The bending of thin vertical rod and Bessel functions

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Abstract: *This paper is based on physical problem when anyone who has tried holding a long, thin, flexible rod in a vertical position. If the rod is short, and its tip is given a small sideways displacement and released , the rod will perform transverse oscillations until it reaches an equilibrium position in a bent shape because of supporting its own weight. The longer the road, the larger the amplitude of these oscillations and the greater the bending under its own weight when in equilibrium, until at some critical length the rod will bend until its tip just touches the ground, after which it will remain in that position.*

Introduction: An idealization of this phenomenon can be modelled by a long, thin, flexible, flag pole of uniform cross-section, the base of which is clamped in the ground so the pole is vertical. We than ask at what length will the pole became unstable so that any displacement of the top of the pole will cause it to bend under its own weight until the top of the pole touches and remains in the contact with the ground. This paper can be posed in the mathematical terms, and it is the one that will be answered here. The solution of this problem will involve the use of Bessel functions, but the linear differential equation involved will have to satisfy a two point boundary condition instead of the initial conditions we have consider so far. This means that the existence and uniqueness of solutions to initial value problems guaranteed.

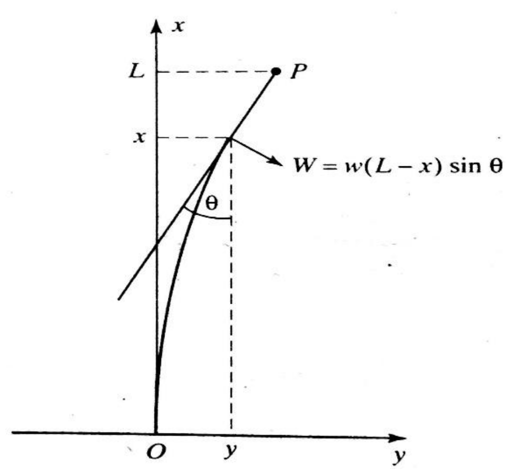
Definitions:
The Bessel differential equation is the linear second-order ordinary differential equation given by

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + \left(x^2 - n^2\right) y = 0.$$

Bessel functions of the first kind, denoted as Jn(x), are solutions of Bessel's differential equation that are finite at the origin (x = 0) for integer or positive α, and diverge as x approaches zero for negative non-integer n and given by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n-r+1)!} \left(\frac{x}{2}\right)^{2n-r}$$

Result :Let us mode the problem by considering a thin uniform flexible rod of length L with a constant cross section that is constructed from material with a Young's modules of elasticity E, with the moment of inertia of a cross section about the diameter normal to the plane of bending equal to I. The line density along the rod will be assumed to be constant and equal to w. the x-axis will be taken to be vertical and to coincide with the undistorted axis of the rod, with its origin located at the base of the rod. Thehorizontal displacement of the rod at a position x will be taken to be y, as shown in figure



It is known that if the moment acting on the rod at a position x is M(x), the equation governing its transverse deflection y when in equilibrium is

$$EI\frac{d^2y}{dx^2} = M(x) \qquad \text{…(a)}$$

The shear on the rod at point x is the force exerted perpendicular to the axis of the rod at x due to the weight of the rod extending from x to the top at P. As the length of this part of the rod is L-x, and its line density is w, the weight of this section is given by w(L-x), so the component W of this force normal to the axis of the rod at x is simply

$$W=w(L-x)\sin\theta \qquad \text{…(b)}$$

Where θ is the angle of deflection of the rod from vertical point x, as shown in figure.
It is known that the shear on a rod is in terms of moment M(x) by

$$\frac{dM}{dx} = -W(x) \qquad \text{…(c)}$$

Now make the approximation that the deflection at point x on the rod is small, so sinθ = tanθ = $\frac{dy}{dx}$
From (a),(b),(c) we arrive at third order differential equation

$$EI\frac{d^3y}{dx^3} + w(L-x)\frac{dy}{dx} = 0 \quad \ldots(d)$$

Take z=L-x then (d) becomes $\frac{d^3y}{dz^3} + \frac{w}{EI}z\frac{dy}{dz} = 0$ …(e)
Now to find the appropriate boundary conditions to be applied at the base and top of the rod. Due to clamping the pole in a vertical position at the origin. $(\frac{dy}{dx})_{x=0}=(\frac{dy}{dz})_{z=L}=0$. When the rod is bent and in equilibrium, there can be no bending moment at the top of the rod, so no curvature at that point. There is no curvature at x=L(z=0) when ρ=∞,
$(\frac{d^2y}{dx^2})_{x=L}=(\frac{d^2y}{dz^2})_{z=0}=0$ …(f)
Put $u=\frac{dy}{dz}$, boundary condition become u(L)=0 & $(\frac{du}{dz})_{z=0}=0$ …(g)
Equation (e) is third order but in terms of u it is second order with conditions (g)
We have $\frac{d^2y}{dz^2} + \frac{w}{EI}zu = 0$
will provide sufficient information for us to find the critical length at which bending occurs.

Identifying equation with x replaced by z, shows that
$1-2a=0,\quad 2c-2=1,\quad a^2\text{-}v^2c^2=0$ *and* $b^2c^2=w/EI$,

So

$$a=\frac{1}{2},\quad c=\frac{3}{2},\quad v=\frac{1}{3},\quad \text{and } b=\frac{2}{3}\sqrt{\frac{w}{EI}}$$

Using this information in the solution to equation gives

$$u(z)=C_1\sqrt{z}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}z^{3/2}\right)+C_2\sqrt{z}\,J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}z^{3/2}\right)$$

Noticing from that for small z

$$J_v(z)\approx\frac{1}{\Gamma(1+v)(\frac{z}{2})^{v}} \quad \text{and} \quad J_{-v}(z)\approx\frac{1}{\Gamma(1-v)(\frac{z}{2})^{-v}}$$

We see that close to the top of the rod , that is, for small z, u(z) can be approximated by

$$u(z)\approx C_1\frac{z}{\Gamma(4/3)}\left(\frac{1}{3}\sqrt{\frac{w}{EI}}\right)^{1/3}+C_2\frac{1}{\Gamma(2/3)}\left(\frac{1}{3}\sqrt{\frac{w}{EI}}\right)^{-1/3}$$

Differentiation of this result gives

$$u(z)\approx C_1\frac{1}{\Gamma(4/3)}\left(\frac{1}{3}\sqrt{\frac{w}{EI}}\right)^{1/3}$$

But to satisfy the second boundary condition $\left(\frac{du}{dz}\right)z=0$, we must set C_1=0, causing solution to reduce to

$$u(z)=C_2\sqrt{z}\,J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}z^{3/2}\right)$$

Applying the remaining boundary condition $u(L)=0$ to gives

$$0 = C_2\sqrt{L}\,J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}L^{3/2}\right),$$

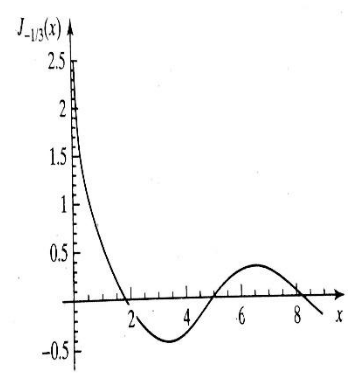
And this will be satisfied if either C_2=0 or $J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}L^{3/2}\right) = 0$. The first condition C_2=0 corresponds to the unstable equilibrium configuration in which the rod is vertical, and so must be rejected, whereas the second condition corresponds to the required critical bending condition, and it will be satisfied when $L$is such that it causes $J_{-1/3}$ to vanish.

It is at this stage that we discover the boundary value problem does not have a unique solution, because the asymptotic behavior of $J_{-1/3}$ shows that it has infinitely many zeros. To resolve this difficulty, and to find the length at which critical bending occurs, we must seek a selection criterion for the length from outside the description of the physical situation provided by the differential equation,

Such a criterion is not hard to find, because criterion bending must occur at the smallest value of Lsay that L_c , that satisfies the condition

$$J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{w}{EI}}L_c^{3/2}\right) = 0,$$

Because if critical bending occurs when L=L_c, it will certainly occur at any larger value of L.
A graph of $J_{-1/3}$(x) is shown, from which it can be seen that the first zero α of $J_{-1/3}$(x) occurs at around the value α≈ 1.87, through numerical calculation provides the more accurate value α=1.86635……… However, this accuracy is unnecessary, because the approximations made when modeling the physical situation introduce errors of sufficient magnitude that the value α≈ 1.87 is adequate.



Using the valueα≈ 1.87 shows that the length L_c for the critical bending must satisfy the formula

$$\left(\frac{2}{3}\sqrt{\frac{w}{EI}}L_c^{3/2}\right) \approx 1.87,$$

Which is equivalent to

$$L_c \approx 1.99\left(\sqrt{\frac{EI}{w}}\right)^{1/3}$$

Conclusion: This shows, as would be expected, that if the rod is not cylindrically symmetric about its axis, the critical length L_c will depend on the plane in which bending occurs, because the moment of inertia will depend on the direction in which the rod bends. Thus, for example, the critical length of the rod with a rectangular cross section that bends in a plane parallel to one pair of its faces will differ from the critical length when bending occurs in a plane parallel to its other pair of its faces. In such cases the model used is to simple because twisting will be likely to occur, causing the rod always to buckle in such a way that L_c assumes its smallest possible value.

The simplest case arrises when the rod has a circular cross section of radius a for then the moment of inertia of the cross section about the any diameter is I=πa^4/4. When this expression is substituted into the approximation for L_c. We obtain approximation $L_c \approx 1.25(\frac{Ea^4}{w})^{\frac{1}{3}}$

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