On Strongly Inversely-$\mathcal{I}$-open and closed maps

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August 26, 2017

Abstract

We will introduce strongly inversely-$\mathcal{I}$-open and strongly inversely-$\mathcal{I}$-closed maps and give characterizations of these maps. We also give the relationship of these maps with pointwise-$\mathcal{I}$-continuous maps.

2010 Mathematics Subject Classification. 54C08, 54C10
Keywords. Strongly inversely-$\mathcal{I}$-open, strongly inversely-$\mathcal{I}$-closed, pointwise-$\mathcal{I}$-continuous, ideal.

1 Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski[2] and Vaidyanathaswamy[6]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X which satisfies that (i) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X known as ideal topological space and $(.)^* : \wp(X) \to \wp(X)$, called a local function[2] of A with respect to $\mathcal{I}$ and τ, is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open nhd. U of x in $X\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology, finer than τ, is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$[5]. When there is no chance of confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and $\tau^*(\mathcal{I})$ for $\tau^*(\mathcal{I}, \tau)$.

Throughout this paper (X, τ) will denote topological space on which no separation axioms are assumed. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. For a subset A of X, $cl(A)$ and $int(A)$ will denote the closure of A, interior of A in (X, τ), respectively, $cl^*(A)$ and $int^*(A)$ will denote the closure of A, interior of A in (X, τ^*), respectively, and A^C will denote the complement of A in X.

We will also make use of the following results:

Lemma 1.1. *[3] For any sets X and Y, let $f : X \to Y$ be any map and E be any subset of X. Then:*

(a) $f^\#(E) = \{y \in Y : f^{-1}(y) \subseteq E\}$.

(b) $f^\#(E^C) = (f(E))^C$ *and so* $f^\#(E) = (f(E^C))^C$ *and* $f(E) = (f^\#(E^C))^C$.

Definition 1.1. [1] A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be pointwise-$\mathcal{I}$-continuous if the inverse image of every open set in Y is $\tau^*(\mathcal{I})$-open in X. Equivalently, $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is pointwise-$\mathcal{I}$-continuous if and only if $f : (X, \tau^*(\mathcal{I})) \to (Y, \sigma)$ is continuous.

Definition 1.2. [4] A map $f : (X, \tau) \to (Y, \sigma)$ is said to be inversely open if $int(f(A)) \subseteq f(int(A))$ for any subset A of X.

Definition 1.3. [4] A map $f : (X, \tau) \to (Y, \sigma)$ is said to be inversely closed if for any subset A of X, A is closed in X whenever $f(A)$ is closed in Y.

2 Results

We begin by introducing the following definitions and results:

Definition 2.1. A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ with $\mathcal{J} = f(\mathcal{I})$ is said to be strongly inversely-$\mathcal{I}$-open if for any subset A of X, $int^*(f(A)) \subseteq f(int(A))$.

Remark 2.1. (a) It can be easily seen that every strongly inversely-$\mathcal{I}$-open map is inversely open, since $int(f(A)) \subseteq int^*(f(A))$ for any subset A of X but converse is not true.

(b) From the definition of strongly inversely-$\mathcal{I}$-open maps, it follows immediately that if a space (X, τ) is discrete topological space then every map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is strongly inversely-$\mathcal{I}$-open.

The following theorem gives various characterizations of strongly inversely-$\mathcal{I}$-open maps.

Theorem 2.1. *For any map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ with $\mathcal{J} = f(\mathcal{I})$, the following conditions are equivalent:*

(a) f is strongly inversely-$\mathcal{I}$-open i.e. for each subset A of X, $int^*(f(A)) \subseteq f(int(A))$,

(b) $f^\#(cl(A)) \subseteq cl^*(f^\#(A))$,

(c) if W is σ^*-open subset of Y and $W \subseteq f(X)$, then each set consisting of exactly one point and so at least one point from each fiber $f^{-1}(y)$, where $y \in W$, is open in X,

(d) for any subset A of X, A is open in X, whenever $f(A)$ is σ^*-open subset of Y,

(e) for any subset A of X, A is closed in X, whenever $f^\#(A)$ is σ^*-closed subset of Y.

Proof. (a)⇔(b): It follows from the Lemma 1.1(b) and using $(cl(A))^C = int(A^C)$.

(a)⇒(c): Let f be strongly inversely-$\mathcal{I}$-open and W is σ^*-open subset of Y such that $W \subseteq f(X)$. Let $V = \bigcup_{y\in W} f^{-1}(y)$ such that $f^{-1}(y)$ is singleton for all $y \in W$ then we will prove that V is open in X. Now $f(V) = W$ and $f(S) \subset W$ for any proper subset S of V. If possible, let V is not open in X, then $int(V) \subset V$ but $int(V) \neq V$ and so $f(int(V)) \subset W$ but $f(int(V)) \neq W$. Therefore, by (a) $int^*(f(V)) \subseteq f(int(V)) \subset W$ but $f(int(V)) \neq W$, contradicting the fact that $W = int^*(f(V))$, since W is σ^*-open in Y. Hence (c) holds.

(c)⇒(d): Let $W = f(A)$ be σ^*-open subset of Y. We have to prove that A is open in X. Consider the collection $\{V_y\}_{y\in W}$, where each $V_y = f^{-1}(y) \cap A$ and so $A = \cup_{y\in W} V_y$ and each V_y contains at least one point from each fiber $f^{-1}(y)$. Hence by (c), the set A is open in X.

(d)⇒(e): For any set A in X, let $f^\#(A)$ be σ^*-closed in Y, then by Lemma 1.1(b), $(f(A^C))^C$ is σ^*-closed in Y and so $f(A^C)$ is σ^*-open in Y. Therefore, by (d), A^C is open and so A is closed in X. Hence (e) holds.

(e)⇒(d): If $f(A)$ is σ^*-open in Y, then by Lemma 1.1(b), $f^\#(A^C)$ is σ^*-closed in Y. Therefore, by (e), A^C is closed and so A is open in X. Hence (d) holds.

(d)⇒(a): Since for any set A in X, $int^*(f(A)) \subseteq f(A)$ and so there exists $S \subseteq A$ such that $f(S) = int^*(f(A))$. This implies that $f(S)$ is σ^*-open subset of Y and by (d), S is open in X. Therefore, $S \subseteq int(A)$ and so $f(S) \subseteq f(int(A))$. Hence $int^*(f(A)) \subseteq f(int(A))$, i.e., f is strongly inversely-$\mathcal{I}$-open. □

The following corollary gives the sufficient condition for a strongly inversely-$\mathcal{I}$-open map to be pointwise-$\mathcal{I}$-continuous.

Corollary 2.1. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be strongly inversely-$\mathcal{I}$-open. Then f is pointwise-$\mathcal{I}$-continuous if $f(X)$ is σ^*-open in Y.*

Proof. Let W be open in Y. Then $f(f^{-1}(W)) = W \cap f(X)$ implies that $f(f^{-1}(W))$ is σ^*-open in Y, since $f(X)$ is σ^*-open in Y and $\sigma \subseteq \sigma^*$ and so $f^{-1}(W)$ is open in X by Theorem 2.1. Hence f is pointwise-$\mathcal{I}$-continuous. □

The proof of the following corollary is immediate and hence is omitted.

Corollary 2.2. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be surjective and strongly inversely-$\mathcal{I}$-open. Then f is pointwise-$\mathcal{I}$-continuous.*

Remark 2.2. A pointwise-$\mathcal{I}$-continuous injective map need not be strongly inversely-$\mathcal{I}$-open.

Corollary 2.3. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be strongly inversely-$\mathcal{I}$-open map and $f^{-1}(y)$ contains at least two points for all $y \in Y$. Then (X, τ) is T_1.*

Proof. Let $x \in X$ be any element, we will prove that $\{x\}$ is closed. Now $f(\{x\}^C) = Y$, since $f^{-1}(y)$ contains at least two points for each $y \in Y$. Also $f(\{x\}^C) = Y$ is σ^*-open in Y and so Theorem 2.1(c), $\{x\}^C$ is open in X. Therefore, $\{x\}$ is closed in X and hence (X, τ) is T_1. □

Corollary 2.4. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any constant map i.e. $f(x) = b$ for all $x \in X$ and for some $b \in Y$. Then*

(a) (X, τ) is discrete topological space if f is strongly inversely-$\mathcal{I}$-open and $\{b\}$ is σ^*-open in Y.

(b) f is strongly inversely-$\mathcal{I}$-open if $\{b\}$ is not σ^*-open in Y.

Proof. (a): Follows from the fact that for any subset A of X, $f(A) = \{b\}$ is σ^*-open in Y.
(b): f is vacuously strongly inversely-$\mathcal{I}$-open, since for any subset A of X, $f(A) = \{b\}$ is never σ^*-open in Y. □

For our next results we introduce strongly inversely-$\mathcal{I}$ closed maps.

Definition 2.2. A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be strongly inversely-$\mathcal{I}$-closed if for any subset A of X, A is closed in X, whenever $f(A)$ is σ^*-closed subset of Y.

Remark 2.3. (a) It can be seen easily that every strongly inversely-$\mathcal{I}$-closed map is inversely closed, since every closed subset of Y is σ^*-closed. But converse is not true.

(b) From the definition of strongly inversely-$\mathcal{I}$-closed maps, it follows immediately that if a space (X, τ) is discrete topological space then every map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is strongly inversely-$\mathcal{I}$-closed.

The following theorem gives different characterizations of strongly inversely-$\mathcal{I}$-closed maps.

Theorem 2.2. *For any map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following conditions are equivalent:*

(a) f is strongly inversely-$\mathcal{I}$-closed i.e. for each subset A of X, A is closed in X, whenever $f(A)$ is σ^*-closed subset of Y.

(b) for any subset A of X, A is open in X, whenever $f^{\#}(A)$ is σ^*-open subset of Y.

(c) if F is σ^*-closed subset of Y and $F \subseteq f(X)$, then $T = \bigcup_{y \in F} f^{-1}(y)$ (where each $f^{-1}(y)$ contains at least one point) is closed in X.

Proof. (a)⇒(b): It follows from the Lemma 1.1(b).

(b)⇒(c): Let F be σ^*-closed subset of Y such that $F \subseteq f(X)$ and T be any set consisting of at least one point from each fiber $f^{-1}(y)$, $y \in F$ then $f(T) = F$. We will prove that T is closed in X. Now, by Lemma 1.1(b) it follows that $(f^{\#}(T^C))^C = F$ is σ^*-closed and so $f^{\#}(T^C)$ is σ^*-open in Y. Therefore, (b) implies that T^C is open and so T is closed in X. Hence (c) holds.

(c)⇒(a): Let $f(A)$ be σ^*-closed subset of Y. Then by (c) $A = \cup_{y \in f(A)} f^{-1}(y) \cap A$ is closed in X. Hence (a) holds. □

The next corollary gives the sufficient condition for a strongly inversely-$\mathcal{I}$-closed map to be pointwise-$\mathcal{I}$-continuous.

Corollary 2.5. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be strongly inversely-$\mathcal{I}$-closed. Then f is pointwise-$\mathcal{I}$-continuous if $f(X)$ is σ^*-closed subset of Y.*

Proof. Proof is similar to that of Corollary 2.1 and hence is omitted. □

Corollary 2.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be surjective and strongly inversely-$\mathcal{I}$-closed. Then f is pointwise-$\mathcal{I}$-continuous.*

Remark 2.4. A pointwise-$\mathcal{I}$-continuous injective map need not be strongly inversely-$\mathcal{I}$-closed.

Corollary 2.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be strongly inversely-$\mathcal{I}$-closed map and $f^{-1}(y)$ contains at least two points for each $y \in Y$. Then (X, τ) is discrete topological space.*

Proof. Let $x \in X$ be any element then we will prove that $\{x\}$ is open in X. Now $f(\{x\}^C) = Y$, since $f^{-1}(y)$ contains at least two points for each $y \in Y$. Also $f(\{x\}^C) = Y$ is σ^*-closed in Y and so by Theorem 2.2(c), $\{x\}^C$ is closed in X. Therefore, $\{x\}$ is open in X and hence (X, τ) is discrete topological space. □

Corollary 2.8. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any constant map i.e. $f(x) = b$ for all $x \in X$ and for some $b \in Y$. Then the following holds:*

(a) (X, τ) is discrete topological space if f is strongly inversely-$\mathcal{I}$-closed and $\{b\}$ is σ^*-closed in Y.

(b) f is strongly inversely-$\mathcal{I}$-closed if $\{b\}$ is not σ^*-closed in Y.

3 Examples

The following example shows that inversely open(inversely closed) map need not be strongly inversely-$\mathcal{I}$- open(strongly inversely-$\mathcal{I}$-closed).

Example 3.1. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $Y = \{0, 1\}$, $\sigma = \{\emptyset, \{0\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 1, f(b) = 0$. So $\mathcal{J} = \{\emptyset, \{0\}\}$ and $\sigma^* = \wp(Y)$. Then it can be easily checked that f is inversely open(inversely closed) map but not strongly inversely-$\mathcal{I}$-open(strongly inversely-$\mathcal{I}$-closed). Since $f(\{a\}) = \{1\}(f(\{b\}) = \{0\})$ is σ^*-open(σ^*-closed) in Y but $\{a\}(\{b\})$ is not open(closed) in X.

The following example shows that strongly inversely-$\mathcal{I}$-closed map need not be strongly inversely-$\mathcal{I}$-open or pointwise-$\mathcal{I}$-continuous.

Example 3.2. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\mathcal{I} = \{\emptyset\}$. And $Y = \{0, 1, 2\}$, $\sigma = \{\emptyset, \{0\}, \{0, 1\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 1, f(b) = 0$. So $\mathcal{J} = \{\emptyset\}$ and $\sigma = \sigma^*$. Then $f(X)$ has no σ^*-closed subsets and therefore, f is vacuously strongly inversely-$\mathcal{I}$-closed. But f is neither strongly inversely-$\mathcal{I}$-open nor pointwise-$\mathcal{I}$-continuous, since $f(\{b\}) = \{0\}$ is open(σ^*-open) in Y, but $\{b\} = f^{-1}(\{0\})$ is not open in X.

The following Example shows that strongly inversely-$\mathcal{I}$-open map need not be strongly inversely-$\mathcal{I}$-closed or pointwise-$\mathcal{I}$-continuous.

Example 3.3. In the above Example 3.2, if we define the map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ as $f(a) = 2, f(b) = 1$. Then $f(X)$ has no τ^*-open subsets and therefore, f is vacuously strongly inversely-$\mathcal{I}$-open. But f is neither strongly inversely-$\mathcal{I}$-closed nor pointwise-$\mathcal{I}$-continuous.

The following Example shows that a pointwise-$\mathcal{I}$-continuous injective map need not be strongly inversely-$\mathcal{I}$-open or strongly inversely-$\mathcal{I}$-closed.

Example 3.4. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$. So $\tau = \wp(X)$. And $Y = \{0, 1, 2\}$, $\sigma = \{\emptyset, \{0\}, \{0, 1\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 1, f(b) = 0$. Therefore, $\mathcal{J} = \{\emptyset, \{1\}\}$ and so $\sigma^* = \{\emptyset, \{0\}, \{0, 1\}, \{0, 2\}, Y\}$. Then f is vacuously pointwise-$\mathcal{I}$-continuous. But f is neither strongly inversely-$\mathcal{I}$-open nor strongly inversely-$\mathcal{I}$-closed , since $f(\{b\}) = \{0\}(f(\{a\}) = \{1\})$ is σ^*-open(σ^*-closed) in Y, but $\{b\}(\{a\})$ is not open(closed) in X.

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