Proper Lucky Number of Hexagonal Mesh and Honeycomb Network

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***Abstract*:** *Let $G(V,E)$ be a graph with vertex set V and edge set E. Let f be a labeling defined in G. Define the sum of neighbourhood of vertex v by $s(v)=\sum_{u\in N(v)} f(u)$, where $N(v)$ denotes the open neighbourhood of vertex $v\in V$. A labelingf is a proper lucky labeling if $f(u)\neq f(v)$ and $s(u)\neq s(v)$ for all $(u,v)\in E(G)$. The proper lucky number of G, denoted by $\eta_p(G)$ is the least positive integer k such that G has a proper lucky labeling with $\{1,2,\dots,k\}$ as the set of labels. In this paper we determine proper lucky number of Hexagonal mesh and Honeycomb network.*

Keywords - *Lucky labeling, Proper lucky number, Hexagonal mesh and Honeycomb network.*

1. Introduction

For the basic definitions of graph theory refer Douglas B.West [9] and Harary [13]. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions [11]. Rosa, in the year 1967 introduced the concept of labeling, by Graham and Sloane in 1980. The concept of labeling has much importance in graph theory as it is being used in various fields such as communication networks, coding theory, astronomy etc.

Graph coloring is one of the most studied subjects in graph theory. It is an assignment of labels called colors to the elements of a graph, subject to certain constraints. Karonski, Luczak and Thomason [8] initiated the study of proper labeling. The problem of proper labeling offers numerous variants and established great significance at recent times. Graph coloring is used in various research areas of computer science such as networking, image segmentation, clustering, image capturing and data mining.

The lucky labeling of graphs were studied by A. Ahai et al [2] and S. Akbari et al [3]. Suppose the vertices of a graph G were labeled arbitrarily by positive integers and let $s(v)$ denote the sum of labels over all neighbours of vertex v. A labeling is *lucky* if the function s is a proper coloringof G. The least positive integer k for which a graph G has a *lucky labeling* from the set $\{1,2,\dots,k\}$ is the *lucky number* of$G$, denoted by $\eta(G)$. Kins et al [8] obtained the lower bound of proper lucky number for any connected graph G using clique number ω. The chromatic number of complete graph $K_n$is $\chi(K_n)=n$. In this paper, we determine the proper lucky number of Hexagonal mesh and Honeycomb network.

Definition 1.1[8]:Let $G(V,E)$ be a graph with vertex set V and edge set E. Let f be a labeling defined in G. Define the sum of neighbourhood of vertex v by $s(v)=\sum_{u\in N(v)} f(u)$, where $N(v)$ denotes the open neighbourhood of vertex of $v\in V$. A labeling f is a *proper lucky labeling* if $f(u)\neq f(v)$ and $s(u)\neq s(v)$ for all $(u,v)\in E(G)$. The *proper lucky number* of G, denoted by $\eta_p(G)$ is the least positive integer k such that G has a proper lucky labeling with $\{1,2,\dots,k\}$ as the set of labels.

Result 1.2 [8]**:** For any connected graph G, the chromatic number is less than or equal to proper lucky number i.e. $\chi\leq\eta_p$.

For any connected graph G, let η_p be its proper lucky number and ω be its clique number, then $\omega\leq\eta_p$.

2.Proper Lucky Number of Hexagonal Mesh

The triangular tessellation is used to define a hexagonal mesh and this is widely studied. A hexagonal mesh of dimension n, denoted by HX_n it has $3n^2-3n+1$ vertices and $9n^2-15n+6$ edges. There are six vertices of degree three which we call as *corner vertices*. There is exactly onevertex v at distance $n-1$from each of thecorner vertices. This vertex is called the centre of HX_n.

For n dimensional of the hexagonal meshHX_n, there are $2n$-1 vertical lines. In this paper we name vertical lines as X lines as follows. The middle vertices of the hexagonal mesh as X_0 and call as the spine of the hexagonal mesh. Left of X_0 we call as $X_1, X_2, \ldots X_{n-1}$ lines and the right side as X_{-1}, $X_{-2}, \ldots X_{-n+1}$ lines as shown in Fig 1. We label the vertices on X_0 from top to bottom as $v_{1,1}, v_{1,2}, \ldots v_{1,2n-1}$, the vertices on X_1 from top to bottom as $v_{2,1}, v_{2,2}, \ldots, v_{2,2n-2}$, and so on , finally the vertices on X_{n-1} from top to bottom as $v_{n,1}, v_{n,2}, \ldots v_{n,n}$. Similarly, we name X_{-1}as $v_{-1,1}, v_{-1,2}, \ldots v_{-1,2n-1}, \ldots$ and X_{-n+1} as $v_{-n+1,1}, v_{-n+1,2}, \ldots v_{-n+1,n}$. The diameter of $HX_n$is $2n-2$.

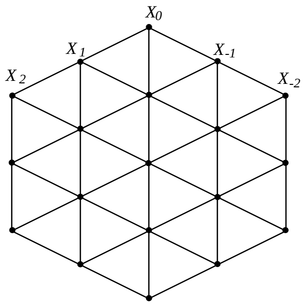

Fig 1. Hexagonal mesh of dim n=3 and its X lines

Theorem 2.1: Let G be a hexagonal mesh HX_n. Then the proper lucky number of G is $\eta_p(G)$= 3.

Proof.Let G be a hexagonal mesh HX_n of dimension n.

Define a mapping $f: V(G) \to \{1,2,3\}, \forall v_{ij} \in V$ as follows.

Case 1:For $n \equiv 0 \; mod(3)$

$f(v_{3i-2,\ 3j-2}) = 1.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - (i-1).$

$f(v_{3i-1,\ 3j-1}) = 1.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - i.$

$f(v_{3i,\ 3j}) = 1.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - i.$

$f(v_{3i-2,\ 3j-1}) = 3.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - (i-1).$

$f(v_{3i-1,\ 3j}) = 3.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - i.$

$f(v_{3i,\ 3j}) = 3.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - i.$

$f(v_{3i-2,\ 3j}) = 2.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - i.$

$f(v_{3i-1,\ 3j-2}) = 2.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - (i-1).$

$f(v_{3i,\ 3j-1}) = 2.\quad i = 1,2,\ldots\frac{n}{3},\ j = 1,2,\ldots\left(\frac{2n}{3}\right) - i.$

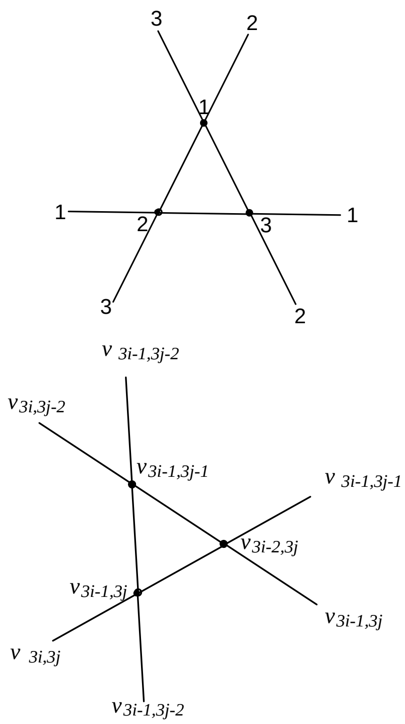

Fig 2. A general proper labeling of vertices in Hexagonal Mesh HX_n

From the above mapping we obtained values for the each neighbourhood of v_{ij}.

Case 2: For $n \equiv 1 \; mod\ (3)$

$f(v_{3i-2,\ 3j-2}) = 1.\quad i = 1,2,\ldots\left\lceil\frac{n+1}{3}\right\rceil,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - (i-1).$

$f(v_{3i-1,\ 3j-1}) = 1.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - i.$

$f(v_{3i,\ 3j}) = 1.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - (i+1).$

$f(v_{3i-2,\ 3j-1}) = 3.\quad i = 1,2,\ldots\left\lceil\frac{n+1}{3}\right\rceil,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - i.$

$f(v_{3i-1,\ 3j}) = 3.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - i.$

$f(v_{3i,\ 3j-2}) = 3.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - i.$

$f(v_{3i-2,\ 3j}) = 2.\quad i = 1,2,\ldots\left\lceil\frac{n+1}{3}\right\rceil,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - i.$

$f(v_{3i-1,\ 3j-2}) = 2.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - i.$

$f(v_{3i,\ 3j-1}) = 2.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n+1}{3}\right) - i.$

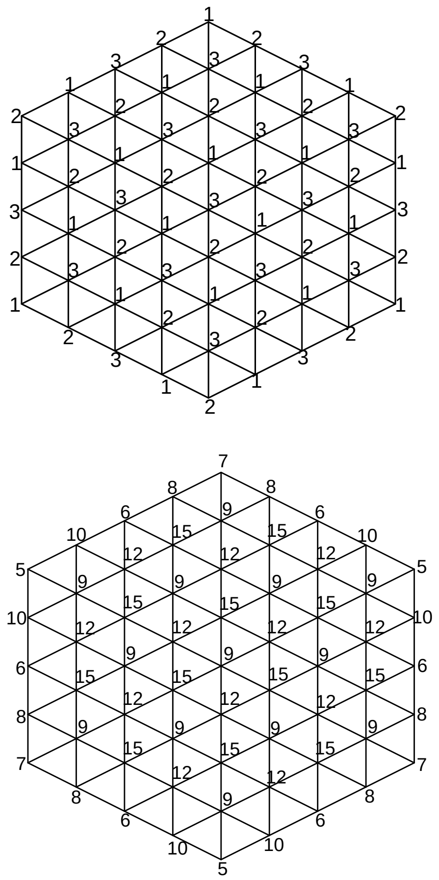

Fig 3. Proper Lucky labeling of Hexagonal Mesh HX_5 and its sum of neighbourhood

Case 3: For $n \equiv 2 mod(3)$

$f(v_{3i-2,\ 3j-2}) = 1.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n-1}{3}\right) - (i-1).$

$f(v_{3i-1,\ 3j-1}) = 1.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n-1}{3}\right) - (i-1).$

$f(v_{3i,\ 3j}) = 1.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor, j = 1,2,\ldots\left(\frac{2n-1}{3}\right) - i.$

$f(v_{3i-2,\ 3j-1}) = 3.\quad i = 1,2,\ldots\left\lfloor\frac{n}{2}\right\rfloor,\ j = 1,2,\ldots\left\lceil\frac{n}{2}\right\rceil.$

$f(v_{3i-1,\ 3j}) = 3.\quad i = 1,2,\ldots\left\lfloor\frac{n}{2}\right\rfloor - 1,\ j = 1,2,\ldots\left\lceil\frac{n}{2}\right\rceil - 1.$

$f(v_{3i,\ 3j-2}) = 3.\quad i = 1,2,\ldots\left\lfloor\frac{n}{2}\right\rfloor - 1,\ j = 1,2,\ldots\left\lceil\frac{n}{2}\right\rceil.$

$f(v_{3i-2,\ 3j}) = 2.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n-1}{3}\right) - (i-1).$

$f(v_{3i-1,\ 3j-2}) = 2.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor,\ j = 1,2,\ldots\left(\frac{2n-1}{3}\right) - (i-1).$

$f(v_{3i,\ 3j-1}) = 2.\quad i = 1,2,\ldots\left\lfloor\frac{n}{3}\right\rfloor, j = 1,2,\ldots\left(\frac{2n-1}{3}\right) - i.$

Similarly, the symmetrical part of the graph $X_{-i}, i = 1,2,\ldots n+1$ is labeled as X_i lines.

Claim: To prove that $f(u) \neq f(v)$.

Subcase (i): If $v_{ij} \in V$ is not a boundary vertex then it is adjacent to the vertices v_{ij-1}, v_{ij+1}, v_{i-1j}, v_{i-1j+1}, v_{i+1j-1}, v_{i+1j}.

Let $f(v_{ij}) = 1$, then its adjacent vertices receives the map 2 or 3 under f alternatively.

i.e. $f(v_{i-1j}) = 2$, $f(v_{i-1j+1}) = 3$, $f(v_{ij+1}) = 2$, $f(v_{i+1j}) = 3$, $f(v_{i+1j-1}) = 2$, $f(v_{ij-1}) = 3$. Clearly the adjacent vertices of v_{ij} which are adjacent to each other does not receive the same map under f. Similarly if $f(v_{ij}) = 2$ or 3 then its adjacent vertices receives the map 1 and 3 or 1 and 2 under f alternatively as discussed above.

Subcase (ii): If $v_{ij} \in V$ is a boundary vertex then it is adjacent to the vertices v_{i-1j}, v_{i-1j+1}, v_{ij+1} or v_{i-1j}, v_{i-1j+1}, v_{ij+1} .

Let $f(v_{ij}) = 1$, then its adjacent vertices receives the map 2 or 3 under f alternatively.

i.e. $f(v_{i-1j}) = 2$, $f(v_{i-1j+1}) = 3$, $f(v_{ij+1}) = 2$, $f(v_{i+1j}) = 3$ or $f(v_{i-1j}) = 2$, $f(v_{i-1j}) = 2$. Clearly the adjacent vertices of v_{ij} which are adjacent to each other does not receive the same map under f. Similarly if $f(v_{ij}) = 2$ or 3 then its adjacent vertices receives the map 1 and 3 or 1 and 2 under f alternatively as discussed above.

Clearly $f(u) \neq f(v)$, for all $(u, v) \in E(G)$. Hence the given labeling is a proper labeling.

Next we claim that the given mapping is a lucky labeling. That is, to prove $s(u) \neq s(v)$

for all $(u, v) \in E(G)$.

We obtain $s(v_{ij})$, the inner sum of labels over all neighbours of vertex v_{ij} .

Consider any vertex of HX_n. Let $v_{(3i,\ 3j)}$ be the vertex with six adjacent vertices say $v_{(3i,\ 3j-1)}$, $v_{(3i-1,\ 3j)}$, $v_{(3i-1,\ 3j-2)}$, $v_{(3i,\ 3j-2)}$, $v_{(3i-2,\ 3j)}$ and $v_{(3i-2,\ 3j-1)}$.

Fig 4.Sum of neighbourhood of $v_{(3i,\ 3j)}$

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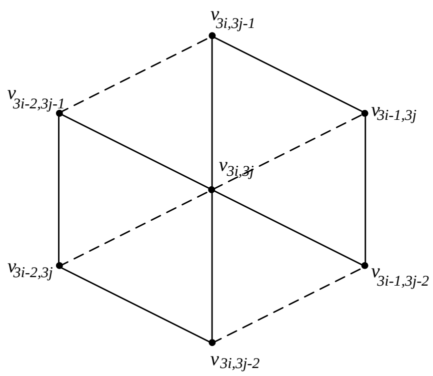

Fig 4. Sum of neighbourhood of $v_{(3i,\ 3j)}$

Hence its sum of neighbourhood are

$$
\begin{aligned}
s(v_{3i,\ 3j}) &= f(v_{3i,\ 3j-1}) + f(v_{3i-1,\ 3j}) \\
&\quad + f(v_{3i-1,\ 3j-2}) + f(v_{3i,\ 3j-2}) \\
&\quad + f(v_{3i-2,\ 3j}) + f(v_{3i-2,\ 3j-1}) \\
&= 2+3+2+3+2+3 \\
&= 15
\end{aligned}
$$

Here we are taking $v_{(3i-2,\ 3j)}$ the adjacent vertices of $(v_{3i,\ 3j})$.

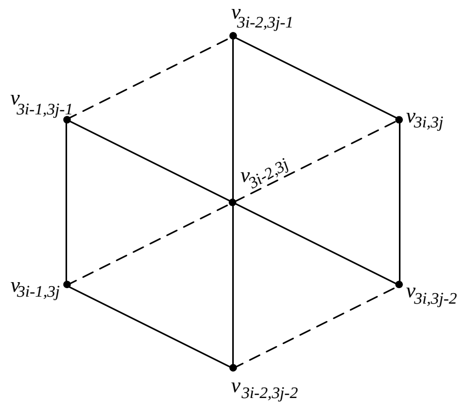

Fig 5. Sum of neighbourhood of $v_{(3i-2,\ 3j)}$

$$
\begin{aligned}
s(v_{3i-2,\ 3j}) &= f(v_{3i-2,\ 3j-1}) + f(v_{3i,\ 3j}) \\
&\quad + f(v_{3i,\ 3j-2}) + f(v_{3i-2,\ 3j-2}) \\
&\quad + f(v_{3i-1,\ 3j}) + f(v_{3i-1,\ 3j-1}) \\
&= 3+1+3+1+3+1 \\
&= 12
\end{aligned}
$$

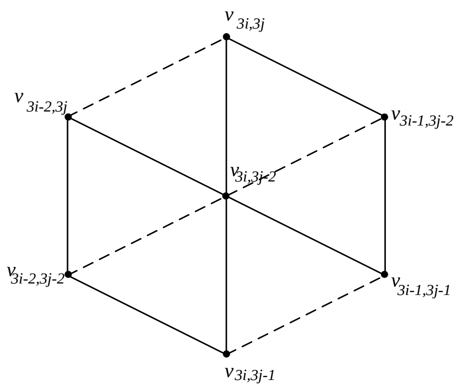

Fig 6. Sum of neighbourhood of $v_{(3i,\ 3j-2)}$

Similarly, we can show that $s(v_{3i,\ 3j-2}) = 9$, $s(v_{3i-2,\ 3j-1}) = 9$,

$s(v_{3i,\ 3j-1}) = 12$, $s(v_{3i-1,\ 3j}) = 9$, $s(v_{3i-1,\ 3j-2}) = 12$

From the above cases we see that $s(u) \neq s(v)$ for all $uv \in E(G)$.

Similarly, we can prove other cases.

Therefore $\eta_p \leq 3$.

Since the clique number of HX_n is 3, and by the theorem 1.2 $\eta_p \geq 3$. Therefore $\eta_p(G) = 3$.

3. Proper Lucky Number of Honeycomb Networks

A high level honeycomb network can be constructed from a low level one. A unit honeycomb network is a hexagon, denoted by *HC*(1). Honeycomb network of size 2 denoted *HC*(2), can be obtained by adding six hexagons around the boundary edges of *HC*(1). Inductively, honeycomb network *HC*(*n*) can be obtained from *HC*(*n – 1*) by adding a layer of hexagons around the boundary edges of *HC*(*n – 1*). Alternatively, the size d of *HC*(*n*) is determined as the number of hexagons between the center and boundary of *HC*(*n*) (inclusive) and the number of vertices and edges of *HC*(*n*) are $6n^2$ and $9n^2 - 3n$ respectively. We use the level numbering scheme proposed by the Sharieh et al.[14] for the honeycomb networks. Each node in *HC*(*n*) is identified by a v_{ij}, where *i*denotes the line number in which the node exists, and *j* denotes the location of the node in the line. A node with the address 1,1 is the first node that exists at line number 1. The node 1,2 refers to the second node that exists at line number 1, and so on. See Fig 7.

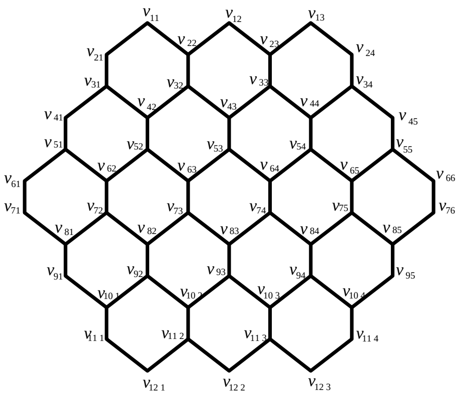

Fig 7. Honeycomb network with addressing HC(3)

Theorem 3.1: Let G be a honeycomb $HC(n)$. Then the proper lucky number of G is $\eta_p(G)$= 2.

Proof. Let G be a honeycomb graph $HC(n)$ of dimension n.

Define a mapping $f: V(G) \to \{1,2\}, \forall v_{ij} \in V$ as follows

$$f(v_{ij}) = \begin{cases} 1, & i \text{ is odd} \\ 2, & i \text{ is even} \end{cases}$$

Subcase (i): If $v_{ij} \in V$ is not a boundary vertex then it is adjacent to the even line vertices v_{i-1j-1}, v_{i-1j}, $v_{i+1\,j}$, upto $2n$ lines and it is adjacent to the odd line vertices v_{i-1j}, $v_{i+1\,j}$, $v_{i+1\,j+1}$, upto $2n-1$. Below the line $2n$, v_{ij} is adjacent to even line vertices v_{i-1j}, $v_{i-1\,j+1}$, v_{i+1j} upto 4n-2 and it is adjacent to the odd line vertices v_{i-1j}, v_{i+1j-1}, $v_{i+1\,j}$ upto $2n-1$.

Let $f(v_{ij}) = 1$, then its adjacent vertices receives the map 2 under f alternatively.

Clearly the adjacent vertices of v_{ij} which are adjacent to each other does not receive the same map under f. Similarly if $f(v_{ij}) = 2$ then its adjacent vertices receives the map 1 under f alternatively.

Subcase (ii): If $v_{ij} \in V$ is a boundary vertex then the first line of $HC(n)$ is adjacent to the vertices v_{i+1j}, $v_{i+1\,j+1}$ and the n^{th} line is adjacent to the vertices v_{i-1j}, $v_{i-1\,j+1}$. The left side of the boundary is adjacent to the vertices v_{i-1j}, $v_{i+1\,j}$ and the right side of v_{ij} is adjacent to even line vertices $v_{i-1\,j-1}$, v_{i+1j} and the odd line vertices v_{i-1j}, v_{i+1j+1} upto $2n$ lines and below the $2n$ lines v_{ij} is adjacent to even line vertices $v_{i-1\,j+1}$, v_{i+1j} and the odd line vertices v_{i-1j}, v_{i+1j-1}.

Let $f(v_{ij}) = 1$, then its adjacent vertices receives the map 2 under f alternatively.

Clearly the adjacent vertices of v_{ij} which are adjacent to each other does not receive the same map under$f$. Similarly if $f(v_{ij}) = 2$ then its adjacent vertices receives the map 1 under f alternatively as discussed above. See Fig 8.

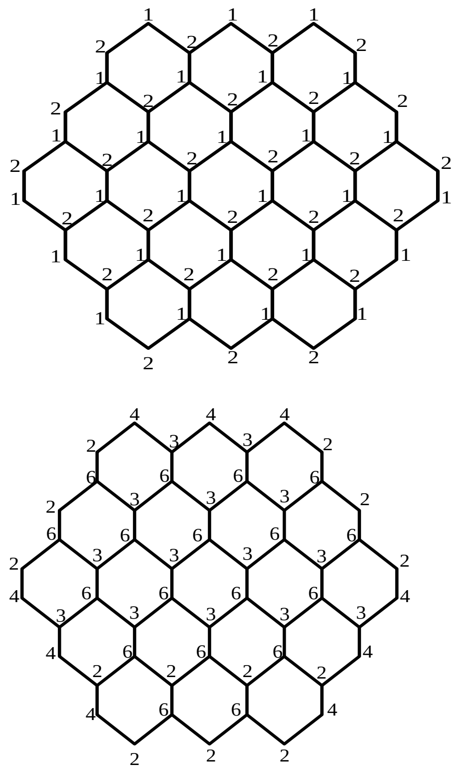

Fig 8. Proper Lucky labeling of Honeycomb $HC(3)$ and its sum of neighbourhood

Proof is similar to theorem 2.1.

Conclusion:

In this paper, we obtained the proper lucky number for Hexagonal mesh and Honeycomb network. Futher, we investigate the problems in various interconnection networks such as butterfly, benes, torus etc.

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