongly $\psi^*\alpha$ -quotient, $f^{-1}(V)$ is a $\psi^*\alpha$ -open set in (X, τ). Hence f is $\psi^*\alpha$ -continuous, Let $f^{-1}(V)$ be a open set in (X, τ). Then $f^{-1}(V)$ is a $\psi^*\alpha$ -open set in (X, τ). Since f is strongly $\psi^*\alpha$ -quotient, V is open in (Y, σ). Hence f is $\psi^*\alpha$ -quotient.

Example 3.21 Let X = {a, b, c, d}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X}, Y = {l, m, n}, σ = { ϕ, {l}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n, f(c) = f(d) = m. Then the map f is $\psi^*\alpha$ - quotient but not strongly $\psi^*\alpha$ -quotient, since for the $\psi^*\alpha$ -open set {l, m} in (Y, σ). f^{-1}({l, m}) = {a, c, d} is open in (X, τ) but the set {l, m} is not open in (Y, σ).

Proposition 3.22 Every $(\psi^*\alpha)^*$ -quotient map is strongly $\psi^*\alpha$ -quotient map but not conversely.

Proof: Let V be an open set in (Y, σ). Then it is $\psi^*\alpha$ -open in (Y, σ). Since f is $(\psi^*\alpha)^*$ -quotient, $f^{-1}(V)$ is a $\psi^*\alpha$ -open set in (X, τ). If $f^{-1}(V)$ is $\psi^*\alpha$ -open set in (X, τ). Then V is open in (Y, σ) as f is $(\psi^*\alpha)^*$ -quotient. Hence f is a strongly $\psi^*\alpha$ -quotient map.

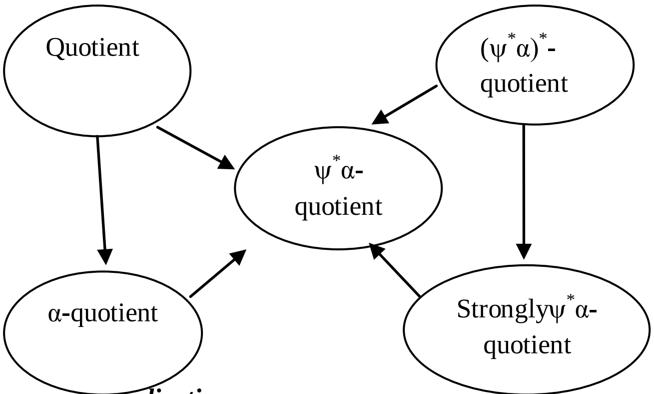
Example 3.23 Let X = {a, b, c, d}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n = f(c), f(d) = m. Then the map f is strongly $\psi^*\alpha$ - quotient but not $(\psi^*\alpha)^*$ -quotient, since for the $\psi^*\alpha$ -open set {l, n} in (Y, σ) f^{-1}({l, n}) = {a, b, c} is $\psi^*\alpha$ -open in (X, τ) but the set {l, n} is not open in (Y, σ).

Proposition 3.24 Every $(\psi^*\alpha)^*$ -quotient map is a $\psi^*\alpha$ -quotient map but not conversely.

Proof: Let f be a $(\psi^*\alpha)^*$ -quotient map. Then f is $\psi^*\alpha$ -irresolute, by **theorem 3.4[4]** f is $\psi^*\alpha$ -continuous. Let $f^{-1}(V)$ be an open set in (X, τ). Then $f^{-1}(V)$ is a $\psi^*\alpha$ -open set in (X, τ). As f is $(\psi^*\alpha)^*$ -quotient, V is a open set in (Y, σ). It implies that V is a $\psi^*\alpha$ -open set in (Y, σ). Hence f is a $\psi^*\alpha$ -quotient map.

Example 3.25 Let X = {a, b, c, d}, τ = {ϕ, {a}, {a, b}, X}, Y = {l, m, n}, σ = { ϕ, {l}, Y}, $\psi^*\alpha O(X)$ = {ϕ, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X} and $\psi^*\alpha O(Y)$ = { ϕ, {l}, {l, m}, {l, n}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = l, f(b) = n = f(c), f(d) = m. Then the map f is $\psi^*\alpha$ - quotient but not $(\psi^*\alpha)^*$ -quotient, since for the $\psi^*\alpha$ -open set {l, m} in (Y, σ) f^{-1}({l, m}) = {a, d} is $\psi^*\alpha$ -open in (X, τ) but the set {l, m} is not open in (Y, σ).

Remark 3.26 From the above observations we have the following diagram, where A → B represents A implies B but not conversely



4. Applications

Proposition 4.1 If a map $f : (X, \tau) \to (Y, \sigma)$ is open, surjective and $\psi^*\alpha$ -irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is a $\psi^*\alpha$ -quotient map. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is a $\psi^*\alpha$ -quotient map.

Proof: Let V be any open set in (Z, η). Then $g^{-1}(V)$ is a $\psi^*\alpha$ -open set in (Y, σ), since g is a $\psi^*\alpha$ -quotient map. Since f is $\psi^*\alpha$ -irresolute, $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$ -open set in (X, τ) which implies that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$ -open in (X, τ). This shows that $g \circ f$ is $\psi^*\alpha$ -continuous.

Now assume that $(g \circ f)^{-1}(V)$ is open in (X, τ) for a subset $V \subseteq Z$. Since f is open, $f(f^{-1}(g^{-1}(V)))$ is open in (Y, σ). This implies that $g^{-1}(V)$ is open in (Y, σ), as f is surjective. Since g is a $\psi^*\alpha$ -quotient map, V is a $\psi^*\alpha$ -open set in (Z, η). Hence $g \circ f$ is a $\psi^*\alpha$ -quotient map.

Proposition 4.2 If $h : (X, \tau) \to (Y, \sigma)$ is a $\psi^*\alpha$ -quotient map and $g : (X, \tau) \to (Z, \eta)$ is a continuous map where Z is a space that is constant on each set $h^{-1}(\{y\})$ for each $y \in Y$, then g induces $\psi^*\alpha$ -continuous map $f : (Y, \sigma) \to (Z, \eta)$ such that $f \circ h = g$.

Proof: Since g is constant on $h^{-1}(\{y\})$ for each $y \in Y$, the set $g(h^{-1}(\{y\}))$ is an one point set in (Z, η). If we let f(y) to denote this point, then it is clear that f is well defined and for each $x \in X$, $f(h(x)) = g(x)$. We prove that f is $\psi^*\alpha$ -continuous. For if we let V be any open set in (Z, η),then $g^{-1}(V)$ is an open set as g is continuous. But $g^{-1}(V) = h^{-1}(f^{-1}(V))$ is open in (X, τ).Since h is a $\psi^*\alpha$ - quotient map, $f^{-1}(V)$ is a $\psi^*\alpha$ -open set in (Y, σ). Hence f is $\psi^*\alpha$ -continuous.

Proposition 4.3 Let $f : (X, \tau) \to (Y, \sigma)$ be strongly $\psi^*\alpha$ -open, surjective and $\psi^*\alpha$ -irresolute map and $g : (Y, \sigma) \to (Z, \eta)$ be a strongly $\psi^*\alpha$ -quotient map. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is a strongly $\psi^*\alpha$ -quotient map.

Proof: Let V be any open set in (Z, η). Then $g^{-1}(V)$ is a $\psi^*\alpha$ -open set in (Y, σ), since g is strongly $\psi^*\alpha$ -quotient. Since f is $\psi^*\alpha$ -irresolute, $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$ -open set in (X, τ) which implies that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$ -open in (X, τ).

Now assume that $(g \circ f)^{-1}(V)$ is a $\psi^*\alpha$ -open set in (X, τ) for a subset $V \subseteq Z$. Then $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$ -open set in (X, τ). Since f is strongly $\psi^*\alpha$ -open, $f(f^{-1}(g^{-1}(V)))$ is $\psi^*\alpha$ -open in (Y, σ). This implies that $g^{-1}(V)$ is a $\psi^*\alpha$ -open set in (Y, σ), as f is surjective. This gives that V is an open set in (Y, σ), since g is a strongly $\psi^*\alpha$ -quotient map. Hence $g \circ f$ is a strongly $\psi^*\alpha$ -quotient quotient map.

Proposition 4.4 Let $p : (X, \tau) \to (Y, \sigma)$ be a $\psi^*\alpha$ -quotient map where (X, τ) and (Y, σ) are ${}_{\psi^*\alpha}T_c$ -spaces. A map $g : (Y, \sigma) \to (Z, \eta)$ is quasi $\psi^*\alpha$ -continuous if and only if the composite map $g \circ p : (X, \tau) \to (Z, \eta)$ is a quasi $\psi^*\alpha$ -continuous map.

Proof: Let g be quasi $\psi^*\alpha$ -continuous and U be any $\psi^*\alpha$ -open set in (Z, η). Then $g^{-1}(U)$ is open in (Y, σ). Since p is $\psi^*\alpha$ -quotient, $p^{-1}(g^{-1}(U)) =(g \circ p)^{-1}(U)$ is $\psi^*\alpha$ -open in (X, τ). Since (X, τ) is a ${}_{\psi^*\alpha}T_c$ -space, $p^{-1}(g^{-1}(U))$ is in open in (X, τ). Thus $(g \circ p)$ is quasi $\psi^*\alpha$ -continuous

Conversely, assume that $(g \circ p)$ is quasi $\psi^*\alpha$ -continuous. Let U be any $\psi^*\alpha$ -open set in (Z, η). $p^{-1}(g^{-1}(U))$ is open in (X, τ). Since p is $\psi^*\alpha$ -quotient, $g^{-1}(U)$ is $\psi^*\alpha$ -open in (Y, σ).Since (Y, σ). is a ${}_{\psi^*\alpha}T_c$ -space, $g^{-1}(U)$ is open in (Y, σ).Hence g is quasi $\psi^*\alpha$ -continuous.

Proposition 4.5 Let $f : (X, \tau) \to (Y, \sigma)$ be strongly $\psi^*\alpha$ -open, surjective and $\psi^*\alpha$ -irresolute map and $g : (Y, \sigma) \to (Z, \eta)$ be a $(\psi^*\alpha)^*$ -quotient map. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is a $(\psi^*\alpha)^*$ -quotient map.

Proof: Since f and g are $\psi^*\alpha$ -irresolute, $g \circ f$ is $\psi^*\alpha$ -irresolute by **theorem 4.16[4]** Suppose that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$ -open in (X, τ) for a subset $V \subseteq Z$, that is $f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$ -open in(X, τ). Since f is strongly $\psi^*\alpha$ -open and surjective, $f(f^{-1}(g^{-1}(V)) = g^{-1}(V)$ is $\psi^*\alpha$ -open in (Y, σ). Since g is $(\psi^*\alpha)^*$ -quotient implies V is open in (Y, σ). Hence $g \circ f$ is a $(\psi^*\alpha)^*$ -quotient map.

Proposition 4.6 Let $f : (X, \tau) \to (Y, \sigma)$ be a strongly $\psi^*\alpha$ - quotient and $\psi^*\alpha$ -irresolute map and $g : (Y, \sigma) \to (Z, \eta)$ be a $(\psi^*\alpha)^*$ -quotient map. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is a $(\psi^*\alpha)^*$ -quotient map.

Proof: Let V be a $\psi^*\alpha$ -open set in (Z, η). Then $g^{-1}(V)$ is a $\psi^*\alpha$ -open set in (Y, σ), since g is $(\psi^*\alpha)^*$ -quotient. Since f is $\psi^*\alpha$ -irresolute, $f^{-1}(g^{-1}(V))$ is a $\psi^*\alpha$ -open set in (X, τ) which implies that $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$ -open in (X, τ). This shows that $g \circ f$ is $\psi^*\alpha$ -irresolute. Let $f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$ -open in (X, τ) for a subset $V \subseteq Z$. Since f is strongly $\psi^*\alpha$ -quotient, $g^{-1}(V)$ is open in (Y, σ). This implies that $g^{-1}(V)$ is a $\psi^*\alpha$ -open set in (Y, σ). Since g is $(\psi^*\alpha)^*$ -quotient, V is open in (Z, η). Hence $(g \circ f)$ is a $(\psi^*\alpha)^*$ -quotient map.

Proposition 4.7 Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $(\psi^*\alpha)^*$ -quotient maps. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is also a $(\psi^*\alpha)^*$ -quotient map.

Proof: Let V be any $\psi^*\alpha$ -open set in (Z, η). Then $g^{-1}(V)$ is $\psi^*\alpha$ -open in (Y, σ), since g is $(\psi^*\alpha)^*$ -quotient. Since f is $(\psi^*\alpha)^*$ -quotient, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is a $\psi^*\alpha$ –open in (X, τ). This shows that $g \circ f$ is $\psi^*\alpha$ -irresolute. Let $(g \circ f)^{-1}(V)$ is $\psi^*\alpha$ -open in (X, τ). Then $f^{-1}(g^{-1}(V))$ $\psi^*\alpha$ -open in (X, τ). Since f is $(\psi^*\alpha)^*$ -quotient, $g^{-1}(V)$ is open in (Y, σ). Since g is $(\psi^*\alpha)^*$ -quotient, V is open in (Z, η). Hence $(g \circ f)$ is $(\psi^*\alpha)^*$ -quotient.

Proposition 4.8 Let $f : (X, \tau) \to (Y, \sigma)$ be a map where (X, τ) and (Y, σ) are ${}_{\psi^*\alpha}T_c$ -spaces. Then the following statements are equivalent.

(i) f is a $(\psi^*\alpha)^*$ -quotient map

(ii) f is strongly $\psi^*\alpha$ -quotient map

(iii) f is a $\psi^*\alpha$ -quotient map

Proof: (i) $\Rightarrow$ (ii) Follows from **Proposition 3.22**

(ii) $\Rightarrow$ (iii) Follows from **Proposition 3.20**

(iii) $\Rightarrow$ (i) Since (Y, σ) is a ${}_{\psi^*\alpha}T_c$ -space, f is $\psi^*\alpha$ -irresolute by **theorem 3.7 [4]** Suppose $f^{-1}(V)$ is $\psi^*\alpha$ -open in (X, τ). Since (X, τ) is a ${}_{\psi^*\alpha}T_c$ -space, $f^{-1}(V)$ is open in (X, τ). By (iii), V is $\psi^*\alpha$ -open in (Y, σ). Since (Y, σ) is a ${}_{\psi^*\alpha}T_c$ -space, V is $\psi^*\alpha$ -open in (Y, σ). Hence f is a $(\psi^*\alpha)^*$ -quotient map.

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